Consistency of an Infinite System of Third Order Three-Point Boundary Value Problem in the bv_0 Space by the Theory of Measure of Noncompactness

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ABSTRACT: Several authors have examined the solvability conditions for an infinite system of differential equations in different Banach spaces using the concept of measure of noncompactness. In all these studies, they have considered differential equations where the boundary conditions are defined on two points. In this paper, we have studied the solvability conditions for an infinite system of third-order three point boundary value problem in the sequence space of bounded variation bv_0 with the help of the theory of measure of noncompactness and have given a suitable example to illustrate the result.

Key Words: Infinite system of differential equations, Meir-Keeler condensing operator, fixed point theory, measure of noncompactness, the sequence space of bounded variation.

Contents

1. Introduction

Infinite system of differential equations for it's solvability and existence of solutions has been studied by several authors over the years. The following system of second order infinite system of differential equations is one such example studied by Mursaleen and Rizvi [14],

$$x_i^{''}(\eta) = -h_i(\eta, x_1, x_2, x_3, \cdots); \ x_i(0) = x_i(U) = 0, \ \eta \in [0, U].$$

These kinds of equations occurs in various topics of non-linear analysis. Several works can be found in the literature for solvability and existence conditions of various kinds of first and second order infinite systems of differential equations in various sequence spaces (cf. [2], [15], [4], [13], [21], [19], [22], [23], [24]). These works originate with the introduction of infinite system of differential equations by Persidskii in 1959 [16] under the name "Countable Systems of differential equations" and later in 1961 [17] and 1976 [18]. The theory of measure of noncompactness (MNC) which was introduced in 1930 by Kuratowski [11] and later by Darbo [7], Goldenštein and Markus [8] and Istrăţescu [9] and the axiomatic definition by Banaś and Goebel [5] helped in working for the solvability conditions of infinite system of differential equations.

In this paper, we address the solvability of infinite system of third-order differential equations of the type

$$\frac{d^3}{d\xi^3} u_i(\theta) + f_i(\theta, u_1, u_2, \cdots) = 0, \ i \in \mathbb{N}, \ \theta \in [a, b]$$

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with three-point boundary value

$$u_i(a) = 0 = u_i^{'}(a), \ u_i(b) = ku_i(\eta), \forall i \in \mathbb{N}, \eta \in (a, b)$$

in the sequence space of bounded variation bv_0. Further, we claim that no work has been done by any author in three-point boundary value problem in this sequence space. For detailed discussion of the system see section 4.

2. Preliminaries and Background

Let $(X, ||.||)$ be a Banach space over $\mathbb{R}$. Also, let $\mathfrak{M}_X$ be the sub-collection of all non-empty and bounded subsets of X and $\mathfrak{M}_X^C \subseteq \mathfrak{M}_X$ be the collection of all closed sets. Further, let $\mathfrak{N}_X$ be the collection of all non-empty and relatively compact subsets of X. If $E \subseteq X$, then $\bar{E}$ and $\text{Conv}(E)$, be the closure and convex closure E respectively.

The following definition of measure of noncompactness is given by Banaś and Goebel [5]:

Definition 2.1. *For a Banach space X, a function $\varphi : \mathfrak{M}_X \to \mathbb{R}$ is called a measure of noncompactness in X if it satisfies the following axioms:*

i) The family $\ker \varphi = \{A \in \mathfrak{M}_X : \varphi(A) = 0\}$ *is non-empty and* $\ker \varphi \subset \mathfrak{N}_X$.

ii) $A_1 \subset A_2 \Rightarrow \varphi(A_1) \leq \varphi(A_2)$.

iii) $\varphi(\overline{A}) = \varphi(A)$.

iv) $\varphi(\text{Conv}(A)) = \varphi(A)$.

v) $\varphi(\theta A + (1 - \theta)B) \leq \theta\varphi(A) + (1 - \theta)\varphi(B)$ *for all* $\theta \in (0, 1)$.

vi) If (F_n) *is a decreasing sequence in* $\mathfrak{M}_X^C$ *and* $\lim\limits_{n \to \infty} \varphi(F_n) = 0$, *then* $\bigcap\limits_{n=1}^{\infty} F_n \neq \emptyset$.

The following definitions and results are used to establish the main results of this paper.

Definition 2.2. (Akhmerov et. al. [3]) *For two arbitrary measures of noncompactness φ_1 and φ_2 on the Banach spaces E_1 and E_2 respectively. A (φ_1, φ_2)-condensing operator is defined as an operator $F : E_1 \to E_2$ such that*

i) F is continuous,

ii) for each set A in E_1 which is bounded as well as non-compact, we have $\varphi_2(F(A)) < \varphi_1(A)$.

It is to be noted that a (φ_1, φ_2)-condensing operator is said to be a φ-condensing operator if $\varphi_1 = \varphi_2 = \varphi$ and is defined on the same Banach space.

Theorem 2.3. (Darbo [7]) *For a Banach space E with an arbitrary measure of noncompactness and $A \in \mathfrak{M}_E^C$, a continuous mapping $F : A \mapsto A$ contains a fixed point in A if*

i) A is convex,

ii) $\varphi(F(A)) \leq k\varphi(A)$ *for some $k \in [0, 1)$.*

Definition 2.4. (Meir and Keeler [12]) *For a metric space (E, d) a Meir-Keeler contraction on E is defined as a mapping F on E such that for any $\epsilon > 0$, there exists $\delta > 0$ with $d(Fu, Fv) < \epsilon$ whenever $\epsilon \leq d(u, v) < \epsilon + \delta$ for all $u, v \in E$.*

Theorem 2.5. (Meir and Keeler [12]) *A Meir-Keeler contraction mapping F on a metric space (E, d) has an unique fixed point if E is complete.*

Definition 2.6. (Aghajani et. al. [1]) *For an arbitrary measure of noncompactness φ on a Banach space E, and C a non-empty subset of E. A Meir-Keeler condensing operator acting upon C is defined as an operator $F : C \mapsto C$ such that for any $\epsilon > 0$, there exists $\delta > 0$ with $\varphi(F(B)) < \epsilon$ whenever $\epsilon \leq \varphi(B) < \epsilon + \delta$ for every bounded subset B of C.*

Theorem 2.7. (Aghajani et. al. [1]) *Let E be a Banach space and $A \in \mathfrak{M}_E^C$ with φ an arbitrary measure of noncompactness on E. A continuous mapping $F : A \mapsto A$ contains at least one fixed point and also the set of all fixed points of F in A is compact if*

  i) A is convex,

  ii) F is a Meir-Keeler condensing operator.

3. Hausdorff Measure of Noncompactness(MNC) on Banach Spaces

There are various measure of noncompactness developed over the years. Among such measures is χ-measure defined by

$$\chi(E) = \inf\{\epsilon > 0 : X \text{ contains finite } \epsilon\text{-net of } E\}$$

which is also called Hausdorff MNC. Here finite ϵ-net represents $\bigcup\limits_{i=1}^{n\in\mathbb{N}} B(x_i, r_i)$, where each $B(x_i, r_i)$ being the open ball of radius $r_i < \epsilon$, $x_i \in X$ that covers $E \subset X$. To calculate the Hausdorff measure following result will help.

Theorem 3.1. (Banaś and Mursaleen [6]) *Let X be a Banach sequence space with continuous coordinates such that:*

  i) X satisfies AK,

  ii) X has a monotone norm $||.||$,

  iii) $E \in \mathfrak{M}_X$, and P_n be the projection mapping defined by

$$P_n(y_1, y_2, \cdots) = y^{[n]} = (y_1, y_2, \cdots, y_n, 0, 0, \cdots)$$

  mapped from X into itself for all $(y_1, y_2, \cdots) \in X$,

then

$$\chi(E) = \lim_{n \to \infty} \left(\sup_{y \in E} ||(I - P_n)y|| \right).$$

4. Infinite System of Third-order Differential Equations

In this study, we consider the following infinite system of differential equations

$$\frac{d^3}{d\xi^3}u_i(\theta) + f_i(\theta, u_1, u_2, \cdots) = 0, \ i \in \mathbb{N}, \ \theta \in [a, b] \tag{4.1}$$

$$u_i(a) = 0 = u_i'(a), \ u_i(b) = ku_i(\eta), \forall i \in \mathbb{N}, \eta \in (a, b) \tag{4.2}$$

where, $k \in \mathbb{R}$ such that $k(a - \eta)^2 \neq (a - b)^2$ and $f_i \in C([a, b] \times \mathbb{R}, \mathbb{R})$ with $f_i(\theta, u_1, u_2, \cdots) \neq 0$. For the infinite system of differential equations (4.1) and the boundary conditions (4.2) the Green's function is given by,

$$\mathcal{G}(\theta, \tau) = \begin{cases} \frac{(a-\theta)^2(\tau-b)^2}{2(a-b)^2} - \frac{(\tau-\theta)^2}{2} + \frac{k(a-\theta)^2}{(a-b)^2 - k(a-\eta)^2}\mathcal{R}(\eta, \tau), & a \leq \tau \leq \theta \leq b, \\ \frac{(a-\theta)^2(\tau-b)^2}{2(a-b)^2} + \frac{k(a-\theta)^2}{(a-b)^2 - k(a-\eta)^2}\mathcal{R}(\eta, \tau), & a \leq \theta \leq \tau \leq b. \end{cases} \tag{4.3}$$

where, $\mathcal{R}(\theta, \tau)$ is Green's function associated with the differential equation

$$u'''(\theta) + f(\theta) = 0; \ \theta \in [a, b]$$

$$u(a) = 0 = u'(a); u(b) = 0,$$

given by

$$\mathcal{R}(\theta, \tau) = \begin{cases} \frac{(a-\theta)^2(\tau-b)^2}{2(a-b)^2} - \frac{(\tau-\theta)^2}{2}, & a \leq \tau \leq \theta \leq b, \\ \frac{(a-\theta)^2(\tau-b)^2}{2(a-b)^2} & a \leq \theta \leq \tau \leq b. \end{cases}$$

For the detailed formulation of above Green's function, one can refer to [20]. The following bound for $\mathcal{G}(\theta, \tau)$ shall serve later for achieving the results.

$$\int_a^b |\mathcal{G}(\theta, \tau)| d\tau \leq \frac{(b-a)^3}{3} + \frac{|k|}{3} \frac{(b-a)^5}{|(b-a)^2 - k(\eta-a)^2|} \tag{4.4}$$

5. Sequence Space of bounded variation bv and it's subspace bv_0

The sequence space of bounded variation

$$bv := \left\{ x = (x_n) : \sum_{n=1}^{\infty} |x_n - x_{n+1}| < \infty \right\}$$

forms a BK-Space with the norm $||x||_{bv} = |x_0| + \sum_{k=1}^{\infty} |x_k - x_{k-1}|$. And the space bv_0 denotes $bv \cap c_0$. This will also further mean that $bv = bv_0 + \{e\}$, where e denotes the sequence $(1, 1, 1, \cdots)$ and $bv_0 \subset bv$ (details in [10]). Hence,

$$bv_0 := \left\{ x = (x_n) : \sum_{n=1}^{\infty} |x_n - x_{n+1}| < \infty \text{ and } \lim_{n \to \infty} x_n \right\}$$

Also bv_0 forms a BK-Space with AK under the norm $||x||_{bv_0} = \sum_{n=1}^{\infty} |\Delta x_n|$. This norm can easily be verified to be monotonic as for each $x, y \in bv_0$ with $|x_k| \leq |y_k|$ for all k, then $||x|| \leq ||y||$. Hence with the use of Theorem 3.1, we can formulate the Hausdorff measure of noncompactness for bv_0 as follows:

$$\chi(B) = \lim_{n \to \infty} \left\{ \sup_{u \in B} \left[\sum_{k=n}^{\infty} |u_k - u_{k+1}| \right] \right\}. \tag{5.1}$$

For the rest of the paper we shall discuss the solvability of infinite system of differential equations (4.1) along with the three-point boundary conditions (4.2).

6. Solvability in bv_0

For the solvability in the sequence space of bounded variation bv_0, we take the following assumptions;

(H1) Collection $((fu)(\theta))_{\theta \in I}$ at each $u \in bv_0$ holds equicontinuity. Where, f is the operator defined on the space $[a, b] \times bv_0$ by

$$(\theta, u) \rightarrowtail (fu)(\theta) = (f_1(\theta, u), f_2(\theta, u), f_3(\theta, u)...);$$

$$f_i(\theta, u_1, u_2, u_3, ...) \in ([a, b] \times \mathbb{R}^{\infty}, \mathbb{R})$$

(H2) $p_i(\theta)$ and $q_i(\theta)$ are continuous functions defined on interval $[a, b]$ such that following inequality holds:

$$|f_n(\theta, u) - f_{n+1}(\theta, u)| \leq \{|u_n(\theta) - u_{n+1}(\theta)|\} q_n(\theta) + p_n(\theta).$$

$\sum_{k \geq n} p_k(\theta)$ shows uniform convergence on $[a, b]$ and the collection $(q_n(\theta))$ is equibounded on interval $[a, b]$.

Now, we introduce the following notations:

- $p(\theta) = \sum_{k \geq n} p_k(\theta), \forall \theta \in [a, b]$.

- $P = \sup\{p(\theta) : \theta \in [a, b]\}$.

- $Q = \sup\{q_n(\theta) : \theta \in [a, b], n \in \mathbb{R}\}$.

Theorem 6.1. *Under the hypotheses (H1)-(H2), infinite system of differential equations (4.1) along with the boundary conditions (4.2) has at least one solution $u(\theta) = u_i(\theta)$ whenever*

$$Q\left(\frac{(b-a)^3}{3} + \frac{|k|}{3}\frac{(b-a)^5}{|(b-a)^2 - k(\eta-a)^2|}\right) < 1$$

such that $u(\theta) \in bv_0$, for all $\theta \in [a, b]$.

Proof. Using the Green's function (4.3) along with the hypothesis (H1), we get

$$||u(\theta)||_{bv_0} = \sum_{i=1}^{\infty}\left|\int_a^b \mathcal{G}(\tau,\theta)(f_i(\tau,u(\tau)) - f_{i+1}(\tau,u(\tau)))d\tau\right|$$

$$\leq \sum_{i=1}^{\infty}\int_a^b |\mathcal{G}(\theta,\tau)||f_i(\tau,u(\tau)) - f_{i+1}(\tau,u(\tau))|d\tau$$

$$\leq \sum_{i=1}^{\infty}\int_a^b |\mathcal{G}(\theta,\tau)|(|u_i(\tau) - u_{i+1}(\tau)|q_i(\tau) + p_i(\tau))d\tau$$

$$= \sum_{i=1}^{\infty}\int_a^b |\mathcal{G}(\theta,\tau)||u_i - u_{i+1}|q_i(\tau)d\tau + \sum_{i=1}^{\infty}\int_a^b |\mathcal{G}(\theta,\tau)|p_i(\tau)d\tau$$

$$\leq \left(\frac{(b-a)^3}{3} + \frac{|k|}{3}\frac{(b-a)^5}{|(b-a)^2 - k(\eta-a)^2|}\right)(Q||u||_{bv_0} + P).$$

Hence, we get

$$||u||_{bv_0} < \frac{(|(b-a)^2 - k(\eta-a)^2|(b-a)^3 + |k|(b-a)^5)P}{|(b-a)^2 - k(\eta-a)^2|(3 - (b-a)^3Q) + |k|(b-a)^5Q} = r. \tag{6.1}$$

Let a non-empty ball $B = B(u^0, r_1)$ with centre $u^0(\theta) = (u_i^0(\theta))$ and radius $r_1 \leq r$, where $u_i^0(\theta) = 0$, is such that $\overline{B}$ remains bounded and is convex in bv_0. Then we consider $u(\theta) = (u_i(\theta)) \in \overline{B}$ and an operator $\mathcal{F} = (\mathcal{F}_i)$ defined on $C([a,b], \overline{B})$ given by

$$(\mathcal{F}u)(\theta) = \{(\mathcal{F}_i u)(\theta)\} = \left\{\int_a^b \mathcal{G}(\theta,\tau)f_i(\tau,u(\tau))d\tau\right\}; \quad \theta \in [a,b], \ u_i(\theta) \in C([a,b], \mathbb{R}). \tag{6.2}$$

With the use of the fact that $(f_i(\theta, u(\theta))) \in bv_0$, we have for $\forall\, \theta \in [a,b]$,

$$\sum_{k=1}^{\infty}|(\mathcal{F}_k u)(\theta) - (\mathcal{F}_{k+1}u)(\theta)| = \sum_{k=1}^{\infty}|\int_a^b \mathcal{G}(\theta,\tau)f_k(\tau,u(\tau))d\tau$$

$$- \int_a^b \mathcal{G}(\theta,\tau)f_{k+1}(\tau,u(\tau))d\tau|$$

$$= \sum_{k=1}^{\infty}|\int_a^b \mathcal{G}(\theta,\tau)(f_k(\tau,u(\tau) - f_{k+1}(\tau,u(\tau)))d\tau|$$

$$\leq \sum_{k=1}^{\infty}\int_a^b |\mathcal{G}(\theta,\tau)||f_k(\tau,u(\tau) - f_{k+1}(\tau,u(\tau))|d\tau$$

$$\leq \sum_{k=1}^{\infty}\int_a^b |\mathcal{G}(\theta,\tau)|(|u_k(\tau) - u_{k+1}(\tau)|q_k(\tau) + p_k(\tau))d\tau$$

$$\leq \left(\frac{(b-a)^3}{3} + \frac{|k|}{3}\frac{(b-a)^5}{|(b-a)^2 - k(\eta-a)^2|}\right)(Q||u||_{bv_0} + P)$$

$$< \infty$$

Thus, we find that for all $\theta \in [a, b]$, $(\mathcal{F}u)(\theta) = \{(\mathcal{F}_i u)(\theta)\} \in bv_0$.

Now,

$$\mathcal{G}(\theta, \tau) = \frac{(a-a)^2(\tau-b)^2}{2(a-b)^2} + \frac{k(a-a)^2}{(a-b)^2 - k(a-\eta)^2} R(\eta, \tau) = 0.$$

Therefore,

$$(\mathcal{F}_i)(a) = \int_a^b \mathcal{G}(a, \tau) f_i(\tau, u(\tau)) d\tau$$

$$= \int_a^b 0 . f_i(\tau, u(\tau)) d\tau$$

$$= 0.$$

Also, $(\mathcal{F}_i')(\theta)$ at $\theta = a$ denoted by

$$(\mathcal{F}_i')(\theta)\Big|_{\theta=a} = \int_a^b \left\{ \frac{-2(a-\theta)(\tau-b)^2}{2(a-b)^2} - \frac{-2k(a-\theta)}{(a-b)^2 - k(a-\eta)} R(\eta, \tau) \right\}\Big|_{\theta=a} f_i(\tau, u(\tau)) \, d\tau$$

and so,

$$(\mathcal{F})_i'(a) = \int_a^b 0 . f_i(\tau, u(\tau)) d\tau$$

$$= 0.$$

Also,

$$(\mathcal{F}_i)(b)$$

$$= \int_a^b \left\{ \frac{(a-b)^2(\tau-b)^2}{2(a-b)^2} - \frac{(\tau-b)^2}{2} + \frac{k(a-b)^2}{(a-b)^2 - k(a-\eta)^2} R(\eta, \tau) \right\} f_i(\tau, u(\tau)) \, d\tau$$

$$= \int_a^b \frac{k(a-b)^2}{(a-b)^2 - k(a-\eta)^2} R(\eta, \tau) f_i(\tau, u(\tau)) \, d\tau.$$

But,

$$(\mathcal{F}_i)(\eta) = \int_a^b G(\eta, s) . f_i(\tau, u(\tau)) \, d\tau$$

$$= \int_a^b \left\{ R(\eta, \tau) + \frac{k(a-\eta)^2}{(a-b)^2 - k(a-\eta)^2} R(\eta, \tau) \right\} f_i(\tau, u(\tau)) \, d\tau$$

$$= \int_a^b \frac{(a-b)^2 - k(a-\eta)^2 + k(a-\eta)^2}{(a-b)^2 - k(a-\eta)^2} R(\eta, \tau) \, d\tau$$

$$= \int_a^b \frac{(a-b)^2}{(a-b)^2 - k(a-\eta)^2} R(\eta, \tau) . f_i(\tau, u(\tau)) \, d\tau.$$

Hence,

$$(\mathcal{F}_i)(b) = k \int_a^b \frac{(a-b)^2}{(a-b^2) - k(a-\eta)^2} R(\eta, \tau) . f_i(\tau, u(\tau)) \, d\tau$$

$$= k(\mathcal{F}_i)(\eta).$$

Therefore, each $(\mathcal{F}_i u)(\theta)$ satisfies boundary condition given in (4.2).

Since, $||(\mathcal{F}u)(\theta) - u^0(\theta)||_{bv_0} = ||(\mathcal{F}u)(\theta)||_{bv_0} \leq r$, thus $\mathcal{F}$ on the closed ball $\overline{B}$ self-maps. Now, we will find $\delta_\epsilon > 0$ for every $\epsilon > 0$ such that $\chi(\overline{B}) \in [\epsilon, \epsilon + \delta)$ implies $\chi(\mathcal{F}(\overline{B})) < \epsilon$. Using (5.1), we proceed as follows

$$\chi(\mathcal{F}\overline{B}) = \lim_{n \to \infty} (\sup_{u(\theta) \in \overline{B}} [\sum_{k=1}^{\infty} |\int_a^b \mathcal{G}(\theta, \tau)(f_k - f_{k+1})d\tau|])$$

$$\leq \lim_{n \to \infty} (\sup_{u(\theta) \in \overline{B}} [\sum_{k \geq n} \int_a^b |\mathcal{G}(\theta, \tau)||f_k(\tau, u(\tau)) - f_{k+1}(\tau, u(\tau))|d\tau])$$

$$= \lim_{n \to \infty} (\sup_{u(\theta) \in \overline{B}} \sum_{k \geq n} \int_a^b |\mathcal{G}(\theta, \tau)|(p_k(\tau) + q_k(\tau)|u_k(\tau) - u_{k+1}(\tau)|d\tau))$$

$$= \lim_{n \to \infty} (\sup_{u(\theta) \in \overline{B}} \sum_{k \geq n} \int_a^b |\mathcal{G}(\theta, \tau)|p_k(\tau)d\tau)$$

$$+ \lim_{n \to \infty} (\sup_{u(\theta) \in \overline{B}} \sum_{k \geq n} \int_a^b |\mathcal{G}(\theta, \tau)|q_k(\tau)|u_k(\tau) - u_{k+1}(\tau)|d\tau)$$

$$\leq Q \left(\frac{(b-a)^3}{3} + \frac{|k|}{3} \frac{(b-a)^5}{|(b-a)^2 - k(\eta-a)^2|} \right) \chi(\overline{B}) < \epsilon,$$

which implies

$$\chi(\overline{B}) \leq \frac{\epsilon}{Q} \left(\frac{1}{\frac{(b-a)^3}{3} + \frac{|k|}{3} \frac{(b-a)^5}{|(b-a)^2 - k(\eta-a)^2|}} \right)$$

$$= \frac{\epsilon}{Q} \left(\frac{3|(b-a)^2 - k(\eta-a)^2|}{|(b-a)^2 - k(\eta-a)^2|(b-a)^3 + |k|(b-a)^5} \right).$$

Let us suppose

$$\delta = \left[\frac{1}{Q} \left(\frac{3|(b-a)^2 - k(\eta-a)^2|}{|(b-a)^2 - k(\eta-a)^2|(b-a)^3 + |k|(b-a)^5} \right) - 1 \right] \epsilon. \tag{6.3}$$

Using equation (6.3), we found that $\chi(\overline{B}) \in [\epsilon, \epsilon + \delta)$. Assumption (H1) ensures the continuity of $\mathcal{F}$ in $C([a, b], \overline{B})$. Combining all the results achieved above for the operator $\mathcal{F}$, one can establish $\mathcal{F}$ as a Meir-Keeler condensing operator acting on subset $\overline{B} \subset bv_0$. Further, we also observe that $\mathcal{F}$ also satisfies the conditions given in Theorem(2.7). Therefore, $\overline{B}$ contains a fixed point of $\overline{B}$, which acts as a solution for the system (4.1) along with the three-point boundary conditions (4.2). $\qquad \square$

Note 1. *The fixed point $u(\theta)$ is such that $(u_i(\theta)) \in \ker \chi, \forall \theta \in [a, b]$.*

Example 6.2. *As an example we consider an infinite system of third-order differential equations*

$$u_n'''(\theta) + \frac{(-1)^n \sqrt{\theta}}{2^n n^3} + \sum_{m=n}^{\infty} \frac{\theta \sin(\theta)}{\pi^3 2^{\exp(-\theta)}} \frac{(u_m(\theta) - u_{m+1}(\theta))}{(m+1)^2} = 0, \ \theta \in \left[\frac{1}{2}, \frac{3}{2} \right], \ n \in \mathbb{N} \tag{6.4}$$

along with the boundary conditions

$$u_n \left(\frac{1}{2} \right) = 0 = u_n' \left(\frac{1}{2} \right), \ u_n \left(\frac{3}{2} \right) = \pi u_n(1). \tag{6.5}$$

Clearly, $\frac{(-1)^n \sqrt{\theta}}{2^n n^3}$ and $\sum_{m=n}^{\infty} \frac{\theta \sin(\theta)}{\pi^3 2^{\exp(-\theta)}} \frac{(u_m(\theta) - u_{m+1}(\theta))}{(m+1)^2}$ are continuous on the interval $[\frac{1}{2}, \frac{3}{2}]$, for each $n \in \mathbb{N}$.

Notice that for any $\theta \in [\frac{1}{2}, \frac{3}{2}]$, $(f_n(\theta, u(\theta))) \in bv_0$ if $(u_n(\theta)) \in bv_0$. Also,

$$
\sum_{k=1}^{\infty} |f_k(\theta, u(\theta)) - f_{k+1}(\theta, u(\theta))|
$$

$$
= \sum_{k=1}^{\infty} \left| \frac{(-1)^k \sqrt{\theta}}{2^k k^3} + \sum_{m=k}^{\infty} \frac{\theta \sin(\theta)}{\pi^3 2^{\exp(-\theta)}} \frac{(u_m(\theta) - u_{m+1}(\theta))}{(m+1)^2} \right.
$$
$$
\left. - \frac{(-1)^{k+1} \sqrt{\theta}}{2^{k+1}(k+1)^3} - \sum_{m=k+1}^{\infty} \frac{\theta \sin(\theta)}{\pi^3 2^{\exp(-\theta)}} \frac{(u_m(\theta) - u_{m+1}(\theta))}{(m+1)^2} \right|
$$

$$
\leq \sum_{k=1}^{\infty} \left| \frac{(-1)^k \sqrt{\theta}}{2^k k^3} - \frac{(-1)^{k+1} \sqrt{\theta}}{2^{k+1}(k+1)^3} \right|
$$
$$
+ \sum_{k=1}^{\infty} \left| \sum_{m=k}^{\infty} \frac{\theta \sin(\theta)}{\pi^3 2^{\exp(-\theta)}} \left[\frac{(u_m(\theta) - u_{m+1}(\theta))}{(m+1)^2} - \frac{(u_{m+1}(\theta) - u_{m+2}(\theta))}{(m+2)^2} \right] \right|
$$

$$
\leq \sum_{k=1}^{\infty} \left| \frac{(-1)^k 2(k+1)^3 \sqrt{\theta} - (-1)^{k+1} k^3 \sqrt{\theta}}{2^{k+1}(k+1)^3 k^3} \right|
$$
$$
+ \sum_{k=1}^{\infty} \frac{\theta}{\pi^3} |u_k(\theta) - u_{k+1}(\theta)|
$$

$$
\leq \sum_{k=1}^{\infty} \left| \frac{(-1)^k 2(k+1)^3 \sqrt{\theta} + (-1)^k k^3 \sqrt{\theta}}{2^{k+1}(k+1)^3 k^3} \right| + \frac{3/2}{\pi^3} ||u(\theta)||_{bv_0}
$$

$$
\leq \sum_{k=1}^{\infty} \frac{2(k+1)^3 + k^3}{2^{k-1}(k+1)^3 k^3} |(-1)^k| |\sqrt{\theta}| + \frac{3}{2\pi^3} ||u(\theta)||_{bv_0}
$$

$$
\leq \frac{\sqrt{3}}{\sqrt{2}} \left(-\frac{1}{2} + 2 \operatorname{Li}_3 \left(\frac{1}{2} \right) \right) + \frac{3}{2\pi^3} ||u(\theta)||_{bv_0} < \infty.
$$

We now show that (H1) is satisfied for which we consider $v(\theta) = (v_n(\theta)) \in bv_0$ and fix $u(\theta) = (u_n(\theta)) \in bv_0$, $\epsilon > 0$ arbitrarily. Then the relation

$$
||u(\theta) - v(\theta)|| \leq \delta(\epsilon) := \frac{2\pi^3}{3} \epsilon,
$$

yields

$$
|f_n(\theta, u(\theta)) - f_n(\theta, v(\theta))| = \left| \sum_{m=n}^{\infty} \frac{\theta \sin(\theta)}{\pi^3 2^{\exp(-\theta)}} \frac{(u_m(\theta) - u_{m+1}(\theta)) - (v_m(\theta) - v_{m+1}(\theta))}{(m+1)^2} \right|
$$

$$
\leq \sum_{m=n}^{\infty} \left| \frac{\theta \sin(\theta)}{\pi^3 2^{\exp(-\theta)}} \right| \frac{|(u_m(\theta) - u_{m+1}(\theta)) - (v_m(\theta) - v_{m+1}(\theta))|}{(m+1)^2}
$$

$$
\leq \frac{3}{2\pi^3} \sum_{k=1}^{\infty} \frac{|(u_k(\theta) - v_k(\theta)) - (u_{k+1}(\theta) - v_{k+1}(\theta))|}{(k+1)^2}
$$

$$
\leq \frac{3}{2\pi^3} ||(u(\theta) - v(\theta)||_{bv_0} < \epsilon
$$

This is the continuity as in (H1).

Now, for (H2)

$$|f_k(\theta, u) - f_{k+1}(\theta, u)| = \left| \frac{(-1)^k \sqrt{\theta}}{2^k k^3} - \frac{(-1)^{k+1} \sqrt{\theta}}{2^{k+1}(k+1)^3} \right.$$

$$+ \sum_{m=n}^{\infty} \frac{\theta \sin(\theta)}{\pi^3 2^{\exp(-\theta)}} \frac{(u_m(\theta) - u_{m+1}(\theta))}{(m+1)^2}$$

$$\left. - \sum_{m=n+1}^{\infty} \frac{\theta \sin(\theta)}{\pi^3 2^{\exp(-\theta)}} \frac{(u_m(\theta) - u_{m+1}(\theta))}{(m+1)^2} \right|$$

$$\leq \frac{2(k+1)^3 + k^3}{2^{k+1} k^3 (k+1)^3} |(-1)^k| |\sqrt{\theta}| + 2 \sum_{n=1}^{\infty} \left| \frac{\theta \sin(\theta)}{\pi^3 2^{\exp(-\theta)}} \right| \frac{|(u_n(\theta) - u_{n+1}(\theta))|}{(n+1)^2}$$

$$\leq \frac{2(k+1)^3 + k^3}{2^{k+1} k^3 (k+1)^3} |\sqrt{\theta}| + \frac{2}{\pi^3} \sum_{n=1}^{\infty} \frac{|\theta| |(u_n(\theta) - u_{n+1}(\theta))|}{(n+1)^2}$$

$$= p_k(\theta) + q_k(\theta) |u_k(\theta) - u_{k+1}(\theta)|.$$

The functions $p_k(\theta) = \frac{2(k+1)^3 + k^3}{2^{k+1} k^3 (k+1)^3} |\sqrt{\theta}|$ and $q_k(\theta) = \frac{\theta}{3\pi}$ are continuous. Also the series $\sum_{n \geq 1} p_n(\theta)$ uniformly converges to $\frac{\sqrt{3}}{\sqrt{2}} \left(-\frac{1}{2} + 2 \operatorname{Li}_3 \left(\frac{1}{2} \right) \right)$ and the sequence $(q_n(\theta))$ is equibounded on $[\frac{1}{2}, \frac{3}{2}]$ by $Q = \frac{1}{2\pi}$. Further, for interval $[a, b]$; $a = \frac{1}{2}, b = \frac{3}{2}$ and $k = \pi, \eta = 1$

$$Q \left(\frac{(b-a)^3}{3} + \frac{|k|}{3} \frac{(b-a)^5}{|(b-a)^2 - k(\eta-a)^2|} \right) < 1$$

is satisfied. This gives $Q \cong 1.591549$ and $P \cong 0.703525$.

Thus, by taking a suitable value of r_1 as in Theorem (6.1) operator $\mathcal{F}$ has a fixed point $u(\theta) = (u_i(\theta)) \in bv_0$ which acts as a solution for the system (6.4)-(6.5).

7. Conclusion

In our present work, we have provided the solvability conditions for an infinite system of third-order three point boundary value problem in the sequence space of bounded variation bv_0. We have also provided an example to further illustrate the result. The concept of measure of noncompactness along with the Meir-Keeler condensing operator served as tools for proving our result. Solvability conditions for similar type of infinite system in various other Banach spaces still remains open for further research.

Acknowledgments

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