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New Composition Theorem for Weighted Stepanov-like Pseudo Almost Periodic Functions on Time Scales and Applications

Mohssine Es-saiydy and Mohamed Zitane

abstract: First, we show a new composition theorems for both Stepanov almost periodic functions and weighted Stepanov-like pseudo almost periodic functions on time scales. Next, under some suitable assumptions, we study the existence and uniqueness of weighted pseudo almost periodic solution to some first-order dynamic equations on time scales with weighted Stepanov-like pseudo almost periodic coefficients.

Key Words: Almost periodicity, composition theorems, time scales, evolution family, dynamic equation.

Contents

1. Introduction

The theory of time scales was introduced by Stefan Hilger in his PhD thesis in order to unify continuous and discrete analysis. However, since there are many other time scales than just the set of real numbers or the set of integers, one has a much more general result so unification and extension can be given as two main features of the theory of time scales. Subjects such as existence and uniqueness of solution, stability, floquet theory, periodicity and boundedness of solution can be studied more precisely and generally by utilizing dynamical systems on time scales [7]. For more details about time scales we refer the reader to [2] and references therein.

In 1926, Stepanov [15] introduced the concept of Stepanov almost periodicity which is a natural generalization of the classical notion of almost periodicity in the sense of Bohr. Since then, this notion has found several developments and has been generalised into different directions. In 2018, C. Tang and H. Li [16] presented a new extension on time scales called Stepanov-like pseudo almost periodicity on time scales. Very recently, in 2020, M. Es-saiydy and M. Zitane [6] introduced another generalization called weighted Stepanov-like pseudo almost periodicity on time scales, their approach consists of enlarging the so-called ergodic component by using the measure theory on time scales. These new functions generalize in a nature fashion the classical notion of Stepanov almost periodicity and its various extensions including the space of Stepanov-like pseudo almost periodic functions on time scales [16].

It is well known that studying the composition of two functions with special properties is important and fundamental for thorough investigations, especially nonlinear problems. In this paper, under a weaker Lipschitz condition, we generalize the composition theorem for Stepanov almost periodic functions on

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time scales given by Y.K. Li and P. Wang in [11]. Also, we recover the composition theorems established in [6,16]. The main result of this paper is Theorem (4.6), in which we clarify a new composition principle for the class of weighted Stepanov-like pseudo almost periodicity on time scales, works also for the class of Stepanov-like pseudo almost periodic functions on time scales and improves over the state-of-the-art, and has immediate applications in several problems studied in the literature.

Another aim of this work consists of using the newly established composition theorems to study the existence and uniqueness of weighted pseudo almost periodic solution to the first-order dynamic equations on time scales defined by :

$$u^{\Delta}(t) = A(t)u(t) + F(t, u(t)), \quad t \in \mathbb{T}, \tag{1.1}$$

where $A(t) : D(A(t)) \subset X \mapsto X$ is a family of closed linear operators on a Banach space X and the forcing term $F : \mathbb{T} \times X \to X$ is a weighted Stepanov-like pseudo almost periodic function.

Moreover, to study the existence and uniqueness of weighted pseudo almost periodic solution to Eq.(1.1), it is necessary to study the existence of weighted pseudo almost periodic solution to the following nonautonomous evolution equation :

$$u^{\Delta}(t) = A(t)u(t) + f(t), \quad t \in \mathbb{T}. \tag{1.2}$$

This paper is organized as follows. In the second and third sections, we give a set of basic definitions on which we will base for the rest of this paper. In Section 4, we establish some new composition theorems for Stepanov almost periodic functions and weighted Stepanov-like pseudo almost periodic functions on time scales, whiches play a crucial role in the study of the existence and uniqueness of weighted pseudo almost periodic solution to a class of evolution equations in a Banach space. In Section 5, by applying those composition theorems, we study the existence and uniqueness of weighted pseudo almost periodic solution to nonautonomous evolution equation (1.2) and to first-order dynamic equation (1.1).

2. Preliminary

In this section, we collect some preliminary facts that will be used in the sequel.

Throughout this paper $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ stand for the sets of positive integer, real and complex numbers (respectively) and $(X, \|.\|)$ stands for Banach space. Let us first recall some necessary definitions and properties of time scales.

2.1. Time scales

Definition 2.1 ([2]). Let $\mathbb{T}$ be a time scales, that is, a closed and nonempty subset of $\mathbb{R}$.

1. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ and the graininess function $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined respectively by

$$\sigma(t) = \inf\{s \in \mathbb{T} \,:\, s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} \,:\, s < t\}, \quad \mu(t) = \sigma(t) - t.$$

2. A point $t \in \mathbb{T}$ is called left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$.

3. A function $f : \mathbb{T} \to \mathbb{R}$ is called right-dense continuous or rd-continuous provided that it is continuous at all right-dense points in $\mathbb{T}$ and its left-side limits exist (finite) at left-dense points in $\mathbb{T}$. A function $f : \mathbb{T} \to \mathbb{R}$ is called continuous if and only if it is both left-dense continuous and right-dense continuous.

4. A function $p : \mathbb{T} \to \mathbb{R}$ is called regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T} \setminus \max(\mathbb{T})$. The set of all regressive and rd-continuous functions $p : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathfrak{R} = \mathfrak{R}(\mathbb{T}) = \mathfrak{R}(\mathbb{T}; \mathbb{R})$.

5. We define the set $\mathfrak{R}^+$ of all positively regressive elements by $\mathfrak{R}^+ = \mathfrak{R}^+(\mathbb{T}) = \mathfrak{R}^+(\mathbb{T}; \mathbb{R}) = \{p \in \mathfrak{R} : 1 + \mu(t)p(t) > 0 \text{ for all } t \in \mathbb{T}\}$.

6. Let $a, b \in \mathbb{T}$, with $a \leq b$, $[a,b]$, $[a,b)$, $(a,b]$ and (a,b) being the usual intervals on the real line. The intervals $[a,a)$, $(a,a]$, (a,a) are understood as the empty set, and we use the following symbols :

$$[a,b]_{\mathbb{T}} = [a,b] \cap \mathbb{T} \quad [a,b)_{\mathbb{T}} = [a,b) \cap \mathbb{T} \quad (a,b]_{\mathbb{T}} = (a,b] \cap \mathbb{T} \quad (a,b)_{\mathbb{T}} = (a,b) \cap \mathbb{T}.$$

Definition 2.2 ([2]). If $p \in \mathfrak{R}$, then we define the exponential function by :

$$e_p(t,s) = exp\left\{ \int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau \right\}, \quad \text{for} \quad s,t \in \mathbb{T},$$

with the cylinder transformation

$$\xi_h(z) = \begin{cases} \frac{\log(1+hz)}{h}, & if\ h \neq 0, \\ z, & if\ h = 0. \end{cases}$$

Definition 2.3 ([2]). If $p, q \in \mathfrak{R}$, then we define a circle plus addition $p \oplus q$ and a circle minus $\ominus p$ by

$$(p \oplus q)(t) := p(t) + q(t) + p(t)q(t)\mu(t), \forall t \in \mathbb{T} \setminus \max(\mathbb{T}), \quad \text{and} \quad \ominus p = -\frac{p}{1+\mu p}.$$

Definition 2.4 ([2]). For $f : \mathbb{T} \to X$ and $s \in \mathbb{T} \setminus \{\max \mathbb{T}\}$, $f^{\Delta}(t) \in X$ is the delta derivative of f at s if for $\varepsilon > 0$, there is a neighborhood V of s such that for $t \in V$,

$$\| f(\sigma(s)) - f(t) - f^{\Delta}(s)(\sigma(s) - t) \| < \varepsilon \mid \sigma(s) - t \mid .$$

Moreover, f is delta differentiable on $\mathbb{T}$ provided that $f^{\Delta}(s)$ exists for $s \in \mathbb{T}$.

Lemma 2.5 ([2,16]). Let $p, q \in \mathfrak{R}$, and $t, s, r \in \mathbb{T}$. Then,

1) $e_0(t,s) = 1$ and $e_p(t,t) = 1$;

2) $e_p(\sigma(t),s) = (1 + p(t)\mu(t))e_p(t,s)$;

3) $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$;

4) $e_p(t,r)e_p(r,s) = e_p(t,s)$;

5) $(e_p(t,s))^{\Delta} = p(t)e_p(t,s)$;

6) If $a, b, c \in \mathbb{T}$. Then, $\int_a^b e_p(c,\sigma(t))p(t)\Delta t = e_p(c,a) - e_p(c,b)$.

7) For $t_0 \in \mathbb{T}$, $e_{\ominus\alpha}(t_0,.)$ is increasing on $(-\infty, t_0]_{\mathbb{T}}$.

Lemma 2.6 ([2]). Assume $p \in \mathfrak{R}$, and $t_0 \in \mathbb{T}$. If $1 + \mu(t)p(t) > 0$ for all $t \in \mathbb{T}$, then $e_p(t,t_0) > 0$ for all $t \in \mathbb{T}$.

Definition 2.7 ([12]). A time scales $\mathbb{T}$ is called invariant under translations if

$$\Pi = \{\tau \in \mathbb{R} : t \pm \tau \in \mathbb{T}; \forall t \in \mathbb{T}\} \neq \{0\}.$$

Example 2.8. *The time scales* $\mathbb{T} = \bigcup_{\infty}^{+\infty}[2n, 2n+1]$ *is translation invariant. In contrary,* $\mathbb{T} = \mathbb{Z} - 3\mathbb{Z}$ *is not translation invariant.*

2.2. Measure theory on time scales

Let $F_1 = \{[t, s[_{\mathbb{T}} :\ t, s \in \mathbb{T} \text{ with } t \leq s\}$. Define a countably additive measure m_1 on F_1 by assigning to every $[t, s[_{\mathbb{T}} \in F_1$ its length, i.e;

$$m_1([t, s[_{\mathbb{T}}) = s - t.$$

Using m_1, we can generate the outer measure m_1^* on $P(\mathbb{T})$ (the power set of $\mathbb{T}$): for $E \in P(\mathbb{T})$,

$$m_1^*(E) = \begin{cases} \inf_{\mathfrak{B}} \{\sum_{i \in I_B} (s_i - t_i)\} \in \mathbb{R}^+, & \beta \notin E, \\ +\infty, & \beta \in E, \end{cases}$$

where $\beta = \sup \mathbb{T}$, and,

$$\mathfrak{B} = \{\{[t_i, s_i[_{\mathbb{T}} \in F_1\}_{i \in I_B} \ : \ I_B \subset \mathbb{N},\ E \subset \cup_{i \in I_B} [t_i, s_i[_{\mathbb{T}}\}.$$

A set $A \subset \mathbb{T}$ is called $\Delta-$measurable if for $E \subset \mathbb{T}$,

$$m_1^*(E) = m_1^*(E \cap A) + m_1^*(E \cap (\mathbb{T} \setminus A)).$$

Let

$$\mathcal{M}^*(m_1^*) = \{A,\ A \,\text{is}\, \Delta - \text{measurable subset in}\, \mathbb{T}\}.$$

Restricting m_1^* to $\mathcal{M}^*(m_1^*)$, we get the Lebesgue $\Delta-$measure, which is denoted by μ_Δ.

Definition 2.9 ([3]). $f : \mathbb{T} \to X$ is a $\Delta-$measurable function if there exists a simple function sequence $\{f_k : k \in \mathbb{N}\}$ such that $f_k(s) \to f(s)$ a.e. in $\mathbb{T}$.

Definition 2.10 ([3]). $f : \mathbb{T} \to X$ is a $\Delta-$integrable function if there exists a simple function sequence $\{f_k : k \in \mathbb{N}\}$ such that $f_k(s) \to f(s)$ a.e. in $\mathbb{T}$ and,

$$\lim_{k \to \infty} \int_{\mathbb{T}} \parallel f_k(s) - f(s) \parallel \Delta s = 0.$$

Then, the integral of f is defined as

$$\int_{\mathbb{T}} f(s) \Delta s = \lim_{k \to \infty} \int_{\mathbb{T}} f_k(s) \Delta s.$$

Definition 2.11 ([3]). For $p \geq 1$, $f : \mathbb{T} \to X$ is called locally L^p $\Delta-$integrable if f is $\Delta-$measurable and for any compact $\Delta-$measurable set $E \subset \mathbb{T}$, the $\Delta-$integral

$$\int_E \parallel f(s) \parallel^p \Delta s < \infty.$$

The set of all L^p $\Delta-$integrable functions is denoted by $L^p_{loc}(\mathbb{T}; X)$.

Theorem 2.12 ([4]). If $a, b \in \mathbb{T}$, with $a \leq b$, then,
$\mu_\Delta([a, b)) = b - a$,
$\mu_\Delta((a, b)) = b - \sigma(a)$.

Theorem 2.13 ([4]). If $a, b \in \mathbb{T} \setminus \{\max \mathbb{T}\}$, with $a \leq b$, then,
$\mu_\Delta((a, b]) = \sigma(b) - \sigma(a)$,
$\mu_\Delta([a, b]) = \sigma(b) - a$.

3. A framework for function spaces

This section is devoted to definitions and important properties of weighted Stepanov-like pseudo almost periodic functions on time scales introduced by M. Es-saiydy and M. Zitane [6].

We set,

$$K = \begin{cases} \inf\{ \mid \tau \mid ; \tau \in \mathbb{T}, \tau \neq 0\}, & if\, \mathbb{T} \neq \mathbb{R}, \\ 1, & if\, \mathbb{T} = \mathbb{R}. \end{cases}$$

Let $f \in L^p_{loc}(\mathbb{T}, X)$, for $1 \leq p < \infty$. Define :

- $\| \,.\, \|_{S^p}: L^p_{loc}(\mathbb{T}, X) \to \mathbb{R}^+$ as : $\| f \|_{S^p} = \sup_{t\in\mathbb{T}} \left(\frac{1}{K} \int_t^{t+k} \mid f(s) \mid^p \Delta s \right)^{\frac{1}{p}}$,
- $C(\mathbb{T}; X) = \{f : \mathbb{T} \to X : f \text{ is continuous}\}$,
- $C(\mathbb{T} \times X; X) = \{f : \mathbb{T} \times X \to X : f \text{ is continuous}\}$,
- $BC(\mathbb{T}; X) = \{f : T \to X : f \text{ is bounded and continuous}\}$,
- $L^p_{loc}(\mathbb{T}; X) = \{f : \mathbb{T} \to X : f \text{ is locally } L^p\ \Delta - \text{integrable}\}$.
- $BS^p(\mathbb{T}; X) = \{f \in L^p_{loc}(\mathbb{T}; X) : \| f \|_{S^p} < \infty\}$.

Next, we recall the Bochner transform for general time scales as in ([17]).

If $\mathbb{T} \neq \mathbb{R}$, we fix a left scattered point $\omega \in \mathbb{T}$, there is a unique $n_t \in \mathbb{Z}$ such that $t - n_t K \in [\omega, \omega + k)_{\mathbb{T}}$. Let

$$N_t = \begin{cases} t, & \mathbb{T} = \mathbb{R}, \\ n_t & \mathbb{T} \neq \mathbb{R}. \end{cases}$$

Definition 3.1 ([17])**.** Let $f \in BS^p(\mathbb{T}, X)$. The Bochner-like transform of f is the function $f^b : \mathbb{T}\times\mathbb{T} \to X$ defined for all $t, s \in \mathbb{T}$ by

$$f^b(t, s) = f(N_t K + s).$$

And we have

$$\| f \|_{S^p} = \| f^b \|_\infty .$$

Definition 3.2 ([16])**.** Let $f \in BS^p(\mathbb{T}, X)$ and $F \in BS^p(\mathbb{T} \times X, X)$.

i) A function f is called S^p-almost periodic on $\mathbb{T}$ if for every $\varepsilon > 0$, the $\varepsilon-$translation set of f :

$$T(f, \varepsilon) = \{\tau \in \Pi; \| f(t + \tau) - f(t) \|_{S^p} < \varepsilon, \ \forall t \in \mathbb{T}\}$$

is relatively dense in Π. The space of all such functions is denoted by $S^pAP(\mathbb{T}, X)$.

ii) A function $F : \mathbb{T} \times \Omega \to X$ with $\Omega \subset X$ is called S^p-almost periodic in $t \in \mathbb{T}$, if $F(.,u) \in S^pAP(\mathbb{T}, X)$ uniformly for each $u \in S$, where S is an arbitrary compact subset of X. That is, for $\varepsilon > 0$, $\cap_{u\in S} T(F(., u), \varepsilon)$ is relatively dense in Π. Denote the set of all such functions by $S^pAP(\mathbb{T}\times X, X)$.

Let $\mathbb{U}$ denote the collection of functions (weights) $\mu : \mathbb{T} \to (0, \infty)$, which are locally integrable over $\mathbb{T}$ such that $\mu > 0$ almost everywhere. Let $\mu \in \mathbb{U}$, for $r \in \Pi$ with $r > 0$, we denote

$$\mu(Q_r) = \int_{Q_r} \mu(t)\Delta t,$$

where $Q_r = [t_0 - r, t_0 + r]_{\mathbb{T}}$ $\left(t_0 = \min\{[0, \infty)_{\mathbb{T}}\}\right)$. Consequently, we define the space of weights by

$$\mathcal{M} = \left\{ \mu \in \mathbb{U} : \inf_{t\in\mathbb{T}} \mu(t) > 0, \lim_{t\to\infty} \mu(Q_r) = \infty \right\}.$$

In addition, we define the set of weights $\mathbb{U}_b$ by

$$\mathbb{U}_b = \left\{ \mu \in \mathcal{M} : \sup_{t\in\mathbb{T}} \mu(t) < \infty \right\}.$$

It's clear that

$$\mathbb{U}_b \subset \mathcal{M} \subset \mathbb{U}.$$

Throughout this paper, we fix $1 \leq p < \infty$ and $\mu \in \mathcal{M}$ such that : for all $\tau \in \Pi$,

$$\overline{\lim_{|t|\to\infty}} \frac{\gamma(t+\tau)}{\mu(t)} < \infty,$$

and,

$$\overline{\lim_{|t|\to\infty}} \frac{\gamma(Q_{t+\tau})}{\gamma(Q_t)} < \infty.$$

Definition 3.3 ([6]). A function $f \in BS^p(\mathbb{T}, X)$ is said to be weighted ergodic in the sense of Stepanov on $\mathbb{T}$ (or S^p-weighted ergodic) if :

$$\lim_{r\to+\infty} \frac{1}{\mu(Q_r)} \int_{Q_r} \left(\frac{1}{K} \int_{[t,t+K]_{\mathbb{T}}} \| f(s) \|^p \Delta s \right)^{\frac{1}{p}} \mu(t)\Delta t = \lim_{r\to+\infty} \frac{1}{\mu(Q_r)} \int_{Q_r} \| f^b(t) \|_\infty \mu(t)\Delta t = 0.$$

The space of all such functions will be denoted by $WS^pPAP_0(\mathbb{T}, X, \mu)$.

In other words, a function $f \in BS^p(\mathbb{T}, X)$ is said to be weighted ergodic in the sense of Stepanov on $\mathbb{T}$, if its Bochne-like transform f^b is weighted ergodic on $\mathbb{T}$, i.e :

$$f \in WS^pPAP_0(\mathbb{T}, X, \mu) \text{ if and only if } f^b \in PAP_0(\mathbb{T}, BS^p(\mathbb{T}, \mathbb{X}), \mu).$$

Now, we define the space of weighted Stepanov-like pseudo almost periodic functions on $\mathbb{T}$ as follows,

Definition 3.4 ([6]). A function $f \in BS^p(\mathbb{T}, X)$ is said to be weighted Stepanov-like pseudo almost periodic on $\mathbb{T}$ or briefly S^p-weighted pseudo almost periodic if f is written in the following form :

$$f = g + \phi,$$

where $g \in S^pAP(\mathbb{T}, X)$ and $\phi \in WS^pPAP_0(\mathbb{T}, X, \mu)$. The space of all such functions will be denoted by $WS^pPAP(\mathbb{T}, X, \mu)$.

Definition 3.5 ([6]). A function $f : \mathbb{T} \times X \to X$ such that $f(., u) \in BS^p(\mathbb{T}, X)$ for each $u \in X$ is said to be weighted Stepanov-like pseudo almost periodic or briefly S^p-weighted pseudo almost periodic if f is written in the following form:

$$f = g + \phi,$$

where $g \in S^pAP(\mathbb{T} \times X, X)$ and $\phi \in WS^pPAP_0(\mathbb{T} \times X, X, \mu)$. The space of all such functions will be denoted by $WS^pPAP(\mathbb{T} \times X, X, \mu)$.

Remark 3.6 ([6]). A function f is said to be weighted Stepanov-like pseudo almost periodic if and only if $f^b \in PAP(\mathbb{T}, BS^p(\mathbb{T}, \mathbb{X}), \mu)$. Consequently,

$$WS^pPAP(\mathbb{T}, X, \mu) = AP(\mathbb{T}, BS^p(\mathbb{T}, \mathbb{X})) + PAP_0(\mathbb{T}, BS^p(\mathbb{T}, \mathbb{X}), \mu).$$

Lemma 3.7 ([6]). 1) If $h, g \in WS^pPAP(\mathbb{T}, X, \mu)$, then $h + g, hg \in WS^pPAP(\mathbb{T}, X, \mu)$.

2) If $h \in WS^pPAP(\mathbb{T}, X, \mu)$ and $g \in S^pAP(\mathbb{T}, X)$, then $hg \in WS^pPAP(\mathbb{T}, X, \mu)$.

Proposition 3.1 ([6]). The decomposition of S^p- weighted pseudo almost periodic functions is unique.

Proposition 3.2 ([6]). $(WS^pPAP(\mathbb{T}, X, \mu), \|\|_{S^p})$ is a Banach space.

Theorem 3.8. If $f \in WS^pPAP(\mathbb{T}, X, \mu)$, then $f(. - \alpha) \in WS^pPAP(\mathbb{T}, X, \mu)$, for all $\alpha \in \Pi$.

4. Main results

This section is devoted to study the composition theorems of Stepanov almost periodic functions and of weighted Stepanov-like pseudo almost periodic functions on time scales. Our composition theorems generalize some known results.

In the rest of this paper, we need the following assumption:

$(\mathbb{A}_1)$: There exists a nonnegative function $L_f(.) \in BS^p(\mathbb{T}, \mathbb{X})$ such that :

$$\| f(t,x) - f(t,y) \| \leq L_f(t) \| x - y \| \text{ for all } t \in \mathbb{T} \text{ and } x, y \in X.$$

The space of all such functions will be denoted by $\mathcal{L}^p(\mathbb{T}, X)$.

4.1. A general composition theorem of Stepanov almost periodic functions on $\mathbb{T}$

In this subsection, we will discuss the composition theorem of Stepanov almost periodicity on time scales.

Definition 4.1. For any compact set $\Omega \subset X$, we denote by $S^p_\Omega AP(\mathbb{T} \times X, X)$ the set of all the functions $f(.,u) \in BS^p(\mathbb{T}, X)$ such that for $\varepsilon > 0$ we have $\cap_{u\in\Omega} I(f(.,u), \varepsilon)$ is relatively dense in Π. Where,

$$I(f(.,u), \varepsilon) = \left\{ \tau \in \Pi; \sup_{t\in\mathbb{T}} \left(\frac{1}{K} \int_{[t,t+K]_{\mathbb{T}}} \sup_{u\in\Omega} \|f(s+\tau,u) - f(s,u)\|^p \, \Delta s \right)^{\frac{1}{p}} < \varepsilon, \, \forall t \in \mathbb{T} \right\}.$$

By using a similar idea to that of (lemma 6, [17]), we can easily prove the following lemma.

Lemma 4.2. *If $u \in AP(\mathbb{T}, X)$ and $f \in S^pAP(\mathbb{T} \times X, X)$. Then, for $\varepsilon_1, \varepsilon_2 > 0$*

$$\mathit{\Lambda} = T(u, \varepsilon_1) \cap (\cap_{x\in\Omega} T(f(.,x), \varepsilon_2)),$$

is relatively dense in $\mathit{\Pi}$.

To prove the next composition theorem, we also need the following lemma :

Lemma 4.3. *Let Ω is an arbitrary compact subset of X. Assume that :*

1) f satisfy $(\mathbb{A}_1)$,

2) $f \in S^pAP(\mathbb{T} \times X, X)$.

Then, $f \in S^p_\Omega AP(\mathbb{T} \times X, X)$.

Proof. Since $f \in S^pAP(\mathbb{T} \times X, X)$, then $f \in BS^p(\mathbb{T}, X)$. As Ω is an arbitrary compact subset of X, so, for all $\varepsilon > 0$ there exists $\{x_i\}_{i=1}^{i=n} \subset \Omega$ such that

$$\Omega \subset \cup_{i=1}^{i=n} B\left(x_i, \frac{\varepsilon}{4 \| L \|_{S^p}}\right),$$

where $B(x,r)$ with $x \in X$ and $r > 0$ denotes an open ball with radius r and center x. Let

$$\mathit{\Lambda} = T\left(u, \frac{\varepsilon}{\| L \|_{S^p}}\right) \cap \left(\cap_{y\in\Omega} T\big(f(.,y), \frac{\varepsilon}{2n}\big)\right).$$

Then by Lemma (4.2), we get $\mathit{\Lambda}$ is relatively dense in $\mathit{\Pi}$. Let $u \in \Omega$, there exist $j(u) \in \{1,...,n\}$ such that $u \subset B\left(x_{j(u)}, \frac{\varepsilon}{4\|L\|_{S^p}}\right)$.
Therefore, for all $\tau \in \mathit{\Lambda}$, $t \in \mathbb{T}$ and $u \in \Omega$ we have,

$$\begin{aligned} \| f(s+\tau, u(t)) - f(s, u(t)) \| \leq & \| f(s+\tau, u(t)) - f(s+\tau, x_{j(u)}) \| \\ & + \| f(s+\tau, x_{j(u)}) - f(s, x_{j(u)}) \| \\ & + \| f(s, x_{j(u)}) - f(s, u(t)) \| . \end{aligned}$$

By assumption $(\mathbb{A}_1)$, there exists a function $L_f \in BS^p(\mathbb{T}, X)$ such that :

$$\begin{aligned} \| f(s+\tau, u(t)) - f(s, u(t)) \| \leq L_f(s+\tau) & \frac{\varepsilon}{4 \| L \|_{S^p}} \\ & + \sum_{j=1}^{j=n} \| f(s+\tau, x_{j(u)}) - f(s, x_{j(u)}) \| \\ & + L_f(s) \frac{\varepsilon}{4 \| L \|_{S^p}}. \end{aligned}$$

Since, $BS^p(\mathbb{T}, X)$ is invariant translation we get, $L(.), L(.+\tau) \in BS^p(\mathbb{T}, X)$ and $\| L(.+\tau) \|_{S^p} = \| L(.) \|_{S^p}$. Then for each $u \in \Omega$, we have,

$$\begin{aligned} \sup_{u\in\Omega} \| f(s+\tau, u(s)) - f(s, u(s)) \| \leq 2L_f(s) & \frac{\varepsilon}{4 \| L \|_{S^p}} \\ & + \sum_{j=1}^{j=n} \| f(s+\tau, x_{j(u)}) - f(s, x_{j(u)}) \| . \end{aligned}$$

Moreover, by Minkowski's inequality, we obtain,

$$\begin{aligned} & \left(\tfrac{1}{K} \int_{[t,t+K]_{\mathbb{T}}} \left(\sup_{u\in\Omega} \|f(s+\tau, u(s+\tau)) - f(s, u(s))\|^p \right) \Delta s \right)^{\frac{1}{p}} \\ & \leq \left(\frac{1}{K} \int_{[t,t+K]_{\mathbb{T}}} \| L_f(s)) \|^p \, \Delta s \right)^{\frac{1}{p}} \frac{\varepsilon}{2 \| L \|_{S^p}} \\ & + \sum_{j=1}^{j=n} \left(\frac{1}{K} \int_{[t,t+K]_{\mathbb{T}}} \| f(s+\tau, x_{j(u)}) - f(s, x_{j(u)}) \|^p \, \Delta s \right)^{\frac{1}{p}}, \\ & \leq \|L_f\|_{S^p} \frac{\varepsilon}{2\|L_f\|_{S^p}} + n \frac{\varepsilon}{2n} = \varepsilon. \end{aligned}$$

It follows that $\Lambda \subset \cap_{u\in\Omega} I(f(., u), \varepsilon)$. Hence, $\cap_{u\in\Omega} I(f(., u), \varepsilon)$ is relatively dense in Π. Finally, $f \in S^p_\Omega AP(\mathbb{T} \times X, X)$. ☐

Next, we are ready to state our main new composition theorem of Stepanov almost periodicity on time scales.

Theorem 4.4. Let f be a function such that $f \in S^p AP(\mathbb{T} \times X, X)$. Assume that :

1) $f \in \mathcal{L}^r(\mathbb{T} \times X, X)$, with $r \geq \max\{p, \frac{p}{p-1}\}$,

2) $u \in S^p AP(\mathbb{T}, X)$ and $\Omega = \overline{u(\mathbb{T})}$ is compact in X.

Then, there exists $q \in [1, p)$ such that $f(., u(.)) \in S^q AP(\mathbb{T}, X)$.

Proof. We have $r \geq \frac{p}{p-1}$, then, there exist $q \in [1, p)$ such that $r = \frac{pq}{p-q}$.
We pose :

$$q' = \frac{p}{q}; \quad p' = \frac{p}{p - q'}.$$

So, $p', q' > 1$ and $\frac{1}{p'} + \frac{1}{q'} = 1$.
By $(\mathbb{A}_1)$ we will get, for all $t \in \mathbb{T}$,

$$\|f(t, u(t))\|_{S^q} \leq \left(\frac{1}{K} \int_{[t,t+K]_{\mathbb{T}}} \|f(s, u(s)) - f(s, 0)\|^q \Delta s \right)^{\frac{1}{q}} + \|f(t, 0)\|_{S^q},$$

$$\leq \left(\frac{1}{K}\int_{[t,t+K]_{\mathbb{T}}} L_f^q(s)\|u(s)\|^q\Delta s\right)^{\frac{1}{q}} + \|f(t,0)\|_{S^q}.$$

Since, $\frac{q}{r}+\frac{q}{p}=1$, from Hölder's inequality we get,

$$\begin{aligned}\| f(t,u(t)) \|_{S^q} &\leq \left(\frac{1}{K}\int_{[t,t+K]_{\mathbb{T}}} L_f^r(s)\Delta s\right)^{\frac{1}{r}}.\left(\frac{1}{K}\int_{[t,t+K]_{\mathbb{T}}} \| u(s) \|^p \Delta s\right)^{\frac{1}{p}}\\ &+ \| f(t,0) \|_{S^q},\\ &\leq \| L_f \|_{S^r} . \| u \|_{S^p} + \| f(t,0) \|_{S^q},\\ &< \infty.\end{aligned}$$

Which gives that $f \in BS^q(\mathbb{T},X)$.
Now, it remains to show that $f(.,u(.)) \in S^qAP(\mathbb{T},X)$. Indeed, by Lemma (4.3) we have, $f \in S^p_\Omega AP(\mathbb{T}\times X,X)$. Then, for all $\varepsilon >$ and $t \in \mathbb{T}$ we set :

$$F = I\Big(u(.+\tau), \frac{\varepsilon}{2\|L_f\|_{S^r}}\Big) \cap \Big(\cap_{y\in\Omega} I(f(.+\tau,y),\frac{\varepsilon}{2})\Big).$$

So, by Lemma (4.2) we have F is relatively dense in Π. Thus, for all $\tau \in F$ and $t \in \mathbb{T}$ we obtain,

$$\left(\tfrac{1}{K}\textstyle\int_{[t,t+K]}\big(\|f(s+\tau,u(s+\tau)) - f(s,u(s))\|^q\big)\Delta s\right)^{\frac{1}{q}}$$

$$\begin{aligned}&\leq \left(\frac{1}{K}\int_{[t,t+K]_{\mathbb{T}}} L_f^q(s+\tau)\|u(s+\tau)-u(s)\|^q\Delta s\right)^{\frac{1}{q}}\\ &+\left(\frac{1}{K}\int_{[t,t+K]_{\mathbb{T}}} (\|f(s+\tau,u(s)) - f(s,u(s))\|^q)\Delta s\right)^{\frac{1}{q}}.\end{aligned}$$

Therefore, by Hölder's inequality we get,

$$\left(\tfrac{1}{K}\textstyle\int_{[t,t+K]_{\mathbb{T}}} \| f\big(s+\tau,u(s+\tau)\big) - f\big(s,u(s)\big) \|^q \Delta s\right)^{\frac{1}{q}}$$

$$\begin{aligned}&\leq\left(\frac{1}{K}\int_{[t,t+K]_{\mathbb{T}}} L_f^r(s+\tau)\Delta s\right)^{\frac{1}{r}}.\left(\frac{1}{K}\int_{[t,t+K]_{\mathbb{T}}} \| u(s+\tau)-u(s) \|^p \Delta s\right)^{\frac{1}{p}}\\ &+\left(\frac{1}{K}\int_{[t,t+K]_{\mathbb{T}}} \| f\big(s+\tau,u(s)\big) - f\big(s,u(s)\big) \|^q \Delta s\right)^{\frac{1}{q}},\\ &\leq \| L_f \|_{S^r} .\frac{\varepsilon}{2 \| L \|_{S^r}} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

It follows that $F \subset T(f(.,u),\varepsilon)$, which implies that $T(f(.,u),\varepsilon)$ is relatively dense in Π. Finally, $f(.,u(.)) \in S^qAP(\mathbb{T},X)$. □

4.2. A general composition theorem of weighted Stepanov-like pseudo almost periodic functions on $\mathbb{T}$

In this subsection, we will make further study on the composition theorem of weighted Stepanov-like pseudo almost periodic functions on time scales.

To obtain the composition theorem of weighted Stepanov-like pseudo almost periodic functions, we also need the following lemma.

Lemma 4.5. If Ω is an arbitrary compact subset of X and $f \in \mathcal{L}^p(\mathbb{T}\times X,X)\cap S^pPAP_0(\mathbb{T}\times X,X,\mu)$. Then, $\check{f} \in PAP_0(\mathbb{T},\mathbb{R},\mu)$. Where,

$$\check{f}(t) = \left\|\sup_{u\in\Omega}\|f(.,u(.))\|\right\|_{S^p}.$$

Proof. Since Ω is an arbitrary compact subset of X, for all $\varepsilon > 0$ there exist $\{x_i\}_{i=1}^{i=n} \subset \Omega$ such that

$$\Omega \subset \cup_{i=1}^{i=n} B\left(x_i, \frac{\varepsilon}{2\|L\|_{S^p}}\right),$$

where, $B(x,r)$ with $x \in X$ and $r > 0$ denotes an open ball with radius r and center x. Then for all $u \in \Omega$, there exist x_i such that

$$\|f(.,u)\| \leq \|f(.,u) - f(.,x_i)\| + \|f(.,x_i)\|.$$

As $f \in \mathcal{L}^p(\mathbb{T} \times X, X)$, we obtain,

$$\sup_{u\in\Omega} \|f(.,u(.))\|_{S^p} \leq L_f(.).\frac{\varepsilon}{2.\|L\|_{S^p}} + \sum_{i=1}^{i=n} \|f(.,x_i)\|,$$

it follows that

$$\check{f}(t) = \left\| \sup_{u\in\Omega} \|f(.,u(.))\| \right\|_{S^p} \leq \|L_f\|_{S^p}.\frac{\varepsilon}{2.\|L\|_{S^p}} + \sum_{i=1}^{i=n} \|f(.,x_i)\|_{S^p}, \quad t \in \mathbb{T}. \tag{4.1}$$

Since $f \in S^pPAP_0(\mathbb{T}, X, \mu)$, what gives that for all $\varepsilon > 0$,

$$\frac{1}{\mu(Q_r)} \int_{Q_r} \|f^b(t,x_i)\|_\infty \mu(t)\Delta t \leq \frac{\varepsilon}{2n}; \quad i = 1, ..., n. \tag{4.2}$$

So, we use inequality (4.1) we find that,

$$\frac{1}{\mu(Q_r)} \int_{Q_r} \check{f}(t)\mu(t)\Delta t \leq \|L_f\|_{S^p}.\frac{\varepsilon}{2}\|L_f\|_{S^p} + n\frac{\varepsilon}{2n} = \varepsilon.$$

Finally, we have $\check{f} \in PAP_0(\mathbb{T}, \mathbb{R}, \mu)$. □

Next, we establish a new composition theorem for weighted Stepanov-like pseudo almost periodic functions on time scales, which generalizes (Theorem (3.23), [6])

Theorem 4.6. Let f be a function such that $f = h + \varphi \in WS^pPAP(\mathbb{T} \times X, X, \mu)$. Assume that :

1) $f, h \in \mathcal{L}^r(\mathbb{T} \times X, X)$, with $r \geq \max\{p, \frac{p}{p-1}\}$.

2) $u = x+y \in WS^pPAP(\mathbb{T}, X, \mu)$, where $x \in S^pAP(\mathbb{T}, X)$, $\overline{x(\mathbb{T})}$ compact and $y \in WS^pPAP_0(\mathbb{T}, X, \mu)$.

Then, there exists $q \in [1, p)$ such that $f(., u(.)) \in WS^qPAP(\mathbb{T}, X, \mu)$.

Proof. Let p, q, p' and q' as in Theorem (4.4) and $f = h + \varphi \in WS^pPAP(\mathbb{T} \times X, X, \mu)$ where $h \in S^pAP(\mathbb{T} \times X, X)$ and $\varphi \in WS^pPAP_0(\mathbb{T} \times X, X, \mu)$. Suppose that, for all $t \in \mathbb{T}$:

$$\psi(t) = h(t, x(t)), \ \varpi(t) = f(t, u(t)) - f(t, x(t)), \quad \Gamma(t) = \varphi(t, x(t)).$$

According to Theorem (4.4) we have, $\psi \in S^qAP(\mathbb{T}, X)$. So, show that $\varpi, \Gamma \in WS^qPAP_0(\mathbb{T}, X, \mu)$, i.e. it remains to show that

$$\varpi^b, \Gamma^b \in PAP_0(\mathbb{T}, BS^q(\mathbb{T}, X), \mu).$$

Indeed, for ϖ^b we have,

$$\begin{aligned} \frac{1}{\mu(Q_r)} \int_{Q_r} \| \varpi(t) \|_{S^q} \mu(t)\Delta t &= \frac{1}{\mu(Q_r)} \int_{Q_r} \left(\frac{1}{K} \int_{[t,t+K]_\mathbb{T}} \| \varpi(s) \|^q \Delta s \right)^{\frac{1}{q}} \mu(t)\Delta t, \\ &\leq \frac{1}{\mu(Q_r)} \int_{Q_r} \left(\frac{1}{K} \int_{[t,t+K]_\mathbb{T}} L_f^q(s) \| y(s) \|^q \Delta s \right)^{\frac{1}{q}} \mu(t)\Delta t, \\ &\leq \| L_f \|_{S^r} \frac{1}{\mu(Q_r)} \int_{Q_r} \| y^b(t) \|_\infty \mu(t)\Delta t. \end{aligned}$$

Since $y^b \in PAP_0(\mathbb{T}, BS^p(\mathbb{T},X),\mu)$, therefore, $\frac{1}{\mu(Q_r)}\int_{Q_r}\|y^b(t)\|_\infty \mu(t)\Delta t \to 0$ as $r\to+\infty$. which means that

$$\lim_{r\to+\infty}\frac{1}{\mu(Q_r)}\int_{Q_r}\|\varpi(t)\|_{S^q}\mu(t)\Delta t = 0.$$

Finally, $\varpi \in WS^qPAP_0(\mathbb{T},X,\mu)$.
For Γ^b, we can write φ in the following form :

$$\varphi = f - h \in \mathcal{L}^r(\mathbb{T}\times X, X) \subset \mathcal{L}^p(\mathbb{T}\times X, X),$$

so, according to Lemma (4.5) we obtain,

$$\lim_{r\to+\infty}\frac{1}{\mu(Q_r)}\int_{Q_r}\left\|\sup_{v\in\Omega}\|\varphi(t+.,v(.))\|\right\|_{S^p} = 0,$$

it follows that

$$\begin{aligned}\frac{1}{\mu(Q_r)}\int_{Q_r}\parallel\Gamma(t)\parallel_{S^q}\mu(t)\Delta t &= \frac{1}{\mu(Q_r)}\int_{Q_r}\left(\frac{1}{K}\int_{[t,t+K]_{\mathbb{T}}}\parallel\varphi(s,x(s))\parallel^p\Delta s\right)^{\frac{1}{p}}\mu(t)\Delta t,\\ &\le \frac{1}{\mu(Q_r)}\int_{Q_r}\left(\frac{1}{K}\int_{[t,t+K]_{\mathbb{T}}}\left(\sup_{v\in\Omega}\parallel\varphi(s,v)\parallel\right)^p\Delta s\right)^{\frac{1}{p}}\mu(t)\Delta t,\end{aligned}$$

since, $\lim_{r\to+\infty}\frac{1}{\mu(Q_r)}\int_{Q_r}\left(\frac{1}{K}\int_{[t,t+K]_{\mathbb{T}}}\left(\sup_{v\in\Omega}\parallel\varphi(s,v)\parallel\right)^p\Delta s\right)^{\frac{1}{p}}\mu(t)\Delta t = 0$,
then,

$$\lim_{r\to+\infty}\frac{1}{\mu(Q_r)}\int_{Q_r}\|\Gamma(t)\|_{S^q}\mu(t)\Delta t = 0.$$

We conclude that

$$\Gamma \in S^qPAP_0(\mathbb{T},X,\mu).$$

Which finishes the proof. □

5. Existence of weighted pseudo almost periodic solution

The aim of this section is to show the existence and uniqueness of a weighted pseudo almost periodic solution to the first-order dynamic equation (1.1) on time scales.

Definition 5.1. *([2])*An evolution family $U(t,s)$ is called hyperbolic or has exponential dichotomy if there are projections $P(t)$ for $t\in\mathbb{T}$, being uniformly bounded and strongly continuous in t, and constants $C,\alpha>0$ such that :

1) $U(t,s)P(s) = P(t)U(t,s)$ for all $t\ge s$,
2) The restriction $U_Q(t.,s) : Q(s)\to Q(t)X$ is invertible for all $t\ge s$ such that $t,s\in\mathbb{T}$ and we set $U_Q(s.,t) = U_Q(t.,s)^{-1}$,
3) $\|U(t,s)P(s)\| \le Ce_{\ominus\alpha}(t,s)$ for all $t\ge s$ and $\|U_Q(s.,t)Q(s)\| \le Ce_{\ominus\alpha}(t,s)$, for all $s\ge t$ where $Q = I-P$.

Definition 5.2. *([2])* We define the function of Green as a :

$$\|G(t,s)\| = \begin{cases} U(t,s)P(s), & t\ge s \quad t,s\in\mathbb{T},\\ -U_Q(t.,s)Q(s), & t<s \quad t,s\in\mathbb{T},\end{cases}$$

We add the following assumptions :

$(\mathbb{A}_2):$ $f=g+h\in WS^pPAP(\mathbb{T}\times X,X,\mu)\cap\mathcal{L}^r(\mathbb{T}\times X,X)$, where $g\in S^pAP(\mathbb{T}\times X,X)\cap\mathcal{L}^r(\mathbb{T}\times X,X)$ and $h\in WS^pPAP_0(\mathbb{T}\times X,X,\mu)\cap\mathcal{L}^r(\mathbb{T}\times X,X)$, with :

$$r\geq\max\{p,\tfrac{p}{p-1}\}.$$

$(\mathbb{A}_3):$ The evolution family $(U(t,s))_{t\geq s}$ generated by $A(t)$ has an exponential dichotomy with constants $C>0$, $\alpha>0$ and dichotomy projections $P(t)$ for all $t\in\mathbb{T}$ and and Green's function $G(t,s)$.

$(\mathbb{A}_4):$ $G(t,s)$ is bi-almost periodic.

Definition 5.3. *A mild solution to equation (1.2) is a continous function $u:\mathbb{T}\to X$ satisfying*

$$u(t)=U(t,\gamma)u(\gamma)+\int_\gamma^t U\left(t,\sigma\left(s\right)\right)f(s)\Delta s, \tag{5.1}$$

for all $\gamma,t\in\mathbb{T}$ and $t\geq\gamma$.

Now, we can establish the following results.

Theorem 5.4. Suppose that $(\mathbb{A}_3)$-$(\mathbb{A}_4)$ hold. If $f=g+h\in WS^qPAP(\mathbb{T}\times X,X,\mu)\cap C(\mathbb{T},X)$, then the equation (1.2) has a unique weighted pseudo almost periodic solution given by:

$$u(t)=\int_{-\infty}^t U(t,\sigma(s))P(s)f(s)\Delta s-\int_t^{+\infty}U_Q(t,\sigma(s))Q(s)f(s)\Delta s.$$

Proof. **Step 1 :** To show that u satisfies (5.1) for all $\gamma,t\in\mathbb{T}$ and $t\geq\gamma$. For that we pose,

$$u(\gamma)=\int_{-\infty}^\gamma U(\gamma,\sigma(s))P(s)f(s)\Delta s-\int_\gamma^{+\infty}U_Q(\gamma,\sigma(s))Q(s)f(s)\Delta s. \tag{5.2}$$

Multiplying both sides of (5.2) by $U(t,\gamma)$ therefore,

$$\begin{aligned}U(t,\gamma)u(\gamma)&=\int_{-\infty}^\gamma U(t,\sigma(s))P(s)f(s)\Delta s-\int_\gamma^{+\infty}U_Q(t,\sigma(s))Q(s)f(s)\Delta s,\\&=\int_{-\infty}^t U(t,\sigma(s))P(s)f(s)\Delta s-\int_\gamma^t U(t,\sigma(s))P(s)f(s)\Delta s\\&\quad-\int_t^{+\infty}U_Q(t,\sigma(s))Q(s)f(s)\Delta s-\int_\gamma^t U_Q(t,\sigma(s))Q(s)f(s)\Delta s,\\&=u(t)-\int_\gamma^t U\left(t,\sigma\left(s\right)\right)f(s)\Delta s.\end{aligned}$$

Thus, u is a mild solution to Eq.(1.2).
Now let,

$$\begin{aligned}u\left(t\right)=&\int_{-\infty}^t U\left(t,\sigma\left(s\right)\right)P\left(s\right)g\left(s\right)\Delta s+\int_{-\infty}^t U\left(t,\sigma\left(s\right)\right)P\left(s\right)h\left(s\right)\Delta s\\&+\int_{+\infty}^t U_Q\left(t,\sigma\left(s\right)\right)Q\left(s\right)g\left(s\right)\Delta s+\int_{+\infty}^t U_Q\left(t,\sigma\left(s\right)\right)Q\left(s\right)h\left(s\right)\Delta s.\end{aligned}$$

We set,

$$\begin{aligned}\Phi\left(t\right)&=\int_{-\infty}^t U\left(t,\sigma\left(s\right)\right)P\left(s\right)g\left(s\right)\Delta s+\int_{+\infty}^t U_Q\left(t,\sigma\left(s\right)\right)Q\left(s\right)g\left(s\right)\Delta s,\\\Gamma\left(t\right)&=\int_{-\infty}^t U\left(t,\sigma\left(s\right)\right)P\left(s\right)h\left(s\right)\Delta s+\int_{+\infty}^t U_Q\left(t,\sigma\left(s\right)\right)Q\left(s\right)h\left(s\right)\Delta s.\end{aligned}$$

Step 2 : Let us show that $\Phi(.) \in AP(\mathbb{T}, X)$. In fact, for $t \in \mathbb{T}$ and $n = 1, 2, 3...$ we get

$$\Phi_n(t) = \int_{t-nK}^{t-(n-1)K} U\left(t, \sigma\left(s\right)\right) P\left(s\right).g\left(s\right) \Delta s + \int_{t+nK}^{t+(n-1)K} U_Q\left(t, \sigma\left(s\right)\right) Q\left(s\right).g\left(s\right) \Delta s.$$

By Lemma (2.5, (7)), we have

$$e_{\ominus\alpha}\left(t, \sigma\left(t\right) - (n-1)K\right) \leq e_{\ominus\alpha}(t, \sigma(t)) = 1 + \alpha\mu(t) \leq 1 + \alpha\bar{\mu},$$

where $\bar{\mu} = \sup_{t\in\mathbb{T}} \mu(t)$. Let $q^{'} > 1$ such that $\frac{1}{q^{'}} + \frac{1}{p} = 1$. Thus,

$$\begin{aligned}
\| \Phi_n(t) \| &\leq C \int_{t-nK}^{t-(n-1)K} e_{\ominus\alpha}\left(t, \sigma\left(s\right)\right) g(s)\Delta s \\
&+ C \int_{t+(n-1)K}^{t+nK} e_{\alpha}\left(t, \sigma\left(s\right)\right) g(s)\Delta s, \\
&\leq Ce_{\ominus\alpha}\left(t, \sigma\left(t\right) - (n-1)K\right) \int_{t-nK}^{t-(n-1)K} g(s)\Delta s \\
&+ Ce_{\alpha}\left(t, \sigma\left(t\right) + nK\right) \int_{t+(n-1)K}^{t+nK} g(s)\Delta s, \\
&\leq C(1+\alpha\bar{\mu}) \int_{t-nK}^{t-(n-1)K} g(s)\Delta s + C(1+\alpha\bar{\mu}) \int_{t+(n-1)K}^{t+nK} g(s)\Delta s.
\end{aligned}$$

Hölder's inequality implies that

$$\begin{aligned}
\| \Phi_n(t) \| &\leq C(1+\alpha\bar{\mu})K^{\frac{1}{q^{'}}} \left(\int_{t-nK}^{t-(n-1)K} | g(s) |^p \Delta s\right)^{\frac{1}{p}} \\
&+ C(1+\alpha\bar{\mu})K^{\frac{1}{q^{'}}} \left(\int_{t+(n-1)K}^{t+nK} | g(s) |^p \Delta s\right)^{\frac{1}{p}}, \\
&\leq C(1+\alpha\bar{\mu})K \left(\frac{1}{K}\int_{t-nK}^{t-(n-1)K} | g(s) |^p \Delta s\right)^{\frac{1}{p}} \\
&+ C(1+\alpha\bar{\mu})K \left(\frac{1}{K}\int_{t+(n-1)K}^{t+nK} | g(s) |^p \Delta s\right)^{\frac{1}{p}}, \\
&\leq C(1+\alpha\bar{\mu})K \| g \|_{S^p} + C(1+\alpha\bar{\mu})K \| g \|_{S^p}, \\
&\leq 2C(1+\alpha\bar{\mu})K \| g \|_{S^p}.
\end{aligned}$$

As a consequence, the series $\sum_{n=1}^{\infty} \Phi_n(t)$ is uniformly convergent on $\mathbb{T}$.
Further by condition $(\mathbb{A}_4)$, for $\varepsilon_1, \varepsilon_2 > 0$, the following sets

$$\Psi(\varepsilon_1) = \left\{\tau \in \Pi : \| U\left(t+\tau, s+\tau\right) P\left(s+\tau\right) - U\left(t, s\right) P\left(s\right) \| < \varepsilon_1,\ t, s \in \mathbb{T} :\ t \geq \sigma\left(s\right)\right\},$$

$$\Sigma(\varepsilon_2) = \left\{\tau \in \Pi : \| U_Q\left(t+\tau, s+\tau\right) Q\left(s+\tau\right) - U_Q\left(t, s\right) Q\left(s\right) \| < \varepsilon_2,\ t, s \in \mathbb{T} :\ t \geq \sigma\left(s\right)\right\},$$

are relatively dense in Π.
So, $F_1 = \Psi\left(\frac{\varepsilon}{4K\|g\|_{S^p}}\right) \cap \Sigma\left(\frac{\varepsilon}{4K\|g\|_{S^p}}\right) \cap T\left(g, \frac{\varepsilon}{4KC}\right)$ is relatively dense in Π.

Let $\tau \in F_1$ and $q^{'} > 1$ such that $\frac{1}{q^{'}} + \frac{1}{p} = 1$. Using the Hölder's inequality, it follows that

$$\begin{aligned}
\| \Phi_n(t+\tau) - \Phi_n(t) \| \leq & \int_{t-nK}^{t-(n-1)K} \|U(t+\tau, \sigma(s)+\tau)P(s+\tau) - U(t,\sigma(s))P(s+)\|.|g(s+\tau)| \\
& + \int_{t-nK}^{t-(n-1)K} \|U(t,\sigma(s))\|.|g(s+\tau) - g(s)| \\
& + \int_{t+(n-1)K}^{t+nK} \|U_Q(t+\tau,\sigma(s)+\tau)Q(s+\tau) - U_Q(t,\sigma(s))Q(s+)\|.|g(s+\tau)| \\
& + \int_{t+(n-1)K}^{t+nK} \|U_Q(t,\sigma(s))\|.|g(s+\tau) - g(s)|, \\
& \leq \frac{\varepsilon}{4K \| g \|_{S^p}} \int_{t-nK}^{t-(n-1)K} | g(s+\tau) | \Delta s \\
& + C(1+\alpha\bar{\mu}) \int_{t-nK}^{t-(n-1)K} | g(s+\tau) - g(s) | \Delta s \\
& + \frac{\varepsilon}{4K \| g \|_{S^p}} \int_{t+(n-1)K}^{t+nK} | g(\tau+s) | \Delta s \\
& + C(1+\alpha\bar{\mu}) \int_{t+(n-1)K}^{t+nK} | g(\tau+s) - g(s) | \Delta s, \\
& \leq \frac{\varepsilon}{4K \| g \|_{S^p}} K \| g \|_{S^p} + CK(1+\alpha\bar{\mu}) \frac{\varepsilon}{4KC(1+\alpha\bar{\mu})} \\
& + \frac{\varepsilon}{4K \| g \|_{S^p}} K \| g \|_{S^p} + CK(1+\alpha\bar{\mu}) \frac{\varepsilon}{4KC(1+\alpha\bar{\mu})}, \\
& \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.
\end{aligned}$$

Then, $F_1 \subset T(\Phi_n, \varepsilon)$. Thus, $T(\Phi_n, \varepsilon)$ is relatively dense in Π. Which means that $\Phi(u) \in AP(\mathbb{T}, X)$.
Step 3 : It remains to show that $\Gamma(u) \in PAP_0(\mathbb{T}, X, \mu)$. Indeed, by a similar calculation we can show that

$$\| \Gamma_j(t) \| \leq 2C(1+\alpha\bar{\mu})K \| h \|_{S^p}.$$

Where $\Gamma_j(t) = \int_{t-nK}^{t-(n-1)K} U(t,\sigma(s)) P(s) .h(s) \Delta s + \int_{t+nK}^{t+(n-1)K} U_Q(t,\sigma(s)) Q(s) .h(s) \Delta s$,
since, $h^b \in PAP_0(\mathbb{T} \times X, BS^p(\mathbb{T}, X), \mu)$ it follows that $\Gamma_j(.) \in PAP_0(\mathbb{T}, X, \mu)$.
Therefore, from the following inequality :

$$\begin{aligned}
\frac{1}{\mu(Q_r)} \int_{Q_r} \| \Gamma(t) \| \mu(t) \Delta(t) \leq & \frac{1}{\mu(Q_r)} \int_{Q_r} \| \Gamma(t) - \sum_{j=1}^{n} \Gamma_j(t) \| \mu(t) \Delta(t) \\
& + \sum_{j=1}^{n} \frac{1}{\mu(Q_r)} \int_{Q_r} \| \Gamma_j(t) \| \mu(t) \Delta(t),
\end{aligned}$$

it is clear that $\Gamma(.) = \sum_{j=1}^{n} \Gamma_j(.) \in PAP_0(\mathbb{T}, X, \mu)$.
Finally $x(.) \in PAP(\mathbb{T}, X, \mu)$.
Step 4 : Uniqueness. Let $x : \mathbb{T} \to X$ is another weighted pseudo almost periodic solution of (1.2). So, from Definition (5.3) we have x is written in the following form : for all $\gamma, t \in \mathbb{T}$

$$x(t) = U(t,\gamma) x(\gamma) + \int_{\gamma}^{t} U(t,\sigma(s)) f(s) \Delta s.$$

Moreover, hypothesis $(\mathbb{A}_3)$ gives that for $t \in \mathbb{T}$,

$$P(t)x(t) = \int_{-\infty}^{t} U(t,\sigma(s)) P(s) f(s) \Delta s,$$

and,

$$Q(t)x(t)=\int_{-\infty}^{t}U_Q\left(t,\sigma\left(s\right)\right)Q(s)f(s)\Delta s.$$

It is easy to see that
$x(t)=P(t)x(t)+Q(t)x(t)=\int_{-\infty}^{t}U\left(t,\sigma\left(s\right)\right)P\left(s\right)f\left(s\right)-\int_{t}^{+\infty}U_Q\left(t,\sigma\left(s\right)\right)Q\left(s\right)f\left(s\right)=u(t)$. Which ends the proof. □

In the following theorem, we discuss the existence and uniqueness of weighted pseudo-almost periodic solution to evolution equation (1.1) on time scales.

Theorem 5.5. Assume that $(\mathbb{A}_1-\mathbb{A}_4)$ hold. If

$$\| L_f\left(s\right)\|_{S^r}\quad<\quad\begin{cases}\left(2C\left(\frac{1+e^{\alpha r_0}}{\alpha r_0}\right)^{\frac{1}{r_0}}\sum_{n=1}^{\infty}e^{-\alpha n}\right)^{-1}, & if\quad \mathbb{T}=\mathbb{R},\\ \\ \frac{1}{CK}\left(\frac{(1+\alpha\bar{\mu})^2}{\alpha\bar{\mu}}+\frac{\alpha\bar{\mu}+1}{\alpha\bar{\mu}}\right)^{-1}, & if\quad \mathbb{T}\neq\mathbb{R},\end{cases}$$

then, there exists a unique weighted pseudo almost periodic solution u of equation (1.1) such that

$$u(t)=\int_{-\infty}^{t}U(t,\sigma(s))P(s)F(s,u(s))\Delta s-\int_{t}^{+\infty}U_Q(t,\sigma(s))Q(s)F(s,u(s))\Delta s\quad t\in\mathbb{T}.$$

Proof. Let $u=x+y\in PAP(\mathbb{T},X,\mu)$, then $u=x+y\in WS^pPAP(\mathbb{T},X,\mu)$, where $x\in S^pAP(\mathbb{T},X)$, $\overline{x(\mathbb{T})}$ compact and $y\in WS^pPAP_0(\mathbb{T},X,\mu)$. Then, due to Theorem (4.6) there exists $q\in[1,p)$ such that $F(.,u(.))\in WS^qPAP(\mathbb{T},X,\mu)$.
Next, by using the Banach fixed-point theorem, we will prove the existence and uniqueness of solution. Let $u,v\in PAP\left(\mathbb{T},X,\mu\right)$ and consider the nonlinear operator H defined by

$$H(u)(t)=\int_{-\infty}^{t}U(t,\sigma(s))P(s)F(s,u(s))\Delta s-\int_{t}^{+\infty}U_Q(t,\sigma(s))Q(s)F(s,u(s))\Delta s,\quad t\in\mathbb{T}.$$

Step 1 : We will prove that $H(.)\in PAP\left(\mathbb{T},X,\mu\right)$. In fact, applying Theorem (5.4) one can get,

$$H(.)\in PAP\left(\mathbb{T},X,\mu\right).$$

Then, H maps $PAP\left(\mathbb{T},X,\mu\right)$ into $PAP\left(\mathbb{T},X,\mu\right)$.
Step 2 : Now, we will prove the existence and uniqueness of solution. Case 1 : if $\mathbb{T}=\mathbb{R}$ i.e, $K=1$, then

by Hölder's inequality, for all $t \in \mathbb{R}$ and $\frac{1}{r_0} + \frac{1}{r} = 1$, we have

$$\begin{aligned}
\parallel H\left(u\right)\left(t\right) - H\left(v\right)\left(t\right) \parallel &\leq C \parallel u - v \parallel \int_{-\infty}^{t} e^{-\alpha(t-s)} L_f\left(s\right) ds \\
&+ C \parallel u - v \parallel \int_{t}^{+\infty} L_f\left(s\right) e^{-\alpha(s-t)} ds, \\
&\leq C \parallel u - v \parallel \sum_{n=1}^{\infty} \int_{t-n}^{t-(n-1)} e^{-\alpha(t-s)} L_f\left(s\right) ds \\
&+ C \parallel u - v \parallel \sum_{n=1}^{\infty} \int_{t+n-1}^{t+n} e^{\alpha(t-s)} L_f\left(s\right) ds, \\
&\leq C \parallel u - v \parallel \sum_{n=1}^{\infty} \left(\int_{t-n}^{t-(n-1)} e^{-\alpha r_0 (t-s)} ds \right)^{\frac{1}{r_0}} \parallel L_f\left(s\right) \parallel_{S^r} \\
&+ C \parallel u - v \parallel \sum_{n=1}^{\infty} \left(\int_{t+n-1}^{t+n} e^{\alpha r_0 (t-s)} ds \right)^{\frac{1}{r_0}} \parallel L_f\left(s\right) \parallel_{S^r}, \\
&\leq C \parallel u - v \parallel \sum_{n=1}^{\infty} \left(\frac{e^{-\alpha r_0 (n-1)} - e^{-\alpha r_0 n}}{\alpha r_0} \right)^{\frac{1}{r_0}} \parallel L_f\left(s\right) \parallel_{S^r} \\
&+ C \parallel u - v \parallel \sum_{n=1}^{\infty} \left(\frac{e^{-\alpha r_0 n} - e^{-\alpha r_0 (n-1)}}{-\alpha r_0} \right)^{\frac{1}{r_0}} \parallel L_f\left(s\right) \parallel_{S^r}, \\
&\leq 2C \parallel u - v \parallel \left(\frac{1 + e^{\alpha r_0}}{\alpha r_0} \right)^{\frac{1}{r_0}} \sum_{n=1}^{\infty} e^{-\alpha n} \parallel L_f\left(s\right) \parallel_{S^r} . \\
&< 1.
\end{aligned}$$

Case 2 : if $\mathbb{T} \neq \mathbb{R}$. By Hölder's inequality, for all $t \in \mathbb{T}$ and $\frac{1}{r_0} + \frac{1}{r} = 1$, we obtain

$$\begin{aligned}
\parallel H\left(u\right)\left(t\right) - H\left(v\right)\left(t\right) \parallel &\leq C \parallel u - v \parallel \sum_{n=1}^{\infty} \int_{t-nK}^{t-(n-1)K} e_{\ominus\alpha}\left(t, \sigma\left(s\right)\right) L_f\left(s\right) \Delta s \\
&+ C \parallel u - v \parallel \sum_{n=1}^{\infty} \int_{t+(n-1)K}^{t+nK} e_{\alpha}\left(t, \sigma\left(s\right)\right) L_f\left(s\right) \Delta s, \\
&\leq C \parallel u - v \parallel \sum_{n=1}^{\infty} e_{\ominus\alpha}\left(t, \sigma\left(t\right) - (n-1)K\right) \int_{t-nK}^{t-(n-1)K} L_f\left(s\right) \Delta s \\
&+ C \parallel u - v \parallel \sum_{n=1}^{\infty} e_{\alpha}\left(t, \sigma\left(t\right) + nK\right) \int_{t+(n-1)K}^{t+nK} L_f\left(s\right) \Delta s.
\end{aligned}$$

In this case we have $\bar{\mu} = \sup_{t \in \mathbb{T}} \mu(t) \leq K$ and there exists a right-scattered point t_0 such that $\bar{\mu} = \mu(t_0) \leq K$. Thus, the following interval $[\sigma(t) - nK + K, s)_{\mathbb{T}}$ contains at least $n-2$ right-scattered points, for all $t \in \mathbb{T}$ and $n \geq 3$, such that for $r(t) \in \mathbb{Z}$ this right-scattered point is written in the following form : $r(t)K + t_0$

with $\mu(r(t)K+t_0)=\mu(t_0)=\bar{\mu}$. As a consequence, for $n\in\mathbb{N}$ we obtain,

$$\begin{aligned}\sum_{n=1}^{\infty} e_{\ominus\alpha}(t,\sigma(t)-(n-1)K) &\leq \left(e_{\ominus\alpha}(t,\sigma(t))+e_{\ominus\alpha}(t,\sigma(t)-K)+\sum_{n=3}^{\infty} e_{\ominus\alpha}(t,\sigma(t)-(n-1)K)\right),\\ &\leq (e_{\ominus\alpha}(t,\sigma(t))+e_{\ominus\alpha}(t,\sigma(t)-K)+\sum_{n=3}^{\infty}(1+\alpha\bar{\mu})^{2-n},\\ &\leq 1+1+\alpha\bar{\mu}+\frac{1}{\alpha\bar{\mu}}.\end{aligned}$$

Analogously, we prove that,

$$\sum_{n=1}^{\infty} e_{\alpha}(t,\sigma(t)+nK)\leq 1+\frac{1}{\alpha\bar{\mu}}.$$

So, according to this little discussion we have,

$$\begin{aligned}\| H(u)(t)-H(v)(t)\| &\leq CK\| u-v\|\left(1+1+\alpha\bar{\mu}+\frac{1}{\alpha\bar{\mu}}\right)\| L_f(s)\|_{S^r}\\ &+CK\| u-v\|\left(1+\frac{1}{\alpha\bar{\mu}}\right).\| L_f(s)\|_{S^r},\\ &\leq CK\| u-v\|\frac{(1+\alpha\bar{\mu})^2}{\alpha\bar{\mu}}.\| L_f(s)\|_{S^r}\\ &+CK\| u-v\|\frac{\alpha\bar{\mu}+1}{\alpha\bar{\mu}}.\| L_f(s)\|_{S^r},\\ &\leq\| L_f(s)\|_{S^r} CK\| u-v\|\left(\frac{(1+\alpha\bar{\mu})^2}{\alpha\bar{\mu}}+\frac{\alpha\bar{\mu}+1}{\alpha\bar{\mu}}\right),\\ &<1.\end{aligned}$$

Therefore, H has a unique fixed point. Finally, the equation (1.1) has a unique weighted pseudo almost periodic solution on $\mathbb{T}$. □

Corollary 5.6. *Under assumptions* $(\mathbb{A}_1-\mathbb{A}_4)$. *Then, the following dynamic equation with delay:*

$$u^{\Delta}(t)=A(t)u(t)+F(t,u(t-\theta)),\quad t\in\mathbb{T}\quad and\quad \theta\in\Pi, \tag{5.3}$$

has a unique weighted pseudo almost periodic solution given by

$$u(t)=\int_{-\infty}^{t} U(t,\sigma(s))P(s)F(s,u(s-\theta))\Delta s-\int_{t}^{+\infty} U_Q(t,\sigma(s))Q(s)F(s,u(s-\theta))\Delta s.$$

Whenever

$$\| L_f(s)\|_{S^r} < \begin{cases}\left(2C\left(\frac{1+e^{\alpha r_0}}{\alpha r_0}\right)^{\frac{1}{r_0}}\sum_{n=1}^{\infty} e^{-\alpha n}\right)^{-1}, & if\quad \mathbb{T}=\mathbb{R},\\ \frac{1}{CK}\left(\frac{(1+\alpha\bar{\mu})^2}{\alpha\bar{\mu}}+\frac{\alpha\bar{\mu}+1}{\alpha\bar{\mu}}\right)^{-1}, & if\quad \mathbb{T}\neq\mathbb{R}.\end{cases}$$

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Mohssine Es-saiydy,
Moulay Ismaïl University, Faculty of Sciences, Department of Mathematics,
MACS Laboratory, Meknès, Morocco.
E-mail address: essaiydy1995@gmail.com

and

Mohamed Zitane,
Moulay Ismaïl University, Faculty of Sciences, Department of Mathematics,
MACS Laboratory, Meknès, Morocco.
E-mail address: zitanem@gmail.com