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Analytical and Numerical Approach for a Nonlinear Volterra-Fredholm Integro-differential Equation

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ABSTRACT: An approach for Volterra- Fredholm integro-differential equations using appropriate fixed point theorems of existence, uniqueness is presented. The approximation of the solution is performed using Nyström method in conjunction with successive approximations algorithm. Finally, we give a numerical example, in order to verify the effectiveness of the proposed method with respect to the analytical study.

Key Words: Volterra-Fredholm integro-differential equation, fixed point, Nyströme method, successive approximations.

Contents

1. Introduction

Integral and integro-differential equations have a substantial contributions in various fields of science, e.g. plasma physics, radiative transfer, electromagnetism, biological processes, and fluids mechanics [2,8,10,16]. Hence, it becomes crucial to study the solvability of theses equations and develop effective numerical methods to solve them. Many theoretical treatments, as well as numerical methods have been carried out in this context, namely, spline interpolation [1], hat functions [13,3], collocation method [14,3] and multi-step collocation method [4], finite difference [6] and Haar wavelet [5] and also the product integration method for equation with weakly singular kernel [17]. In recent work [9] the authors studied the existence and uniqueness problem for a Volterra type integro-differential equation, when the derivative of the unknown appears inside the integral. In this paper, we shall be concerned with the treatment of an extended type of [9], which is the following mixed Volterra-Fredholm equation of the second kind :

$$u(t) = f(t) + \int_a^t K_V(t,s,u(s),u'(s))\,ds + \int_a^b K_F(t,s,u(s),u'(s))\,ds, \quad -\infty < a \leq t \leq b < +\infty, \tag{1.1}$$

such that

$$(H1) \quad \left\| \quad K_V,\, K_F\, \frac{\partial K_V}{\partial t},\, \frac{\partial K_F}{\partial t} \text{ are in } C^0([a,b]^2 \times \mathbb{R}^2, \mathbb{R}). \right.$$

and f is a given function in $C^1([a,b],\mathbb{R})$.

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2. Analytical study

This section is devoted to the study of the solvability of (1.1), thus, we discuss practical conditions leading to the existence and uniqueness of the solution for kernels satisfying different properties.

2.1. Compactness of the integral operator:

Define the space $B^1 = C^1([a,b])$ of all continuously differentiable functions on $[a,b]$, equipped with the norm :

$$\|u\|_{B^1} = \|u\|_\infty + \|u'\|_\infty = \sup_{t\in[a,b]} \{|u(t)|\} + \sup_{t\in[a,b]} \{|u'(t)|\}. \tag{2.1}$$

which make B^1 a Banach space.
Let $f \in B^1$ and $\sigma : [a,b] \longrightarrow [a,b]$ a continuously differentiable function.
Consider the operator :

$$\begin{aligned} \forall t \in [a,b], \quad T_f : \quad & B^1 \longrightarrow B^1 \\ & u \longmapsto T_f(u)(t) = f(t) + \int_a^{\sigma(t)} K(t,s,u(s),u'(s))\,ds, \end{aligned}$$

with $K, \frac{\partial K}{\partial t} \in C^0([a,b]^2 \times \mathbb{R}^2, \mathbb{R})$.

Proposition 2.1. *For every $u \in B^1$, $t \in [a,b]$, the derivative of $T_f(u)$ is*

$$T_f'(u)(t) = f'(t) + \sigma'(t)K(\sigma(t),\sigma(t),u(\sigma(t)),u'(\sigma(t))) + \int_a^{\sigma(t)} \frac{\partial K}{\partial t}(t,s,u(s),u'(s))\,ds. \tag{2.2}$$

Proof: Let $h \in \mathbb{R}^*$ and $t \in [a,b]$, then

$$\begin{aligned} \frac{T_f(u)(t+h) - T_f(u)(t)}{h} &= \frac{f(t+h)-f(t)}{h} + \\ + \frac{1}{h}\left(\int_a^{\sigma(t+h)} (K(t+h,s,u(s),u'(s)) - K(t,s,u(s),u'(s))\,ds)\right) &+ \omega(K,h), \end{aligned} \tag{2.3}$$

with

$$\omega(K,h) = \frac{1}{h}\left(\int_a^{\sigma(t+h)} K(t,s,u(s),u'(s))\,ds - \int_a^{\sigma(t)} K(t,s,u(s),u'(s))\,ds\right).$$

Remark that if $\sigma(t+h) > \sigma(t)$, then

$$\omega(K,h) = \frac{1}{h}\int_{\sigma(t)}^{\sigma(t+h)} K(t,s,u(s),u'(s))\,ds,$$

and since there exists $t_h \in (t, t+h)$ verifying $\sigma(t+h) - \sigma(t) = \sigma'(t_h)h$, we get

$$\omega(K,h) = \frac{\sigma'(t_h)}{\sigma(t+h)-\sigma(t)}\int_{\sigma(t)}^{\sigma(t+h)} K(t,s,u(s),u'(s))\,ds.$$

Analogously, if $\sigma(t+h) < \sigma(t)$, then, $\sigma(t) - \sigma(t+h) = -\sigma'(t_h)h$ with $t_h \in (t,t+h)$ and

$$\omega(K,h) = \frac{\sigma'(t_h)}{\sigma(t)-\sigma(t+h)}\int_{\sigma(t+h)}^{\sigma(t)} K(t,s,u(s),u'(s))\,ds.$$

Finally, by the mean value theorem, we obtain

$$\omega(K,h) = \sigma'(t_h)K(t,s_h,u(s_h),u'(s_h)), \quad s_h \in (\sigma(t+h),\sigma(t)).$$

Therefore, the formula (2.2) is obtained from (2.3) by taking the limit as h tends to 0. $\square$

The next theorem establish the compactness of the operator T_f.

Theorem 2.2. *T_f is a compact operator.*

Proof: Let $u_0 \in B^1$ then

$$\|T_f(u) - T_f(u_0)\|_{B^1} \leq \sup_{t\in[a,b]} \int_a^{\sigma(t)} |K(t,s,u(s),u'(s)) - K(t,s,u_0(s),u_0'(s))|\, ds +$$

$$+ \sup_{t\in[a,b]} \{|\sigma'(t)||K(\sigma(t),\sigma(t),u(\sigma(t)),u'(\sigma(t))) - K(\sigma(t),\sigma(t),u_0(\sigma(t)),u_0'(\sigma(t)))|\} +$$

$$+ \sup_{t\in[a,b]} \int_a^{\sigma(t)} \left| \frac{\partial K}{\partial t}(t,s,u(s),u'(s)) - \frac{\partial K}{\partial t}(t,s,u_0(s),u_0'(s)) \right| ds.$$

So, from the continuity of K and $\dfrac{\partial K}{\partial t}$, we deduce that

$$\|T_f(u) - T_f(u_0)\|_{B^1} \longrightarrow 0 \text{ when } \|u-u_0\|_{B^1} \longrightarrow 0.$$

Now, we shall prove that T_f is completely continuous.
Let G a bounded set of B^1 i.e.

$$\exists M > 0 \,\text{such that: } \forall \varphi \in B^1,\ \|\varphi\|_{B^1} \leq M\,.$$

Now, consider the set :

$$T_f(G) = \{g = T_f(u),\ u \in G\}.$$

We observe that

$$\forall g \in T_f(G), \quad \|g\|_{B^1} \leq \|f\|_{B^1} + \max_{t\in[a,b]}\{|\sigma'(t)|\}M_0 + (b-a)(M_0+M_1) = L\,, \tag{2.4}$$

such that

$$M_0 = \max_{[a,b]^2\times \bar{B}_{\mathbb{R}^2}(0_{\mathbb{R}^2},M)} |K(t,s,u(s),u'(s))|\,,\ M_1 = \max_{[a,b]^2\times \bar{B}_{\mathbb{R}^2}(0_{\mathbb{R}^2},M)} \left|\frac{\partial K}{\partial t}(t,s,u(s),u'(s))\right|$$

Our proof will be complete, if we show that $T_f(G)$ is a relatively compact set in B^1.
First, we denote by $B^0 = (C^0([a,b]), \|.\|_\infty)$, so, we remark that

$$T_f(G) \subset B^1 = \{(u,u')/u \in B^0,\ u' \in B^0\} \subset (B^0)^2.$$

Since B^0 is a closed subset of $L^\infty([a,b])$ (the space of all measurable and essentially bounded functions on $[a,b]$), then, it is sufficient to prove that $T_f(G)$ is relatively compact set in $(L^\infty([a,b]))^2$, in the other words, we have to show that for every $\epsilon > 0$ exists a relatively compact ϵ-net of $T_f(G)$ in $(L^\infty([a,b]))^2$. In the following, we proceed as in the proof of the sufficient condition of Arzela-Ascoli theorem in B^0 [11] [15].

Let $\epsilon > 0$, so, since K, $\dfrac{\partial K}{\partial t}$ are uniformly continuous on $[a,b]^2 \times \bar{B}_{\mathbb{R}^2}(0_{\mathbb{R}^2},M)$, as well as, f, f' on $[a,b]$, exists δ_0, $\delta_1 > 0$ such that $\forall t,t' \in [a,b]$, $\forall g \in T_f(G)$

$$|t-t'| < \delta_0 \Longrightarrow |g(t) - g(t')| < \frac{\epsilon}{2}\,,$$

$$|t-t'| < \delta_1 \Longrightarrow |g'(t) - g'(t')| < \frac{\epsilon}{2}\,.$$

Take $\delta = \min(\delta_0,\delta_1)$, since $[a,b]$ is a compact set of the complete metric space $\mathbb{R}$, the Housdorff's theorem (see [11] [15]) asserts the existence of a finite $\frac{\delta}{2}$-net $\{t_1,t_2,...,t_n\}$ for $[a,b]$ in $\mathbb{R}$, which means

that $[a,b] \subseteq \bigcup_{i=1}^{n} B_{\mathbb{R}}\left(t_i, \frac{\delta}{2}\right)$. where $B_{\mathbb{R}}$ is the open ball of $\mathbb{R}$.
Otherwise, if we set

$$\begin{aligned} E_1 &= B_{\mathbb{R}}\left(t_1, \frac{\delta}{2}\right), \\ E_2 &= B_{\mathbb{R}}\left(t_2, \frac{\delta}{2}\right) \backslash B_{\mathbb{R}}\left(t_1, \frac{\delta}{2}\right), \\ &\vdots \\ E_n &= B_{\mathbb{R}}\left(t_n, \frac{\delta}{2}\right) \backslash \bigcup_{i=1}^{n-1} B_{\mathbb{R}}\left(t_i, \frac{\delta}{2}\right). \end{aligned}$$

We see that $E_i \cap E_j = \phi \, \forall i \neq j \, (i,j = \{1,2,...,n\})$, and $[a,b] \subseteq \bigcup_{i=1}^{n} E_i$.

Now, let us consider the following application :

$$\begin{array}{rccl} \psi : & D & \longrightarrow & (L^{\infty}([a,b]))^2 \\ & (\lambda, \eta) & \longmapsto & \left(\sum_{i=1}^{n} \lambda_i \chi_i(t), \sum_{i=1}^{n} \eta_i \chi_i(t)\right), \end{array}$$

where

$$D = \left\{(\lambda,\eta) \in \mathbb{R}^n \times \mathbb{R}^n, \, |\lambda_i| + |\eta_j| \leq L \, , \, (i,j = \{1,...,n\})\right\},$$

and χ_k is the characteristic function of the set $E_i \, (i = \{1,...,n\})$.
It's obvious to see that ψ is an isometric isomorphism from the compact set $D \subset (\mathbb{R}^n)^2$ to $H = Im(\psi) \subset (L^{\infty}([a,b]))^2$ which implies the compactness of H in $(L^{\infty}([a,b]))^2$, so, we have only to prove that H is an ϵ-net of $T_f(G)$.
Hence, take an arbitrary element $\overline{t}_i \in E_i \neq \phi$ for all $i = \{1,2,...n\}$ and consider the family of functions : $\forall g \in T_f(G)$

$$v(t) = \sum_{i=1}^{n} g(\overline{t}_i)\chi_i(t) \quad , \quad w(t) = \sum_{i=1}^{n} g'(\overline{t}_i)\chi_i(t)\,,$$

therefore, from (2.4), the functions $\overline{g} = (v,w)$ belongs to H, moreover, $\forall t \in [a,b]$, $\exists! E_i$ such that $|t - \overline{t}_i| < \delta$, thus

$$|g(t) - v(t)| = |g(t) - g(\overline{t}_i)| < \frac{\epsilon}{2} \quad and \quad |g'(t) - w(t)| = |g'(t) - g'(\overline{t}_i)| < \frac{\epsilon}{2},$$

consequently

$$\|g - \overline{g}\|_{(L^{\infty}([a,b]))^2} = \|g - v\|_{\infty} + \|g' - w\|_{\infty} < \epsilon\,,$$

which means, that H constitutes a relatively compact ϵ-net for $T_f(G)$ in $(L^{\infty}([a,b]))^2$. □

2.2. Existence and uniqueness:

We investigate, in this section, the existence and uniqueness results for equation (1.1) for different types of kernels. Two cases are treated and a comparison of results is undertaken.

Case 1 : We give an existence theorem based on the Schauder fixed point theorem, where the kernels K_V, K_F are uniformly bounded i.e $\exists\, M_V, M_F, \overline{M}_V, \overline{M}_F > 0$ such that

$$(H2) \quad \left\| \begin{array}{ll} \max\limits_{\substack{t,s \in [a,b], \\ (x,y) \in \mathbb{R}^2}} |K_V(t,s,x,y)| \leq M_V, & \max\limits_{\substack{t,s \in [a,b], \\ (x,y) \in \mathbb{R}^2}} |K_F(t,s,x,y)| \leq M_F, \\ \max\limits_{\substack{t,s \in [a,b], \\ (x,y) \in \mathbb{R}^2}} \left|\frac{\partial K_V}{\partial t}(t,s,x,y)\right| \leq \overline{M}_V, & \max\limits_{\substack{t,s \in [a,b], \\ (x,y) \in \mathbb{R}^2}} \left|\frac{\partial K_F}{\partial t}(t,s,x,y)\right| \leq \overline{M}_F. \end{array} \right.$$

After that, we discuss the uniqueness in the case where the kernels K_V, K_F satisfy the following properties [9]: $\exists\, A, B, \overline{A}, \overline{B}, C, D, \overline{C}, \overline{D} > 0$ such that $\forall t, s \in [a,b]$, $x_1, y_1, x_2, y_2 \in \mathbb{R}$

$$(H3)\quad \left\|\begin{array}{l} |K_V(t,s,x_1,y_1) - K_V(t,s,x_2,y_2)| \le A|x_1-x_2| + B|y_1-y_2|, \\ \left|\dfrac{\partial K_V}{\partial t}(t,s,x_1,y_1) - \dfrac{\partial K_V}{\partial t}(t,s,x_2,y_2)\right| \le \overline{A}|x_1-x_2| + \overline{B}|y_1-y_2|. \\ |K_F(t,s,x_1,y_1) - K_V(t,s,x_2,y_2)| \le C|x_1-x_2| + D|y_1-y_2|, \\ \left|\dfrac{\partial K_F}{\partial t}(t,s,x_1,y_1) - \dfrac{\partial K_F}{\partial t}(t,s,x_2,y_2)\right| \le \overline{C}|x_1-x_2| + \overline{D}|y_1-y_2|. \end{array}\right.$$

Case 2 : We establish an other existence and uniqueness result, based on the standard Banach fixed point theorem, according to $(H3)$ where the kernels K_V, K_F are not necessary uniformly bounded ($(H2)$ not verified).

Let us consider the first case in details, so, the equation (1.1) is of the form

$$u(t) = S_f(u)(t), \quad t \in [a,b]. \tag{2.5}$$

Thus, from the previous result, we deduce that S is a compact operator (as a linear combination of two compact operators) from B^1 into itself.

Theorem 2.3. *According to the conditions* $(H1)$, $(H2)$, *the equation* (1.1) *has a solution in* B^1.

Proof: Consider the closed bounded convex set $Q = \overline{B}_{B^1}(0_{B^1}, \|f\|_{B^1} + M_V + (b-a)(M_V + M_F + \overline{M}_V + \overline{M}_F))$, so we have $S_f(Q) \subset Q$, therefore, the existence of the solution is ensured by the Schauder fixed point theorem [11] [15]. $\square$

The following theorem gives us the conditions ensuring the uniqueness in B^1.

Theorem 2.4.
By considering the hypothesis $(H3)$ *and if*

$$\min(\theta_1, \theta_2) < \frac{1}{b-a}, \tag{2.6}$$

with

$$\theta_1 = \max\left(A + C + \frac{(B+D)(A+(b-a)(\overline{A}+\overline{C}))}{1 - B - (b-a)(\overline{B}+\overline{D})}\,,\ \frac{B}{b-a} + \overline{B} + \overline{D}\right),$$
$$\theta_2 = \max\left(\frac{(B+D)(A+(b-a)(\overline{A}+\overline{C}))}{1-(A+C)(b-a)} + \frac{B}{b-a} + \overline{B} + \overline{D}\,,\ A + C\right),$$

then, the solution of (1.1) *is unique in* B^1.

Proof: 1- Consider two different solutions $u_1, u_2 \in B^1$, $(u_1 \neq u_2)$ and $t \in [a,b]$, so, by $(H3)$, we get

$$\begin{aligned} |u_1(t) - u_2(t)| \le\ & A\int_a^t |u_1(s)-u_2(s)|\,ds + B\int_a^t |u_1'(s)-u_2'(s)|\,ds + \\ & + C\int_a^b |u_1(s)-u_2(s)|\,ds + D\int_a^b |u_1'(s)-u_2'(s)|\,ds \end{aligned}$$

and

$$\begin{aligned} |u_1'(t) - u_2'(t)| \le\ & A|u_1(t)-u_2(t)| + B|u_1'(t)-u_2'(t)| + \\ & + \overline{A}\int_a^t |u_1(s)-u_2(s)|\,ds + \overline{B}\int_a^t |u_1'(s)-u_2'(s)|\,ds + \\ & + \overline{C}\int_a^b |u_1(s)-u_2(s)|\,ds + \overline{D}\int_a^b |u_1'(s)-u_2'(s)|\,ds, \end{aligned}$$

thus

$$\|u_1-u_2\|_\infty \le (b-a)((A+C)\|u_1-u_2\|_\infty + (B+D)\|u_1'-u_2'\|_\infty), \tag{2.7}$$

and

$$\|u_1'-u_2'\|_\infty \le \left[A+(b-a)(\overline{A}+\overline{C})\right]\|u_1-u_2\|_\infty + \left[B+(b-a)(\overline{B}+\overline{D})\right]\|u_1'-u_2'\|_\infty, \tag{2.8}$$

thus, from (2.8)

$$\|u_1'-u_2'\|_\infty \le \frac{A+(b-a)(\overline{A}+\overline{C})}{1-B-(b-a)(\overline{B}+\overline{D})}\|u_1-u_2\|_\infty,$$

and by substitution in (2.7), we obtain

$$\|u_1-u_2\|_\infty \le (b-a)\theta_1\|u_1-u_2\|_\infty,$$

in the same way, we have from (2.7)

$$\|u_1-u_2\|_\infty \le (b-a)\frac{B+D}{1-(A+C)(b-a)}\|u_1'-u_2'\|_\infty,$$

substituting the previous relation in (2.8) leads to

$$\|u_1'-u_2'\|_\infty \le (b-a)\theta_2\|u_1'-u_2'\|_\infty,$$

finally, by (2.6), we deduce that $u_1=u_2$ in B^1. □

Now, consider the second case; the next theorem show the existence and uniqueness of the solution by direct application of the Banach fixed point theorem [11].

Theorem 2.5. *By considering* $(H3)$ *and if*

$$\eta = \max\left(A\left(1+\frac{1}{b-a}\right)+\overline{A}+C+\overline{C},\, B\left(1+\frac{1}{b-a}\right)+\overline{B}+D+\overline{D}\right) < \frac{1}{b-a}, \tag{2.9}$$

then, the solution exist and unique in B^1.

Proof: Let us consider $u_1,\, u_2 \in B^1$, so

$$\|S_f u_1 - S_f u_2\|_\infty \le (b-a)((A+C)\|u_1-u_2\|_\infty + (B+D)\|u_1'-u_2'\|_\infty),$$

and

$$\|S_f' u_1 - S_f' u_2\|_\infty \le \left[A+(b-a)(\overline{A}+\overline{C})\right]\|u_1-u_2\|_\infty + \left[B+(b-a)(\overline{B}+\overline{D})\right]\|u_1'-u_2'\|_\infty,$$

consequently

$$\|S_f u_1 - S_f u_2\|_{B^1} \le (b-a)\eta\|u_1-u_2\|_{B^1}.$$

The existence of the unique solution is obtained by the Banach fixed point theorem [11]. □

Remark 2.6. *1- The condition* (2.6) *is more general then* (2.9)*. In fact, if* (2.9) *is satisfied, then*

$$\max\left(\frac{B}{b-a}+\overline{B}+\overline{D},\, A+C\right) < \eta,$$

moreover

$$\frac{B+D}{1-B-(b-a)(\overline{B}+\overline{D})} < \frac{1}{b-a}, \quad \text{and} \quad \frac{A+(b-a)(\overline{A}+\overline{C})}{1-(A+C)(b-a)} < 1,$$

which implies that $\theta_{1,2} < \eta$.
Conversely, if $b-a=1$ *and* $A=\frac{1}{24}$, $\overline{A}=\frac{1}{12}$, $B=\frac{1}{16}$, $\overline{B}=\frac{1}{8}$ *and* $C=\frac{1}{2}$, $\overline{C}=\frac{1}{8}$, $D=\frac{3}{4}$, $\overline{D}=\frac{3}{16}$,

then, $\eta = \max\left(\frac{19}{24}, \frac{19}{16}\right) > 1.$

In the other side, we remark that $\theta_1 = \max\left(\frac{13}{24} + \frac{13}{40}, \frac{3}{8}\right) < 1.$

2- The hypothesis $(H3)$ *with the condition* (2.9) *remain general in the case where* $(H2)$ *not satisfied, for example, the linear kernels with respect to the third and fourth variable or kernels of the form* $K(t,s,x,y) = \overline{K}(t,s)\sqrt{c + x^2 + y^2}$, $c > 0$.

In a nutshell, the existence of the solution of (1.1) *is ensured in the case where* $(H1)$, $(H2)$ *are satisfied, furthermore, if the kernels satisfy* $(H3)$ *with* (2.6)*, then, the solution is unique. Otherwise, the unique solution of* (1.1)*, with not necessary uniformly bounded kernels, may be deduced directly from* $(H3)$ *with a strong condition* (2.9) *in comparison with* (2.6).

3. Numerical treatment

In this section, an approximation algorithm is applied to solve equation (1) using the well known Nyström method. Note that, in [18], the authors presented a numerical scheme for extracting approximate solution for the nonlinear Volterra-Fredholm integral equation by an iterative method. For this purpose, we use the successive approximation method to solve a nonlinear algebraic system constructed from the equation (1.1) by introducing the quadrature rules.
First, let $N \in \mathbb{N}^*$ and consider the following partition Δ_N of $[a,b]$:

$$\Delta_N = \{a = t_0 < t_1 < ... < t_N = b\}.$$

Denote for all $i = 1, ..., N$, $u(t_i) = u_i$, $u'(t_i) = u'_i$, $f(t_i) = f_i$, so, we write the equation (1.1) over Δ_N as follows :

$$\forall i = 0, 1, ..., N, \quad u_i = \int_a^{t_i} K_V(t_i, s, u(s), u'(s))\, ds + \int_a^b K_F(t_i, s, u(s), u'(s))\, ds. \tag{3.1}$$

It is easy to observe, that the derivative of the solution u of (1.1) is given implicitly in the form : $\forall t \in [a,b]$

$$u'(t) = f'(t) + K_V(t, t, u(t), u'(t)) + \int_a^t \frac{\partial K_V}{\partial t}(t, s, u(s), u'(s))\, ds + \int_a^b \frac{\partial K_F}{\partial t}(t, s, u(s), u'(s))\, ds, \tag{3.2}$$

then, we can write the equation (3.2) over Δ_N as follows : $\forall i = 0, 1, ...N$

$$u'_i = f'_i + K_V(t_i, t_i, u_i, u'_i) + \int_a^{t_i} \frac{\partial K_V}{\partial t}(t_i, s, u(s), u'(s))\, ds + \int_a^b \frac{\partial K_F}{\partial t}(t_i, s, u(s), u'(s))\, ds, \tag{3.3}$$

The method consists, first, to replace the integrals containing K_F and $\frac{\partial K_F}{\partial t}$ by the following numerical integration scheme : $\forall i = 0, 1, ..., N$

$$\int_a^b K_F(t_i, s, u(s), u'(s))\, ds = \sum_{j=0}^N w_{j,N} K_F(t_i, t_j, u_j, u'_j) + O_F(N, i), \tag{3.4}$$

$$\int_a^b \frac{\partial K_F}{\partial t}(t_i, s, u(s), u'(s))\, ds = \sum_{j=0}^N w_{j,N} \frac{\partial K_F}{\partial t}(t_i, t_j, u_j, u'_j) + \overline{O}_F(N, i), \tag{3.5}$$

where $w_{j,N}$, $j = 0, 1, ..., N$, are the integration weights and $O_V(N,i)$, $O_F(N,i)$ are the local integration errors.
Suppose that the numerical integration schemes (3.4) (3.5) are interpolatory, i.e its accuracy degree is at least N, therefore, it must be exact for all polynomial p of degree $\leq N$, so if we take $p(t) \equiv 1$, we obtain

$$\forall i = 0, ..., N, \quad b - a = \sum_{j=0}^N w_{j,N}.$$

Also, we suppose that $w_{j,N}, > 0$, $\forall j = 0, 1, ..., N$, which correspond to a large variety of rules (the composite trapezoidal or generally Newton-Cotes rules, the Tchebychev and Gauss rules).
After that, in order to approach the integrals containing K_V and $\dfrac{\partial K_V}{\partial t}$, we use a quadrature rule which conserves the same values u_j, $j = 0, 1, ..., N$, in the nodal points of Δ_N (for example the composite trapezoidal rule or generally the quadrature rules based on the interpolation by Splines), i.e $\forall i = 0, ..., N$

$$\int_a^{t_i} K_V(t_i, s, u(s), u'(s))\, ds = \sum_{j=0}^{i} \sigma_{j,i} K_V(t_i, t_j, u_j, u'_j) + O_V(N, i), \tag{3.6}$$

$$\int_a^{t_i} \frac{\partial K_V}{\partial t}(t_i, s, u(s), u'(s))\, ds = \sum_{j=0}^{i} \sigma_{j,i} \frac{\partial K_V}{\partial t}(t_i, t_j, u_j, u'_j) + \overline{O}_V(N, i), \tag{3.7}$$

Assume, also, that $\sigma_{j,i}, > 0$ and verify $t_i - a = \sum_{j=0}^{i} \sigma_{j,i}$ (which is the case for the composite trapezoidal rule).
Finally, using (3.4) (3.5) (3.6) (3.7), we have the following algebraic system : $\forall i = 0, 1, ..., N$

$$u_i = f_i + \sum_{j=0}^{i} \sigma_{j,i} K_V(t_i, t_j, u_j, u'_j) + \sum_{j=0}^{N} w_{j,N} K_F(t_i, t_j, u_j, u'_j) + O_V(N, i) + O_F(N, i), \tag{3.8}$$

$$\begin{aligned} u'_i = \quad & f_i + K_V(t_i, t_i, u_i, u'_i) + \sum_{j=0}^{i} \sigma_{j,i} \frac{\partial K_V}{\partial t}(t_i, t_j, u_j, u'_j) + \sum_{j=0}^{N} w_{j,N} \frac{\partial K_F}{\partial t}(t_i, t_j, u_j, u'_j) + \\ & + \overline{O}_V(N, i) + \overline{O}_F(N, i) \end{aligned} \tag{3.9}$$

Note that the numerical integration rules used in (3.4) (3.5) (3.6) (3.7) are consistent if

$$\lim_{N \longrightarrow +\infty} \left(\max_{i=0,1,...,N} (|O_V(N, i)| + |O_F(N, i)|) + \max_{i=0,1,...,N} \left(|\overline{O}_V(N, i)| + |\overline{O}_F(N, i)|\right) \right) = 0.$$

So, the equations (3.8), (3.9) can be approximated by the following nonlinear algebraic system in $\mathbb{R}^{2N+2}$: $\forall i = 0, 1, ..., N$

$$x_i = f_i + \sum_{j=0}^{i} \sigma_{j,i} K_V(t_i, t_j, x_j, y_j) + \sum_{j=0}^{N} w_{j,N} K_F(t_i, t_j, x_j, y'_j), \tag{3.10}$$

$$y_i = f'_i + K_V(t_i, t_i, x_i, y_i) + \sum_{j=0}^{i} \sigma_{j,i} \frac{\partial K_V}{\partial t}(t_i, t_j, x_j, y_j) + \sum_{j=0}^{N} w_{j,i} \frac{\partial K_F}{\partial t}(t_i, t_j, x_j, y_j). \tag{3.11}$$

with $v = (x_0, ..., x_N, y_0, ..., y_N)$ the unknown vector in $\mathbb{R}^{2N+2}$.

3.1. System study:

The system (3.10), (3.11) can be written as follows :

$$v = \Psi(v) = \begin{cases} \Psi_1(v) = \left(f_i + \displaystyle\sum_{j=0}^{i} \sigma_{j,i} K_V(t_i, t_j, x_j, y_j) + \sum_{j=0}^{N} w_{j,N} K_F(t_i, t_j, x_j, y_j) \right)_{i=0,...,N}, \\ \Psi_2(v) = \begin{pmatrix} f'_i + K_V(t_i, t_i, x_i, y_i) + \displaystyle\sum_{j=0}^{i} \sigma_{j,i} \frac{\partial K_V}{\partial t}(t_i, t_j, x_j, y_j) + \\ + \displaystyle\sum_{j=0}^{N} w_{j,N} \frac{\partial K_F}{\partial t}(t_i, t_j, x_j, y_j). \end{pmatrix}_{i=0,...,N} \end{cases}. \tag{3.12}$$

Our aim is to investigate the condition ensuring the existence and uniqueness of the solution of the previous algebraic system according to the two cases treated in section 1.
Case 1:

Proposition 3.1. *According to hypotheses* $(H1)$, $(H2)$, *the equation* (3.12) *has at least one solution.*

Proof: First, from $(H1)$, we see that Ψ is continuous over $\mathbb{R}^{2N+2}$.
Denote $d_1 = \|f\|_\infty + (b-a)(M_V + M_F)$, $d_2 = \|f'\|_\infty + M_V + (b-a)(\overline{M_V} + \overline{M_F})$, hence, the set $Z = [-d_1, d_1]^{N+1} \times [-d_2, d_2]^{N+1}$ is a compact convex subset of $\mathbb{R}^{2N+2}$, furthermore, $\Psi(Z) \subseteq Z$.
Thus, the existence of the solution is guaranteed by the Brouwer's fixed point theorem [11]. □

The next preposition gives a uniqueness result analogously to which obtained in Theorem 2.3.

Proposition 3.2. *If* $(H3)$, (2.6) *are satisfied, then, the solution of* (3.12) *is unique.*

Proof: Define the norm :

$$\|v\|_{\mathbb{R}^{2N+2}} = \max_{i=0,1,\dots N} |x_i| + \max_{i=0,1,\dots,N} |y_i| = \|x\|_\infty + \|y\|_\infty,$$

and suppose that the system (3.10), (3.11) admits tow different solutions $v_1 \neq v_2$, so, by using $(H3)$, we get

$$\|x_1 - x_2\|_\infty \leq (b-a)\left((A+C)\|x_1 - x_2\|_\infty + (B+D)\|y_1 - y_2\|_\infty\right). \tag{3.13}$$

and

$$\|y_1 - y_2\|_\infty \leq \left(A + (b-a)(\overline{A} + \overline{C})\right)\|x_1 - x_2\|_\infty + \left(B + (b-a)(\overline{B} + \overline{D})\right)\|y_1 - y_2\|_\infty. \tag{3.14}$$

As in theorem 2.3, we deduce from (2.6) that $v_1 = v_2$. □

Case 2 :
The existence and uniqueness of the solution of (3.10), (3.11) according to $(H3)$ and (2.9) can be obtained in a similar way as in Theorem 2.4.

3.2. Convergence analysis:

In what follows, we treat the convergence of the exact solution (x, y) of the nonlinear system (3.10), (3.11) to the continuous solution (u, u') as $N \longrightarrow +\infty$.

Case 1 :
Let us consider the approximation error :

$$E(N) = \max_{i=0,\dots,N} |x_i - u_i| + \max_{i=0,\dots,N} |y_i - u'_i| = E_0(N) + E_1(N). \tag{3.15}$$

Proposition 3.3. *Denote by*

$$O(N) = O_0(N) + O_1(N),$$

with

$$O_0(N) = \max_{i=0,1,\dots,N}\left(|O_V(N,i)| + |O_F(N,i)|\right), \quad O_1(N) = \max_{i=0,1,\dots,N}\left(|\overline{O}_V(N,i)| + |\overline{O}_F(N,i)|\right),$$

so, from the hypothesis $(H3)$ *and condition* (2.6), *we obtain the following error estimation:*
- If $\min(\theta_1, \theta_2) = \theta_1$, *then*

$$E_0(N) \leq \frac{C_1 O(N)}{1 - (b-a)\theta_1}, \tag{3.16}$$

$$E_1(N) \leq \left(\frac{C_1[A + (b-a)(\overline{A} + \overline{C})] + 1 - (b-a)\theta_1}{(1 - (b-a)\theta_1)^2}\right) O(N), \tag{3.17}$$

with $C_1 = \max\left(1, \frac{(b-a)(B+D)}{1-\theta_1(b-a)}\right)$.

- If $\min(\theta_1,\theta_2)=\theta_2$*, then*

$$E_0(N) \leq \left(\frac{C_2(b-a)(B+D)+1-(b-a)\theta_2}{(1-(b-a)\theta_2)^2}\right) O(N), \tag{3.18}$$

$$E_1(N) \leq \frac{C_2 O(N)}{1-(b-a)\theta_2}, \tag{3.19}$$

with $C_2 = \max\left(1, \frac{B+(b-a)(\overline{B}+\overline{D})}{1-\theta_2(b-a)}\right)$.

Proof: Suppose that $\min(\theta_1,\theta_2)=\theta_1$. The relations (3.8), (3.9) and (3.10), (3.11) implies that

$$E_0(N) \leq (b-a)\left((A+C)E_0(N)+(B+D)E_1(N)\right)+O_0(N), \tag{3.20}$$

and

$$E_1(N) \leq (A+(b-a)(\overline{A}+\overline{C}))E_0(N)+(B+(b-a)(\overline{B}+\overline{D}))E_1(N)+O_1(N), \tag{3.21}$$

therefore, from (3.21), we get

$$E_1(N) \leq \left(\frac{A+(b-a)(\overline{A}+\overline{C})}{1-(B+(b-a)(\overline{B}+\overline{D}))}\right) E_0(N) + \frac{O_1(N)}{1-(B+(b-a)(\overline{B}+\overline{D}))}, \tag{3.22}$$

the substitution of (3.22) in (3.20) gives us

$$E_0(N) \leq (b-a)\theta_1 + C_1 O(N), \tag{3.23}$$

consequently

$$E_0(N) \leq \frac{C_1 O(N)}{1-(b-a)\theta_1}. \tag{3.24}$$

Also, by substituting (3.24) in (3.22), we deduce

$$E_1(N) \leq \left(\frac{A+(b-a)(\overline{A}+\overline{C})}{1-\left(B+(b-a)(\overline{B}+\overline{D})\right)}\right)\frac{C_1 O(N)}{1-(b-a)\theta_1} + \frac{O(N)}{1-\left(B+(b-a)(\overline{B}+\overline{D})\right)}, \tag{3.25}$$

and since $1-(b-a)\theta_1 \leq 1-\left(B+(b-a)(\overline{B}+\overline{D})\right)$, we deduce finally that

$$E_1(N) \leq \left(\frac{C_1[A+(b-a)(\overline{A}+\overline{C})]+1-(b-a)\theta_1}{(1-(b-a)\theta_1)^2}\right) O(N).$$

We proceed in an analogous way if $\min(\theta_1,\theta_2)=\theta_2$, so, we get

$$E_1(N) \leq \frac{C_2 O(N)}{1-(b-a)\theta_2},$$

and

$$E_0(N) \leq \left(\frac{C_2(b-a)(B+D)+1-(b-a)\theta_2}{(1-(b-a)\theta_2)^2}\right) O(N).$$

□

Case 2 :

Proposition 3.4. *If the hypothesis* $(H3)$ *and condition* (2.9) *hold, then*

$$E(N) \leq \frac{O(N)}{1-\eta(b-a)}. \tag{3.26}$$

Proof: By summing (3.20) (3.21), we deduce (3.26). □

We have seen in the previous subsection that if the quadrature rules are consistent with the equations (1.1), (3.2) (which mean that $O(N) \longrightarrow 0$ as $N \longrightarrow +\infty$), the exact solution $v = (x, y)$ of (3.10),(3.11) converge to (u, u'). In practice, it is often difficult to obtain explicitly the solution v, consequently, many approximative methods have been developed to resolve this type of equations. In our case, we propose to use the Picard successive approximations.

Take an initial vector $v^{(0)} = (x^{(0)}, y^{(0)})$ and define the sequence $v^{(k)} = (x^{(k)}, y^{(k)})$, for all $k \in \mathbb{N}^*$, as follows : $\forall i = 0, 1, ..., N$

$$x_i^{(k+1)} = f_i + \sum_{j=0}^{i} \sigma_{j,i} K_V(t_i, t_j, x_j^{(k)}, y_j^{(k)}) + \sum_{i=0}^{N} w_{j,N} K_F(t_i, t_j, x_j^{(k)}, y_j^{(k)}), \tag{3.27}$$

$$y_i^{(k+1)} = f_i' + K_V(t_i, t_i, x_i^{(k)}, y_i^{(k)}) + \sum_{j=0}^{i} \sigma_{j,i} \frac{\partial K_V}{\partial t}(t_i, t_j, x_j^{(k)}, y_j^{(k)}) + \sum_{i=0}^{N} w_{j,N} \frac{\partial K_F}{\partial t}(t_i, t_j, x_j^{(k)}, y_j^{(k)}). \tag{3.28}$$

3.2.1. Convergence of the iterative process: In the first case, the study of the convergence of the Picard sequence (3.27) (3.28) to the exact solution (x, y) according to the analytical study seems to be difficult because of the recurrence relation. Although, we show by an numerical example that the sequence (3.27) (3.28) converge under $(H2)$, $(H3)$ and (2.6).

Case 2 :
The convergence of the sequence (3.27), (3.28) under $(H3)$ and (2.9) is shown in the next propositions.

Proposition 3.5. *According to hypotheses $(H3)$ and condition (2.9), $v^{(k)}$ tends to v as $k \longrightarrow +\infty$, for any initial vector v_0.*

Proof: Using (3.10), (3.11) and (3.27), (3.28), we get

$$\|x - x^{(k+1)}\|_\infty \le (b - a) \left((A + C)\|x - x^{(k)}\|_\infty + (B + D)\|y - y^{(k)}\|_\infty \right),$$

$$\|y - y^{(k+1)}\|_\infty \le \left(A + (b - a)(\overline{A} + \overline{C})\right) \|x - x^{(k)}\|_\infty + \left(B + (b - a)(\overline{B} + \overline{D})\right) \|y - y^{(k)}\|_\infty,$$

so

$$\|v - v^{(k+1)}\|_{\mathbb{R}^{2N+2}} \le (b - a)\eta \|v - v^{(k)}\|_{\mathbb{R}^{2N+2}},$$

applying this relation by recurrence to the right hand side, we obtain

$$\|v - v^{(k+1)}\|_{\mathbb{R}^{2N+2}} \le ((b - a)\eta)^{k+1} \|v - v^{(0)}\|_{\mathbb{R}^{2N+2}}.$$

Hence, from (2.9), we deduce that $v^{(k)} \longrightarrow v$ as $k \longrightarrow +\infty$. □

Finally, we have the following convergence result of the approximative solution calculated by the successive approximations (3.27) (3.28).

Corollary 3.6. *Define the error : $\forall k \in \mathbb{N}^*$*

$$E_k(N) = \max_{i=0,\dots,N} |x_i^{(k)} - u_i| + \max_{i=0,\dots,N} |y_i^{(k)} - u_i'|, \tag{3.29}$$

so, from propositions 3.5, 3.6, 3.7, 3.8, we deduce that

$$E_k(N) \longrightarrow 0 \quad as \quad N, k \longrightarrow +\infty.$$

Proof: We see that

$$|x_i^{(k)} - u_i| \le |x^{(k)} - x| + |x - u_i| \quad \text{and} \quad |y_i^{(k)} - u_i'| \le |y^{(k)} - y| + |y - u_i'|,$$

which leads finally to

$$E_k(N) \le \|v^{(k)} - v\|_{\mathbb{R}^{2N+2}} + E(N).$$

□

3.3. Numerical example:

The following numerical example show the suitability of the proposed method with respect to the analytical study above for the first case.

Example 1 :
Given the equation : $\forall t \in [\frac{1}{2}, \frac{3}{2}]$

$$u(t) = f(t) + \int_{\frac{1}{2}}^{t} \frac{\left(t-\frac{1}{2}\right)^2}{48}(\cos^2(u(s)) + \sin^2(su'(s)))\,ds + \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{4\sqrt{2}\,e^{\frac{1}{4}\left(t-\frac{3}{2}\right)}}{3\sqrt{6}\,(1+u^2(s)+s^2u'^2(s))}, \tag{3.30}$$

with

$$f(t) = t - \frac{1}{48}\left(t-\frac{1}{2}\right)^3 - \frac{4}{3\sqrt{6}}e^{\frac{1}{4}\left(t-\frac{3}{2}\right)}\left(\arctan\left(\frac{3}{\sqrt{2}}\right) - \arctan\left(\frac{1}{\sqrt{2}}\right)\right),$$

and the exact solution $u(t) = t$.
Note that $(H1)$, $(H2)$ are satisfied with

$$M_V = \frac{1}{24}, \, \overline{M_V} = \frac{1}{12}, \quad M_F = \frac{4}{3\sqrt{3}}, \, \overline{M_F} = \frac{1}{3\sqrt{3}}.$$

We see also that the previous equation corresponds to the case taken in Remark 2.1, i.e the kernels satisfy $(H3)$ and (2.6) excepting (2.9). Note that the Lipschitz constants in $(H3)$ are obtained by the Mean Value Theorem.

To calculate the $k-order$ approximative solution $v^{(k)}$ of the algebraic system (3.10) (3.11), we use in (3.4) (3.5) and (3.6) (3.7) the composite trapezoidal rule over the following equidistant subdivision :

$$\Delta_N = \left\{ t_i = a + ih, \, i = 0, ..., N, \, h = \frac{b-a}{N}, \, N \in \mathbb{N}^* \right\}.$$

The consistency errors $O_0(N)$, $O_1(N)$ of the quadrature rules described above is shown in the next table,

Table 1. Consistency error.

N	$O_0(N)$	$O_1(N)$
5	1.1781E-3	4.4520E-4
10	4.4355E-4	1.1089E-4
50	1.7720E-5	4.4301E-6
100	4.4299E-6	1.1075E-6
500	1.7719E-7	4.4298E-8

In the following, we give the values of $E_k(N)$ according to the increasing values of $N, k \in \mathbb{N}^*$,

Table 2. Approximation error.

N	$E_k(N)$				
	k=2	k=4	k=6	k=8	k=10
5	1.5817E-2	1.1135E-3	1.7839E-3	1.8134E-3	1.8147E-3
10	1.7195E-2	2.9128E-4	4.2075E-4	4.5051E-4	4.5175E-4
50	1.7634E-2	7.2605E-4	1.3083E-5	1.6746E-5	1.7993E-5
100	1.7648E-2	7.3962E-4	2.6621E-5	3.2548E-6	4.4573E-6
500	1.7651E-2	7.4396E-4	3.0953E-5	1.1217E-6	1.2805E-7

Remark that the minimum value of $E_k(N)$ appears in some iteration $\hat{k} \in \{1, ..., 10\}$. Next, we give the minimum errors $E_{\hat{k}}(N)$ and also the error $E_{\overline{k}}(N)$ where $\overline{k}$ is the iterations number which corresponds to the following stop condition

$$\|v^{(\overline{k})} - v^{(\overline{k}-1)}\|_{\mathbb{R}^{2N+2}} \leq 10^{-7}. \tag{3.31}$$

Table 3. Minimum errors.

N	$E_{\hat{k}}(N)$	$\hat{k}$	$E_{\overline{k}}(N)$	$\overline{k}$
5	1.1135E-3	4	1.8147E-3	11
10	2.9128E-4	4	4.5182E-4	11
50	1.3083E-5	6	1.8059E-5	11
100	3.2548E-6	8	4.5229E-6	11
500	1.2805E-7	10	1.9161E-7	11

4. Conclusion

Besides the analytic treatment of a nonlinear Volterra-Fredholm integro-differential equation, a numerical method have been constructed to approach the solution using the well known Nyström method coupled with the Picard successive approximations. The effectiveness of the approach according to the existence and uniqueness conditions is shown by numerical example. Our future aim is to extend the analytic study to equations with general types of kernels, as well as, the study of other equation type's as fuzzy and delay integro-differential equation with the consideration of adequate numerical methods like the idea developed in [7,12] based on the linearization then discretization, which considered , is some cases, as the favorable technique.

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