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A Topological Approach to Soft Groupoids

Gulay Oguz

ABSTRACT: In this manuscript, the notion of soft topological groupoid is proposed by examining in a different perspective the notion of soft groupoid, which is defined as a parameterized family of the subgroupoids of a soft groupoid. In this context, topological groupoid structures are equipped with soft set theory and the category of soft topological groupoids is formed. Subsequently, the definition of soft topological subgroupoid is presented and all new notions given are strengthened by illustrative examples.

Key Words: Soft set, groupoid, topological groupoid, soft groupoid, soft topological groupoid.

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1. Introduction

The science of mathematics, which is as old and long as the history of human, is of vital importance in the modeling of sophisticated problems encountered in many disciplines such as engineering, economics, medical science and environmental science. However, in order to solve many problems that vary with subjective data over time, the existing mathematical methods have been insufficient due to some genetic difficulties within themselves. This led scientists to search a new theory free of all these difficulties. As a result of these searches, several theories such as fuzzy set theory, soft set theory and rough set theory have emerged [1, 19, 20].

The soft set theory introduced by Molodtsov has taken its place in the world of mathematics as a powerful tool to model uncertainties [1]. This theory, which maintains its rapid development in almost every discipline, attracts both practical and theoretical mathematicians [2 − 11]. After Molodtsov, the first studies on soft sets were done by defining some new operators by Maji and et.al [2]. The introduction of the soft group and its properties by Aktas and Cagman constructed the basis of algebraic studies for this theory [4]. Also, the topological studies about this theory were initiated by Shabir and Naz [8]. By presenting the definition of a soft topological space, they examined the separation axioms in a soft topological space [8]. Other than these, many algebraic and topological studies have been performed [5 − 7, 9 − 11, 21].

The concepts of category and groupoid that have an interdisciplinary character play an important role in many areas of mathematics as well as in the frame study of algebraic topology [12 − 13]. These concepts were transferred to soft set theory [15 − 17]. The definitions of soft category and soft groupoid were given as a natural result of the intersection of these concepts with soft set theory [15 − 17].

This manuscript is based at the interface of soft set theory and topological groupoids. This remainder part of this manuscript is structured as follows. Section 2 deals with some useful preliminaries that are related to the present study. In Section 3, the concept of soft topological groupoid is introduced and its properties are investigated. Also, the category of soft topological groupoids is established, indicated by $STGd$. Conclusion is drawn with the last section.

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2. Preliminaries

This section of the manuscript is included to recall the main definitions and characterizations concerning soft set theory and groupoid theory. For more details, we recommend to [2, 4, 10, 12-15].

Let X be an initial universe set and E be the non-emty set of parameters. Then, a soft set described by Molodtsov is as follows.

Definition 2.1. [1] *Let $P(X)$ denotes the power set of X and $A \subset E$. A soft set over X is a pair (F, A) such that F is a mapping defined by $F : A \longrightarrow P(X)$.*

From the above definition, one realizes that a soft set over X is actually a parameterized family of subsets of the universe X. For later use, the notion of soft set (F, A) over X will simply be denoted by (X, F, A).

Definition 2.2. [2] *A soft set (X, F, A) is called a* ***null*** *soft set indicated by Φ, if $F(a) = \emptyset$ for all $a \in A$.*

Definition 2.3. [2]. *A soft set (X, F, A) is called an* ***absolute*** *soft set indicated by $\tilde{A}$, if $F(a) = X$ for all $a \in A$.*

We now give the concepts of soft topological group and soft topological ring. In the following, G is a group and A is a nonempty set.

Definition 2.4. [10] *Let G be a group which is furnished with a topology τ, let (F, A) be a non-null soft set defined over G. Then, the triplet (F, A, τ) is defined to be a* ***soft topological group*** *over G if the following two conditions hold for all $a \in A$:*

i. *$F(a)$ is a subgroup of G.*

ii. *the mapping $(x, y) \longmapsto x - y$ of the topological space $F(a) \times F(a)$ onto $F(a)$ is continuous.*

Example 2.5. [10] *Assume that $G = S_3 = \{e, (12), (13), (23), (123), (132)\}$, $A = \{e_1, e_2, e_3\}$ and $F(e_1) = \{e\}$, $F(e_2) = \{e, (12)\}$, $F(e_3) = \{e, (123), (132)\}$. Base for the topology τ is described by $\mathfrak{B} = \{\{e\}\{(12)\}\{(123)\}\{(132)\}\}$. It is a straightforward to see that $F(a)$ is a subgroup of G for all $a \in A$. Further, the second condition of above definition is provided. Therefore, (F, A, τ) is a soft topological group.*

Definition 2.6. [10] *Let (F, A, τ) and (F', B, τ') be two soft topological groups over G and G', respectively. Let $f : G \longrightarrow G'$ and $g : A \longrightarrow B$ be two mappings. Then, the pair (f, g) is said to be a* ***soft topological group homomorphism*** *if the following conditions are satisfied:*

i. *f is a group epimorphism and g is surjection.*

ii. *$f(F(a)) = F'(g(a))$ for all $a \in A$.*

iii. *$f_a : (F(a), \tau_{F(a)}) \longrightarrow (F'(g(a)), {\tau'}_{F'(g(a))})$ is continuous for all $a \in A$.*

Note that (F, A, τ) is said to be soft topologically homomorphic to (F', B, τ') and written as $(F, A, \tau) \sim (F', B, \tau')$.

Definition 2.7. [10] *Let R be a commutative ring which is furnished with a topology τ, and let (F, A) be a non-null soft set defined over R. Then, the triplet (F, A, τ) is defined to be a* ***soft topological ring*** *over R if the following conditions are satisfied for all $a \in A$:*

i. *$F(a)$ is a subring of G.*

ii. *the mapping $F(a) \times F(a) \longrightarrow F(a)$ defined by $(x, y) \longmapsto x - y$ is continuous.*

iii. *the mapping $F(a) \times F(a) \longrightarrow F(a)$ defined by $(x, y) \longmapsto x \cdot y$ is continuous.*

Definition 2.8. [10] *Let (F, A, τ) and (F', B, τ') be two soft topological rings over R and R', respectively. Let $f : R \longrightarrow R'$ and $g : A \longrightarrow B$ be two mappings. Then, the pair (f, g) is said to be a* ***soft topological ring homomorphism*** *such that the following conditions hold:*

i. *f is a ring epimorphism and g is surjection.*

ii. *$f(F(a)) = F'(g(a))$ for all $a \in A$.*

iii. *$f_a : (F(a), \tau_{F(a)}) \longrightarrow (F'(g(a)), {\tau'}_{F'(g(a))})$ is continuous for all $a \in A$.*

Then (F, A, τ) is said to be soft topologically homomorphic to (F', B, τ') and, written as $(F, A, \tau) \sim (F', B, \tau')$. Here, we assume familiarity with the definitions of category theory. Below, we give some well known the descriptions of grupoid and topological groupoid.

Definition 2.9. [13] *A groupoid is a structure $\mathcal{G} = (Mor(\mathcal{G}),\ Ob(\mathcal{G}),\ \alpha,\ \beta,\ m,\ \epsilon,\ i)$ defined by a set $Mor(\mathcal{C})$ of morphisms, a set $Ob(\mathcal{C})$ of objects and five structure maps;*

$$Mor(\mathcal{C})\ {}_{\alpha}\times_{\beta} Mor(\mathcal{C}) \xrightarrow{m} Mor(\mathcal{C}) \xrightarrow{\alpha,\ \beta} Ob(\mathcal{C}) \xrightarrow{\epsilon} Mor(\mathcal{C})$$

For any morphism $g \in Mor(\mathcal{C})$, the maps α and β are called the source and target maps, respectively. The map m is defined for any pair of morphisms f, g with $\alpha(f) = \beta(g)$, and assigns to this pair the composition (f, g) also indicated $f \circ g$.
The map ϵ, called inclusion map, assigns to each object $x \in X$ the identity morphism at x, indicated I_x. The map i, called inversion map, assigns to each object $x \in X$ the inverse morphism at x, indicated x^{-1}.

These maps must hold the following identities:

- $\alpha(f \circ g) = \alpha(f) =$ *and* $\beta(f \circ g) = \beta(g)$ *with* $\alpha(g) = \beta(f)$ *for all* $f, g \in Mor(\mathcal{C})$.
- $(f \circ g) \circ h = f \circ (g \circ h)$ *with* $\alpha(g) = \beta(f)$ *and with* $\alpha(h) = \beta(g)$ *for all* $f, g, h \in Mor(\mathcal{C})$.
- $\alpha(I_x) = \beta(I_x) = x$ *for all* $x \in Ob(\mathcal{C})$.
- $f \circ (\beta(f))^{-1} = f$ *and* $(\alpha(f))^{-1} \circ f = f$ *for all* $f \in Mor(\mathcal{C})$.
- *For each* $f \in Mor(\mathcal{C})$, *there exists a inverse* f^{-1} *such that* $\alpha(f^{-1}) = \beta(f)$, $\beta(f^{-1}) = \alpha(f)$ *and* $f \circ f^{-1} = I_{\alpha(f)}$, $f^{-1} \circ f = I_{\beta(f)}$.

Equivalently, notice that a groupoid is a small category which all its morphisms are isomorphisms. The most familiar examples of groupoids are as follows:

Example 2.10. [13] *The category of groups is a groupoid whose the objects are the groups and the morphisms are isomorphisms between them.*

Example 2.11. [13] *The category of topological spaces is a groupoid whose the objects are the topological spaces and the morphisms are homeomorphisms between them.*

Definition 2.12. [13] *Let $\mathcal{G}$ and $\mathcal{H}$ be two grupoids. Then, $\mathcal{H}$ is said to be a subgroupoid of $\mathcal{G}$ if $\mathcal{H}$ consists of a subcollections of $Mor(\mathcal{G})$ and $Ob(\mathcal{G})$ such that it is closed under the composition and inversion in $\mathcal{G}$.*

Moreover, a subgroupoid $\mathcal{H}$ of $\mathcal{G}$ is said to be ***full*** *if $Mor(\mathcal{H}) = Mor(\mathcal{G})$, while it is said to be* ***wide*** *if $Ob(\mathcal{H}) = Ob(\mathcal{G})$.*

Example 2.13. [13] *Given two groupoid $\mathcal{G}$ and $\mathcal{H}$, and suppose $Ob(\mathcal{H}) = Ob(\mathcal{G})$ and $Mor(\mathcal{H}) = \{I_x : x \in Ob(\mathcal{G})\}$. It is easy to see that $\mathcal{H}$ is a subgroupoid of $\mathcal{G}$.*

Definition 2.14. [4] *A topological groupoid is a groupoid $\mathcal{G}$ such that $Mor(\mathcal{C})$ and $Ob(\mathcal{C})$ are sets equipped with a topology and the five structure maps are continuous with respect to these topologies.*

As a result of studying the concept of groupoid with a soft approach, the following important definition was proposed by Oguz et. al [15].

Definition 2.15. [15] *Let A be a set of parameters and let $P(\mathcal{G})$ indicates the set of all subgroupoids of a groupoid $\mathcal{G}$. A soft grupoid over $\mathcal{G}$ is a pair (F, A) such that F is a mapping given by $F : A \longrightarrow P(\mathcal{G})$ and $F(a)$ is a subgrupoid of $\mathcal{G}$ for all $a \in A$.*

Example 2.16. [15] *Every soft group can be seen as a soft groupoid.*

3. Soft Topological Groupoids

In this section, our aim is to introduce the notion of a soft topological groupoid by expanding the soft set theory to topological groupoids. We also establısh the category $STGd$ whose objects are soft topological groupoids and whose morphisms are soft homomorphisms between them. We completed this study by examining soft topological subgroupoids.

Definition 3.1. *Let $\mathcal{G}$ be a groupoid whose object and arrow sets are topological spaces. Let $F : A \longrightarrow P(\mathcal{G})$ be a mapping, where $P(\mathcal{G})$ is the set of all subgroupoids of $\mathcal{G}$, and A is the set of parameters. Then the pair (F, A) is said to be a* ***soft topological groupoid*** *over $\mathcal{G}$ if the following conditions hold for all $a \in A$:*

i. *$F(a)$ is a subgroupoid of $\mathcal{G}$.*
ii. *The structure maps of the subgroupoid $F(a)$ are continuous with respect to the topologies induced by $Ob(\mathcal{G})$ and $Mor(\mathcal{G})$.*

In particular, we emphasized that if $\mathcal{G}$ is a topological groupoid, the provision of the first condition of the above definition is sufficient to call the pair (F, A) as a soft topological groupoid. The generalizing this definition states that the soft topological groupoid (F, A) can be seen as a parameterized family of subgroupoids of the topological groupoid $\mathcal{G}$.

From now on, we often use the notation $(\mathcal{G}, F, A)$ for the topological groupoid (F, A) over $\mathcal{G}$.

Definition 3.2. *Let $(\mathcal{G}, F, A)$ be a soft topological groupoid. If each $F(a)$ is transitive as a topological groupoid then $(\mathcal{G}, F, A)$ is called* ***transitive****. Otherwise, if each $F(a)$ is totally intransitive as a topological groupoid then $(\mathcal{G}, F, A)$ is called* ***totally intransitive****.*

Example 3.3. *Let $\mathcal{G}$ denotes the groupoid of topological groups and isomorphisms between them. Assume $A = \{normal, finite, abelian\}$ is the set of paramaters. $F(a)$ means normal, finite and abelian topological subgroups for all $a \in A$, respectively. Also, these subgroups have the structure of groupoid with isomorphisms between them such that each of them is a topological subgroupoid of $\mathcal{G}$. So, the pair (F, A) is a soft topological groupoid over $\mathcal{G}$.*

In the general case, we have the following.

Example 3.4. *Suppose the pair (F, A) is a soft topological group over the group G with a topology τ. Then, $F(a)$ is a topological subgroup of G for all $a \in A$. Morever, since each group is a groupoid with only one object, it follows that $F(a)$ is a topological subgroupoid of G. Hence, (F, A) is a soft topological groupoid over G.*

In the same way, it is not difficult to check that

Example 3.5. *A soft topological ring (F, A) over the ring R with identity can be seen as a soft topological groupoid over R because each ring is a groupoid with only one object.*

Theorem 3.6. *Every soft groupoid on a topological groupoid is a soft topological groupoid.*

Proof. Assume $\mathcal{G}$ be a topological groupoid and (F, A) be a soft groupoid over $\mathcal{G}$. Notice that $F(a)$ is a subgroupoid of $\mathcal{G}$ for all $a \in A$ such that $F(a)$ is a topological subgroupoid of $\mathcal{G}$ with recpect to the topologies induced by $Mor(\mathcal{G})$ and $Ob(\mathcal{G})$ for all $a \in A$. Thus, (F, A) is a soft topological groupoid over $\mathcal{G}$. □

It is worth pointing out that

Remark 3.7. *Every soft groupoid is not a soft topological groupoid. However, every soft groupoid $\mathcal{G}$ can be transformed into a soft topological groupoid by equipping both $Ob(\mathcal{G})$ and $Mor(\mathcal{G})$ with discrete or indiscrete topology.*

Definition 3.8. *For two soft topological groupoids $(\mathcal{G}, F, A)$ and $(\mathcal{H}, F', B)$, consider the pair $(\mathcal{K}, g)$, where $g : A \longrightarrow B$ is a surjection and $\mathcal{K} : \mathcal{G} \longrightarrow \mathcal{H}$ is a full functor. Then, the pair $(\mathcal{K}, g)$ is said to be a* ***soft topological groupoid homomorphism*** *if the following conditions hold for all $a \in A$:*

i. $\mathfrak{K}(F(a)) = F'(g(a))$.

ii. $\mathfrak{K}_a : F(a) \longrightarrow F'(g(a))$ *is continuous.*

It follows that the soft topological groupoids and the soft topological groupoid homomorphisms between them define a new category, denoted by $STGd$

Example 3.9. *For two soft topological groups (F, A, τ) and (F', B, τ) over G and G', respectively, let (f, g) be a soft topological group homomorphism as defined in Definition 2.6 between these soft topological groups. By Example 3.4, each soft topological group is a soft topological groupoid. So, the soft topological groups (G, F, A) and (G', F', B) are soft topological groupoids such that the soft topological homomorphism (f, g) is also a soft topological groupoid homomorphism.*

Before continuing, we would like to provide another important example for the above definition.

Example 3.10. *Consider (F, A, τ) and (F', B, τ') be two soft topological rings over R and R', respectively. Let (f, g) be a soft topological ring homomorphism as defined in Definition 2.8 between these soft topological rings. Since each soft group is a soft groupoid, the soft topological groups (R, F, A) and (R', F', B) are soft topological groupoids so that the soft topological homomorphism (f, g) is a soft topological groupoid homomorphism.*

Let us examine the product of topological groupoids in the context of soft set theory here.

Definition 3.11. *Let (F, A) and (F', B) be two soft topological groupoids over $\mathcal{G}$ and $\mathcal{H}$, respectively. The product of them is defined as*

$$(\mathcal{G}, F, A) \times (\mathcal{H}, F', B) = (\mathcal{G} \times \mathcal{H}, F'', A \times B)$$

, where $F''(a, b) = F(a) \times F'(b)$ for all $(a, b) \in A \times B$.

From the above definition we have shown:

Theorem 3.12. *The product of any two soft topological groupoids is also a soft topological groupoid.*

Proof. Suppose (F, A) and (F', B) are two soft topological groupoids over $\mathcal{G}$ and $\mathcal{H}$, respectively. Then, we have the mappings

$$\begin{array}{rcl} F : A & \longrightarrow & P(\mathcal{G}) \\ a & \mapsto & F(a) \end{array}$$

and

$$\begin{array}{rcl} F' : B & \longrightarrow & P(\mathcal{H}) \\ a & \mapsto & F'(a) \end{array}$$

such that $F(a)$ is a topological subgroupoid of the groupoid $\mathcal{G}$ for all $a \in A$ and $F(b)$ is a topological subgroupoid of the groupoid $\mathcal{H}$ for all $b \in B$. Then by using these mappings , we can define the mapping F'' as

$$\begin{array}{rcl} F'' : A \times B & \longrightarrow & P(\mathcal{G} \times \mathcal{H}) \\ (a, b) & \mapsto & F''(a, b) = F(a) \times F'(b) \end{array}$$

In general, we can see in the groupoid theory that $F(a) \times F'(b)$ is also a topological subgroupoid of the product groupoid $\mathcal{G} \times \mathcal{H}$ for all $(a, b) \in A \times B$ which means that $(\mathcal{G} \times \mathcal{H}, F'', A \times B)$ is a soft topological groupoid. □

The rest of this study introduces soft topological subgroupoids in terms of some soft notions described so far.

Definition 3.13. *Let (F, A) and (F', B) be two soft topological groupoids over $\mathcal{G}$. Then, (F', B) is said to be a* ***soft topological subgroupoid*** *of (F, A) if the following conditions are satisfied:*

i. *$B \subset A$.*

ii. *$F'(b)$ is a topological subgroupoid of $F(b)$ for all $b \in B$.*

Example 3.14. *Let (F, A) be the soft topological groupoid over $\mathcal{G}$ as given in Example 3.3. Let us consider another soft topological groupoid (F', B) over the same topological groupoid, where $B = \{abelian\}$ such that $F'(abelian)$ means the groupoid of all finite abelian groups and isomorphisms between them. It is easily seen that the pair (F', B) is a soft topological subcategory of (F, A).*

A result from the topological groupoid theory which has a soft analogue is:

Theorem 3.15. *If (F', B) is a soft topological subgroupoid of (F, A) and (F'', C) is a soft topological subgroupoid of (F', B), then (F'', C) is the soft topological subgroupoid of (F, A).*

Proof. This is obviously seen from the definition of soft topological subgroupoid. □

Definition 3.16. *Let (F', B) be soft topological subgroupoid of (F, A) over $\mathcal{G}$.*

i. *If $F'(a)$ is a full subgroupoid of $F(a)$ for all $a \in B$, then (F', B) is called a* ***full*** *soft topological subgroupoid of (F, A).*

ii. *If $F'(a)$ is a wide subgroupoid of $F(a)$ for all $a \in B$, then (F', B) is called a* ***wide*** *soft topological subgroupoid of (F, A).*

ii. *If $F'(a)$ is a normal subgroupoid of $F(a)$ for all $a \in B$, then (F', B) is called a soft topological* ***normal*** *subgroupoid of (F, A).*

Example 3.17. *Let us consider the soft topological subgroupoid (F', B) of (F, A) that is given in Example 3.14. It is easy to verify that the soft topological subgroupoid (F', B) is a full soft topological subgroupoid of (F, A).*

4. Conclusion

In this manuscript, we have introduced the concept of soft topological groupoids by soft approach to topological groupoids. As a new category, we have presented the category of soft topological groupoids. Also, we have defined the concept of soft topological subgroupoids and established some important characterizations. It is to be noted that this manuscript can be the initial point for the studies on soft topological groupoids and since it is a new avenue concerning topological research in the context of soft set theory.

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Gulay Oguz,
Department of Mathematics,
Harran University,
Şanlıurfa, Turkey.
E-mail address: `gulay.oguz@harran.edu.tr`