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A Note on the Seidel and Seidel Laplacian Matrices

Jalal Askari

ABSTRACT: In this paper we investigate the spectrum of the Seidel and Seidel Laplacian matrix of a graph. We generalized the concept of Seidel Laplacian matrix which denoted by Seidel matrix and obtained some results related to them. This can be intuitively understood as a consequence of the relationship between the Seidel and Seidel Laplacian matrix in the graph by Zagreb index. In closing, we mention some alternatives to and generalization of the Seidel and Seidel Laplacian matrices. Also, we obtain the relation between Seidel and Seidel Laplacian energy, related to all graphs with order n.

Key Words: Eigenvalue, energy, Seidel matrix, Seidel Laplacian matrix, Zagreb index.

Contents

1. Introduction

All of graphs considered in this paper are finite, undirected and simple. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Let $A(G)$ be the (0, 1)-adjacency matrix and $\theta_1, \theta_2, \ldots, \theta_n$ are the eigenvalues of G. The Seidel matrix *(Haemers [7]; Van Lint [14])* of a simple graph G with n vertices and m edges, denoted by $S(G)$, is a real symmetric square matrix of order n of size m which is defined as $S(G) = A(\overline{G}) - A(G)$. Let $D_S(G) = diag(n-1-2d_1, n-1-2d_2, \ldots, n-1-2d_n)$ be a diagonal matrix in which d_i is the degree of the vertex v_i. The Seidel Laplacian matrix of G is defined as $SL(G) = DS(G) - S(G)$. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $\lambda_1^L, \lambda_2^L, \ldots, \lambda_n^L$ be the Seidel and Seidel Laplacian eigenvalues of G, respectively. It is obvious that $-S(G)$ is the Seidel matrix of the complement of G. Haemers [7] similar to the normal energy, defined the Seidel energy $E_s(G)$ of G which is the sum of the absolute values of the eigenvalues of the Seidel matrix. Similarly, he defined the Seidel Laplacian energy $E_{SL}(G) = \sum_{i=1}^{n} |\lambda_i^L - \frac{n(n-1)-4m}{n}|$. If we introduce the auxiliary quantities

$$\sigma_i = \lambda_i^L - \frac{n(n-1)-4m}{n}, \ i = 1, 2, \ldots, n. \tag{1.1}$$

Then the expression for Seidel Laplacian energy becomes analogous to the formula for ordinary graph energy, namely: $\sum\limits_{i=1}^{n} \sigma_i = 0$ and

$$E_{SL}(G) = \sum_{i=1}^{n} |\sigma_i|. \tag{1.2}$$

The first Zagreb index is used to analyse the structure-dependency of total π-electron energy on the molecular orbitals, introduced by Gutman and Trinajstć [5]. It is denoted by $Z_1(G) = \sum_{uv \in E(G)} (d_u + d_v)$. For example consider the complete graph K_n, whose Seidel and Seidel Laplacian matrices are $I-J$ and $J-nI$, respectively. Hence the eigenvalues of $S(K_n)$ are $\underbrace{1, 1, \ldots, 1}_{n-1 \ times}, 1-n$ and $SL(K_n)$ are $0, \underbrace{-n, -n, \ldots, -n}_{n-1 \ times}$. So $E_S(K_n) = E_{SL}(K_n) = 2n - 2$. For more information related to eigenvalue and Seidel eigenvalues and their properties, we refer the reader to [2,3,4,6,7,8,13].

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2. Main result

Let G be a simple graph of order n with vertex set $V=\{v_1,v_2,\dots,v_n\}$ and m edges. Let $\lambda_1,\lambda_2,\dots,\lambda_n$ and $\lambda_1^L,\lambda_2^L,\dots,\lambda_n^L$ be the Seidel and Seidel Laplacian eigenvalues of G, respectively. It is well known that the Seidel and Seidel Laplacian eigenvalues satisfies the following relations:

$$\sum_{n=1}^{n}\lambda_i=0,\ \sum_{i=1}^{n}\lambda_i^2=n(n-1),$$

$$\sum_{i=1}^{n}\lambda_i^L=\mathrm{trace}(SL(G))=\sum_{i=1}^{n}[n-1-2d(v_i)]=n(n-1)-4m,$$

$$\sum_{i=1}^{n}(\lambda_i^L)^2=\mathrm{trace}(SL(G)^2)=n^2(n-1)-8m(n-1)+4Z_1(G). \tag{2.1}$$

Where $Z_1(G)=\sum_{i=1}^{n}d^2(v_i)$ is introduced called the first Zagreb index, whose mathematical properties have been studied in due detail [3,8,12,15].

Theorem 2.1. *Let G be a graph of order n. If $\lambda_1,\lambda_2,\dots,\lambda_n$ are the Seidel eigenvalues of a k-regular graph G, then the Seidel Laplacian eigenvalues of G are $\lambda_i^L=n-1-2k-\lambda_i$, $i=1,2,\dots,n$.*

Proof. Assume that the characteristic polynomial of the Seidel matrix $S(G)$ and Seidel Laplacian matrix $SL(G)$ denoted by $\Phi_S(G,\lambda)$ and $\Phi_{SL}(G,\lambda)$, respectively. If G is a k-regular graph, then $SL(G)=(n-1-2k)I-S(G)$, where I is an identity matrix of order n. Therefore,

$$\begin{aligned}\Phi_{SL}(G,\lambda) &= \det(\lambda I-SL(G))=\det(\lambda I-(n-1-2k)I+S(G))\\ &= \det[(\lambda-n+1+2k)I+S(G)]=\det[(\lambda-n+1+2k)I-S(\overline{G})]\\ &= \Phi_S(\overline{G},\lambda-n+1+2k)\\ &= \Phi_S(-G,\lambda-n+1+2k)=\Phi_S(G,n-1-2k-\lambda)\end{aligned}$$

and the proof is complete. □

Corollary 2.2. *If G is a $\left(\frac{n-1}{2}\right)$-regular graph of odd order n, then $SL(G)=S(\overline{G})=-S(G)$.*

Lemma 2.3. *[4] Let G be a k-regular graph of order n with the adjacency matrix of $A(G)$ and Seidel matrix of $S(G)$. If $k,\theta_2,\theta_3,\dots,\theta_n$ and $\lambda_1,\dots,\lambda_n$ are the adjacency and Seidel eigenvalues, respectively, then the Seidel eigenvalues are $\lambda_1=n-2k-1,\lambda_2=-1-2\theta_2,\lambda_3=-1-2\theta_3,\dots,\lambda_n=-1-2\theta_n$.*

Proposition 2.4. *Let G be a k-regular graph of order n with the adjacency matrix of $A(G)$ and Seidel Laplacian matrix of $SL(G)$. If $k,\theta_2,\theta_3,\dots,\theta_n$ and $\lambda_1^L,\dots,\lambda_n^L$ are the adjacency and Seidel Laplacian eigenvalues, respectively, then the Seidel Laplacian eigenvalues are $\lambda_1^L=0,\lambda_2^L=n-2k+2\theta_2,\lambda_3^L=n-2k+2\theta_3,\dots,\lambda_n^L=n-2k+2\theta_n$.*

Proof. By Theorem 2.1 and Lemma 2.3 the proof is straightforward. □

Corollary 2.5. *Let G be a graph of odd order n. If $\frac{n-1}{2},\theta_2,\dots,\theta_n$ are the eigenvalues of a $\left(\frac{n-1}{2}\right)$-regular graph G, then the Seidel and Seidel Laplacian eigenvalues of G are $\lambda_1=\lambda_1^L=0,\ \lambda_i=-\lambda_i^L=-1-2\theta_i$, $i=2,3,\dots,n$.*

Lemma 2.6. [9] *Let $x_1,x_2,\dots,x_N$ be non negative numbers, and let*

$$\alpha=\frac{1}{N}\sum_{i=1}^{N}x_i \qquad \text{and} \qquad \gamma=\left(\prod_{i=1}^{N}x_i\right)^{1/N}$$

be their arithmetic and geometric means. Then

$$\frac{1}{N(N-1)}\sum_{i<j}\left(\sqrt{x_i}-\sqrt{x_j}\right)^2\le\alpha-\gamma\le\frac{1}{N}\sum_{i<j}\left(\sqrt{x_i}-\sqrt{x_j}\right)^2 .$$

Moreover, equality holds if and only if $x_1=x_2=\cdots=x_n$.

Theorem 2.7. *Let G be a graph of order n with size m. If $|\lambda_1^L| \geq \cdots \geq |\lambda_n^L|$ is the eigenvalues of the Seidel Laplacian matrix $SL(G)$. Then for all k, $\lambda_k^L \leq \frac{E_{SL}(G)}{2} + \frac{n(n-1)-4m}{n}$. Moreover, if $G \cong K_n$, then $\lambda_n^L = \frac{E_{SL}(G)}{2} + \frac{n(n-1)-4m}{n} = 0$.*

Proof. Our proof follows from the ideas came in [1]. Let $|\lambda_1^L| \geq \cdots \geq |\lambda_n^L|$ be the eigenvalues of the Seidel Laplacian matrix $SL(G)$. Thus by formula (1.2), $\sigma_k = -\sum_{i=1, i\neq k}^{n} \sigma_i$. Therefore we find that $|\sigma_k| \leq \sum_{i=1, i\neq k}^{n} |\sigma_i|$. This implies that $2|\sigma_k| \leq E_{SL}(G)$, that is, $|\sigma_k| \leq \frac{E_{SL}(G)}{2}$. So, $\lambda_k^L \leq \frac{E_{SL}(G)}{2} + \frac{n(n-1)-4m}{n}$.

One can easily see that $E_{SL}(K_n) = 2(n-1)$ and $\lambda_n^L = \frac{E_{SL}(G)}{2} + \frac{n(n-1)-4m}{n} = 0$ for K_n and the proof is completed. $\square$

Lemma 2.8. *Let G be a graph of order n. Then 0 is an eigenvalue for the Seidel Laplacian matrix of G.*

Proof. If $S(G)$ is a Seidel matrix of graph G, then $S(G) = J - I - 2A(G)$. So, the sum of i^{th} row is equal to $n - 1 - 2d_i$, $i = 1, \ldots, n$. We have,

$$S(G)\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} n-1-2d_1 \\ n-1-2d_2 \\ \vdots \\ n-1-2d_n \end{pmatrix}.$$

Then one can see that

$$\begin{aligned}
(SL(G))\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} &= (D(G) - S(G))\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} (n-1-2d_1) - (n-1-2d_1) \\ (n-1-2d_2) - (n-1-2d_2) \\ \vdots \\ (n-1-2d_n) - (n-1-2d_n) \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 0\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}
\end{aligned}$$

and hence we obtained the desired result. $\square$

Lemma 2.9. *[13] Let G be a graph of order n, size m and σ_i be as defined above by Eq.(1.1), then*

$$\sum_{i=1}^{n} \sigma_i^2 = n(n-1) + 4Z_1(G) - \frac{16m^2}{n}.$$

Theorem 2.10. If G is a connected graph with order n and size m, then inequality $E_{SL} \leq |n-1-\frac{4m}{n}| + \sqrt{(n-1)(M - (n-1-\frac{4m}{n})^2)}$ holds, where $M = n^2(n-1) - 8m(n-1) + 4Z_1(G) - \left(\frac{(n(n-1)-4m)^2}{n}\right) = n(n-1) + 4Z_1(G) - \frac{16m^2}{n}$.

Proof. Our proof follows from the ideas of Koolen and Moulton in [10], used for obtaining an analogous upper bound for the ordinary graph energy $E(G)$. Let $\sigma_i = \lambda_i^L - \frac{n(n-1)-4m}{n}$, $i = 1, 2, \ldots, n$ be as defined above by Eq.(1.1). Suppose that $|\sigma_1| \geq |\sigma_2| \geq \cdots \geq |\sigma_n|$, by Rayleigh quotient for largest eigenvalues, we get $|\sigma_1| \geq |n - 1 - \frac{4m}{n}|$, by Lemma 2.9, $\sum\limits_{i=1}^{n} |\sigma_i|^2 = M$. Thus, $\sum\limits_{i=2}^{n} |\sigma_i|^2 = M - |\sigma_1|^2$. By using the Cauchy-Schwartz inequality to the vectors $(|\sigma_2|, \cdots, |\sigma_n|)$ and $(1, \ldots, 1)$ with $n-1$ entries, we obtain the inequality $\sum\limits_{i=2}^{n} |\sigma_i| \leq \sqrt{(n-1)(M - \sigma_1^2)}$. Thus, we have $E_{SL} \leq |\sigma_1| + \sqrt{(n-1)(M - \sigma_1^2)}$. Now, since the function $f(x) = x + \sqrt{(n-1)(M - x^2)}$ decreases on the interval $|n - 1 - \frac{4m}{n}| \leq x \leq \sqrt{n(n-1)}$. We have $f(|\sigma_1|) \leq f(|n - 1 - \frac{4m}{n}|)$ and hence the inequality follows. $\square$

Theorem 2.11. *If G is a graph with order n and size m and $\lambda_1^L, \ldots, \lambda_n^L$ are the eigenvalues of the Seidel Laplacian matrix $SL(G)$. Then*

$$(n-2)\left(\frac{16m^2}{n} - n(n-1) - 4Z_1\right) + n(n-1)(\det\widehat{SL}(G))^{\frac{2}{n}} \leq (E_{SL}(G))^2$$

and

$$(E_{SL}(G))^2 \leq n(\det\widehat{SL}(G))^{\frac{2}{n}} + (n-1)\left(n(n-1) + 4Z_1 - \frac{16m^2}{n}\right)$$

where $\det\widehat{SL}(G) = \prod\limits_{i=1}^{n} \sigma_i$ *and* $\sigma_i = \lambda_i^L - \frac{n(n-1)-4m}{n}$.

Proof. Suppose that $\sigma_i = \lambda_i^L - \frac{n(n-1)-4m}{n}$. Then by Lemma 2.9, we have

$$\sum_{i=1}^{n} \sigma_i = 0, \ \sum_{i=0}^{n} \sigma_i^2 = n(n-1) + 4Z_1 - \frac{16m^2}{n}. \tag{2.2}$$

So, by formula (1.2), $E_{SL}(G) = \sum_{i=1}^{n} |\sigma_i|$. If $a_i = \sigma_i^2$, then by the right inequality in Lemma 2.6 and Lemma 2.9, we have:

$$\begin{aligned} n\sum_{i=1}^{n}\sigma_i^2 - \left(\sum_{i=1}^{n}|\sigma_i|\right)^2 &\leq \\ n(n-1)\left[\frac{1}{n}\sum_{i=1}^{n}\sigma_i^2 - \left(\prod_{i=1}^{n}\sigma_i^2\right)^{1/n}\right] &\leq \\ 4(n-1)Z_1 + n(n-1)^2 - \frac{16m^2(n-1)}{n} - n(n-1)\left(\det\widehat{SL}(G)\right)^{\frac{2}{n}}&. \end{aligned}$$

By formula (2.2), we get, $n\sum\limits_{i=1}^{n}\sigma_i^2 - \left(\sum\limits_{i=1}^{n}|\sigma_i|\right)^2 = n(n-1) + 4Z_1 - \frac{16m^2}{n} - (E_{SL}(G))^2$. So,

$$\begin{aligned} n(n-1) + 4Z_1 - \frac{16m^2}{n} - (E_{SL}(G))^2 &\leq \\ 4(n-1)Z_1 + n(n-1)^2 - \frac{16m^2(n-1)}{n} - n(n-1)\left(\det\widehat{SL}(G)\right)^{\frac{2}{n}}&. \end{aligned}$$

Hence, we find that,

$$\begin{aligned} (E_{SL}(G))^2 &\geq \frac{16m^2}{n}(n-2) - n(n-1)(n-2) - 4Z_1(n-2) + n(n-1)(\det\widehat{SL}(G))^{\frac{2}{n}} \\ &= (n-2)\left(\frac{16m^2}{n} - n(n-1) - 4Z_1\right) + n(n-1)(\det\widehat{SL}(G))^{\frac{2}{n}}. \end{aligned}$$

To obtain the upper bound, we use the left inequality in Lemma 2.6.

$$\begin{aligned} n\sum_{i=1}^{n}\sigma_i^2-\left(\sum_{i=1}^{n}|\sigma_i|\right)^2 &\geq n\left[1/n\sum_{i=1}^{n}\sigma_i^2-\left(\prod_{i=1}^{n}\sigma_i^2\right)^{\frac{1}{n}}\right]\\ &= n\left[1/n\left(n(n-1)+4Z_1-\frac{16m^2}{n}\right)-\left(\det\widehat{SL}(G)\right)^{\frac{2}{n}}\right]\\ &= n(n-1)+4Z_1-\frac{16m^2}{n}-n\left(\det\widehat{SL}(G)\right)^{\frac{2}{n}}. \end{aligned}$$

So, we find that,

$$\begin{aligned} E_{SL}^2(G) &\leq n(\det\widehat{SL}(G))^{\frac{2}{n}}+4(n-1)Z_1-n(n-1)^2-\frac{16m^2(n-1)}{n}\\ &= n(\det\widehat{SL}(G))^{\frac{2}{n}}+(n-1)\left(n(n-1)+4Z_1-\frac{16m^2}{n}\right) \end{aligned}$$

and the proof is completed. □

Below, follow Theorem 1.3. the ideas of [11].

Lemma 2.12. *If G_1 and G_2 are two components of a disconnected graph G with vertices n_1 and n_2, respectively. Then the Seidel Laplacian energy of G, $E_{SL}(G)$ has the following inequality,*

$$E_{SL}(G_1)+E_{SL}(G_2)\leq E_{SL}(G)\leq E_{SL}(G_1)+E_{SL}(G_2)+2\sqrt{n_1n_2}$$

Proof. Hear the Seidel Laplacian matrix $SL(G)=\begin{pmatrix} SL_1 & -J\\ -J^T & SL_2\end{pmatrix}$ where SL_1 is the Seidel Laplasian matrix G_1 and SL_2 is the Seidel Laplacian matrix G_2 and J is a non square matrix with entries as unity. Obviously, $E_{SL}(G)\geq E_{SL}(G_1)+E_{SL}(G_2)$. Also $\begin{pmatrix} SL_1 & -J\\ -J^T & SL_2\end{pmatrix}=\begin{pmatrix} SL_1 & 0\\ 0 & SL_2\end{pmatrix}+\begin{pmatrix} 0 & -J\\ -J^T & 0\end{pmatrix}$. Applying the Courant-Weyl inequality for real symmetric matrices, we obtain $E_{SL}(G)\leq E_{SL}(G_1)+E_{SL}(G_2)+2\sqrt{n_1n_2}$. □

Corollary 2.13. *If G_1 and G_2 are two components of a disconnected graph G with order n, then the Seidel Laplacian energy of G has the following inequality, $E_{SL}(G)\leq E_{SL}(G_1)+E_{SL}(G_2)+2n$.*

Definition 2.14. *The **Cocktail party** graph is denoted by $K_{n\times 2}$, is a simple graph of order $2n$, also called the hyper octahedral graph is the graph consisting of two rows of paired nodes in which all nodes but the paired ones are connected with a graph edge. It is the graph complement of the ladder rung graph L_n and the dual graph of the hypercube graph Q_n.*

Definition 2.15. *A **Crown** graph on $2n$ vertices is an undirected graph with two sets of vertices u_i and v_i and with an edge from u_i to v_j whenever $i\neq j$. The crown graph can be viewed as a complete bipartite graph from which the edges of a perfect matching have been removed.*

Theorem 2.16. *Let G be a Cocktail party graph $K_{n\times 2}$ Figure 1 or a Crown graph of order n, Figure 2. For $n\geqslant 2$, the Seidel energy $E_S(G)=6n-6$.*

Proof. By the Lemma 2.3 and the Seidel energy formula the proof is straightforward. □

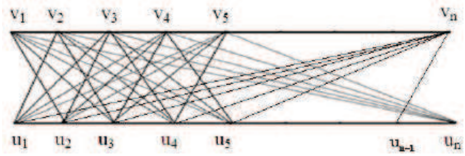

Figure 1: Cocktail party graph

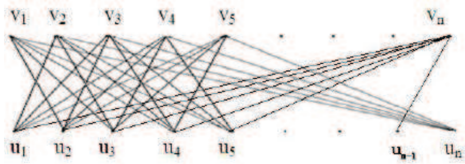

Figure 2: Crown graph

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Jalal Askari,
Department of Applied Mathematics, Faculty of Mathematical Sciences,
University of Kashan, P. O. Box 87317-53153,
Kashan, Iran.
E-mail address: askari@kashanu.ac.ir