Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 2023 (41)** : 1–8.
ISSN-0037-8712 in press
doi:10.5269/bspm.51014

About Weak π-rings

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ABSTRACT: As in [11], a ring is called a weak π-ring if every regular principal ideal is a finite product of prime ideals. In this paper, we establish some characterizations for weak π-rings. Also, we translate the properties weak π-ring and $(*)$-ring of $A \propto E$ in terms of a commutative ring A and an A-module E.

Key Words: Weak π-rings, π-rings, $(*)$-rings, trivial ring extension.

Contents

1. Introduction

All rings in this paper are commutative with identity and all modules are unital. We denote respectively by $Z(R)$, $Reg(R)$ and $Max(R)$ the set of zero-divisors of R, the set of all regular elements of R and the set of all maximal ideals of R. Dedekind domains are integral domains in which every ideal is a finite product of prime ideals. Dedekind rings are defined by the fact that every regular ideal is a finite product of prime ideals. More generally, a ring R with zero divisors is said to be general ZPI-ring if every ideal is a finite product of prime ideals. These rings has the property that every prime ideal (or equivalently, every ideal) is finitely generated and locally principal. General ZPI-rings are also characterized by the property that R is a finite direct product of Dedekind domains and special principal ideal rings (SPIRs), that are, local principal ideal rings, not a field, whose maximal ideal is nilpotent. A ring R has the property that every principal ideal is a finite product of prime ideals if and only if R is a finite direct product of (1) π-domains, (2) SPIRs, and (3) fields. In this case, R called a π-ring.

In [11], Jayaram defined two generalizations of the above-mentioned class of rings, namely, weak π-rings in which every regular principal ideal is a finite product of prime ideals, and rings satisfying the condition $(*)$ (in the sequel, they will be noted by $(*)$-rings) in which every regular principal ideal is a finite intersection of primary ideals and he proved that weak π-rings are $(*)$-rings. He also defined a ring R to be an almost weak π-ring if for each regular principal ideal I, I_M is a finite product of prime ideals in R_M for all maximal ideals M containing I. The author mention that weak π-rings are almost weak π-rings but the converse need not be true. By Theorem 2.1, we state a necessary and sufficient condition for an almost weak π-ring to be a weak π-ring. In the same paper, Jayaram characterizes weak π-rings inside the class of quasi-regular rings and gives necessary and sufficient conditions for a ring R to be a Dedekind ring. For more informations, the reader may consult [3,5,11].

The following diagram summarizes the relations between all these class of rings where the implications cannot be reversed in general.

2010 *Mathematics Subject Classification:* 13D05, 13D02.
Submitted November 16, 2019. Published November 04, 2020

$$\begin{tikzcd}
& \textbf{General ZPI-ring} \arrow[d, shift right] & \\
& \textbf{Dedekind-ring} \arrow[u, shift right, "\text{Domain}"'] \arrow[d, shift right] & \\
& \pi\textbf{-ring} \arrow[u, shift right, "\text{Principal}"'] \arrow[d, shift right] & \\
& \textbf{Weak } \pi\textbf{-ring} \arrow[u, shift right, "\text{Domain}"'] \arrow[dl] \arrow[dr] & \\
(*)\textbf{-ring} & & \textbf{Almost weak } \pi\textbf{-ring}
\end{tikzcd}$$

In this paper, we provide an example of a weak π-ring which is neither a Dedekind ring nor a π-ring (see Example 3.5), and by Example 3.6, we show that $(*)$-rings are not necessary weak π-rings.

Let E be an R-module and $R \propto E$ be the set of pairs (r, e) with pairwise addition and multiplication given by $(r, e)(q, f) = (rq, rf + qe)$. $R \propto E$ is called the trivial ring extension of R by E (also called the idealization of E over R). For instance, the reader is referred to [3,4,7,12]

We will be using the following definitions. A regular element r of R is any element of $R \backslash Z(R)$; An ideal I of R is an invertible ideal if $II^{-1} = R$, where $I^{-1} = (R : I) = \{x \in qf(R) | xI \subseteq R\}$; I is called a multiplication ideal if for every ideal $J \subseteq I$, there exists an ideal K with $J = KI$ (invertible ideals are multiplication ideals); An R-module E is said to be divisible if, for each $e \in E$ and each regular element r of R, there exists $f \in E$ such that $e = rf$.

Our aims in this paper is to extend the result of Jayaram about characterization of weak π-rings [11, Theorem 1] to the general case and to study the transfer of the weak π-ring property and the $(*)$-ring property to the trivial ring extension.

2. General results

We start this section by the following characterizations of a weak π-ring.

Theorem 2.1. *Let R be a commutative ring. The following statements are equivalent:*

1. *R is a weak π-ring*
2. *Every regular prime ideal contains an invertible prime ideal.*
3. *Every invertible ideal is a finite product of invertible prime ideals.*
4. *Every prime ideal minimal over an invertible ideal is invertible.*
5. *R is an almost weak π-ring in which every prime ideal minimal over an invertible ideal is finitely generated.*

The proof of this theorem based on the following result. We are inspired by the proof of [11, Lemma 13].

Lemma 2.2. *Every regular prime ideal contains an invertible prime ideal if and only if every invertible ideal is a finite product of invertible prime ideals.*

Proof: Suppose that every invertible ideal is a finite product of invertible prime ideals. Let P be a regular prime ideal, so there exists $r \in P$ a regular element such that $< r >= P_1P_2...P_n$ where P_k's are invertible prime ideals. Since P is a prime ideal we have $P_k \subseteq P$ for some k. Conversely, assume that every regular prime ideal of R contains an invertible prime ideal and let I be an invertible ideal. We prove that I contains a finite product of invertible prime ideals. Let $S_0 = \{J \in L(R) | J\ is\ a\ finite\ product\ of\ invertible\ prime\ ideals\ of\ R\}$ and $S = \{J \in L(R) | I \subseteq J\ and\ S_0 \cap (J] = \emptyset\}$ where $(J] = \{K \in L(R) | K \subseteq J\}$.. Suppose $S_0 \cap (I] = \emptyset$ then $S \neq \emptyset$. Since every element of S_0 is finitely generated, by Zorn's lemma there exists a prime ideal P such that $I \subseteq P$ and $S_0 \cap (P] = \emptyset$. Since $I \subseteq P$, it follows that P is regular, so $S_0 \cap (P] \neq \emptyset$, a contradiction. Then $S_0 \cap (I] \neq \emptyset$. Thus I contains a finite product of invertible prime ideal. Let $P_1P_2...P_n \subseteq I$ where P_k's are invertible prime ideals. Since I is finitely generated and locally principal, it follows that I is a multiplication ideal and so $IJ = P_1P_2...P_n$ for some $J \in L(R)$. By [11, Lemma 12], I is a finite product of invertible prime ideals. □

Proof of Theorem 2.1.

(1) ⇒ (2) Suppose that R is a weak π-ring and let P be a regular prime ideal of R. Let $r \in P$ be a regular element, we have $< r >= P_1P_2...P_n$ where every P_k is a regular, finitely generated and a locally principal ideal, so an invertible prime ideal. Hence P contains an invertible prime ideal.

(2) ⇒ (3) follows from Lemma 2.2.

(3) ⇒ (4) Suppose (3) holds. Let P be a prime ideal minimal over an invertible ideal I. By hypothesis $I = P_1P_2...P_n$ where P_k's are invertible prime ideals. As P is a prime ideal, it follows that $P_k \subseteq P$, then $P = P_k$ and hence P is invertible.

(4) ⇒ (1). Suppose (4) holds. Let I be a regular principal ideal of R, so I is invertible. By hypothesis and [9, Lemma 5], I contains a finite product of invertible prime ideals minimal over I. Let $P_1P_2...P_n \subseteq I$, $IJ = P_1P_2...P_n$ for some $J \in L(R)$ as I is a multiplication ideal . By [11, Lemma 12], I is a finite product of invertible prime ideals.

(4) ⇒ (5) is obvious since (4) ⇔ (1).

(5) ⇒ (2) Assume that (5) holds. Let P be a regular prime ideal and r a regular element of P. We may assume that P is not minimal over $< r >$, and let Q be a minimal prime over $< r >$ such that $Q \subseteq P$, so by hypothesis Q is finitely generated. Now remains to show that Q is locally principal. Let $Q \subseteq M$ for some maximal ideal M of R. Since R is an almost weak π-ring, $< r >_M = Q_{1M}Q_{2M}...Q_{nM}$ for some prime ideals $Q_1, Q_2, ..., Q_n$ of R. As Q is a prime minimal over $< r >$, it follows that $Q_M = Q_{iM}$ for some i. By [1, Lemma 2.3] and [13, Theorem 2], Q_M is principal in R_M, so a locally principal ideal. Therefore Q is invertible, as desired.

Proposition 2.3. *Let R be a weak π-ring, then:*

1. *Each invertible ideal is principal.*
2. *R is an almost weak π-ring in which every regular rank one prime ideal is invertible.*

Proof:

1. Let I be an invertible ideal of R. By [8][Lemma 18.1, p. 110], I contains a regular element r. Then $< r >= P_1^{\alpha_1}P_2^{\alpha_2}...P_n^{\alpha_n}$ where P_k's are distinct invertible prime ideals by [1, Lemma 2.3], as R is a weak π-ring. By [11, Lemma 12], and since I is a multiplication ideal we obtain $I = P_1^{\alpha_1}P_2^{\alpha_2}...P_n^{\alpha_n} =< r >$, as desired.
2. It's clear that R is an almost weak π-ring. So the result holds by hypothesis and [11, Lemma 5].

□

Next, we establish the transfer of weak π-ring, almost weak π-ring and $(*)$-ring properties to the direct product of rings.

Proposition 2.4. *Let $R = \prod_{i=1}^{n} R_i$ be a direct product of rings. Then the following results hold:*

1. *R is a weak π-ring if and only if so are R_i's.*
2. *R is an almost weak π-ring if and only if so are R_i's.*
3. *R is a $(*)$-ring if and only if so are R_i's.*

Proof: It suffices to study the case of a pairs of rings R and S.

1. Assume that $R \times S$ is a weak π-ring. Let $I =< i >$ and $J =< j >$ be two regular principal ideals respectively of R and S. By our assumption, $I \times J =< (i,j) >= (P_1 \times S)...(P_n \times S)(R \times Q_1)...(R \times Q_m)$ where P_l and Q_l are respectively prime ideals of R and S with $P_1 \neq R$ and $Q_1 \neq S$. Hence $I = P_1...P_n$ and $J = Q_1...Q_m$.
   Conversely, let $H =< (i,j) >$ be a regular principal ideal of $R \times S$. Set $I =< i >$ and $J =< j >$, clearly I(resp. J) is a regular principal ideal of R(resp. S). By hypothesis, $I = P_1...P_n$ and $J = Q_1...Q_m$, thus $H = (P_1 \times S)...(P_n \times S)(R \times Q_1)...(R \times Q_m)$.
2. The result holds since for all $I =< i >$ and $J =< j >$ be two regular principal ideals respectively of R and S we have $(I \times J)_{M \times S} \cong I_M$ for all $M \in Max(R, I)$ and $(I \times J)_{R \times N} \cong J_N$ for all $N \in Max(S, J)$.
3. Clearly, for a pairs of rings R and S, the primary ideals of $R \times S$ have the form $P \times S$ or the form $R \times Q$ where P and Q are primary ideals of R and S respectively. So assume that $R \times S$ is a $(*)$-ring. Let $I =< i >$ and $J =< j >$ be two regular principal ideals respectively of R and S. By our assumption, $I \times J =< (i,j) >= (P_1 \times S) \cap ... \cap (P_n \times S) \cap (R \times Q_1) \cap ... \cap (R \times Q_m)$ where P_l's and Q_l's are respectively primary ideals of R and S with $P_1 \neq R$ and $Q_1 \neq S$. Hence $I = P_1 \cap ... \cap P_n$ and $J = Q_1 \cap ... \cap Q_m$.
   Conversely, let $K =< (i,j) >$ be a regular principal ideal of $R \times S$. Set $I =< i >$ and $J =< j >$, clearly I(resp. J) is a regular principal ideal of R(resp. S). By hypothesis, $I = P_1 \cap ... \cap P_n$ and $J = Q_1 \cap ... \cap Q_m$ where P_l's and Q_l's are respectively primary ideals of R and S, thus $K = (P_1 \times S) \cap ... \cap (P_n \times S) \cap (R \times Q_1) \cap ... \cap (R \times Q_m)$.

□

By the following results, we identified a context in which one can see when a localization of a ring R is a weak π-ring, an almost weak π-ring and a $(*)$-ring.

Proposition 2.5. *Let R be a ring and S a multiplicative set of R such that $S \subseteq Reg(R)$.*

1. *If R is a weak π-ring, then so is $S^{-1}R$.*
2. *If R is an almost weak π-ring, then so is $S^{-1}R$.*
3. *If R is a $(*)$-ring, then so is $S^{-1}R$.*

Proof:

1. Let J be a regular principal ideal of $S^{-1}R$, so $J^c = \varphi^{-1}(J)$ is a regular ideal, where $\varphi : R \to S^{-1}R$ is a ring homomorphism defined by $\varphi(r) = r/1$, indeed:
   Let $\frac{r}{s}$ be a regular element of J, clearly $\frac{r}{1}$ is a regular element of J, then $r \in J^c$. Assume that r is not regular, so there exists $a \in R$ such that $ra = 0$, and so $(\frac{r}{s})(\frac{a}{1}) = \frac{0}{1}$, contradiction.
   As R is a weak π-ring, we deduced that $< r >= P_1...P_n$. Therefore, $S^{-1}(P_1...P_n) = S^{-1}P_1...S^{-1}P_n$ $\subseteq S^{-1}(J^c) = J$. Since $r \in < r >= P_1...P_n$, it follows that $\frac{r}{s} \in S^{-1}P_1...S^{-1}P_n$. Hence $J = S^{-1}P_1...S^{-1}P_n$, as desired.

2. Straightforward.

3. Let J be a regular principal ideal of $S^{-1}R$. Similarly to the proof of Proposition 2.5, $J^c = \varphi^{-1}(J)$ is a regular ideal of R. Since R is a $(*)$-ring, we deduced that $< r >= P_1 \cap ... \cap P_n$ where P_l's are primary ideals of R. Therefore, $S^{-1}(P_1 \cap ... \cap P_n) = S^{-1}P_1 \cap ... \cap S^{-1}P_n \subseteq S^{-1}(J^c) = J$. Since $r \in < r >= P_1 \cap ... \cap P_n$, it follows that $\frac{r}{s} \in S^{-1}P_1 \cap ... \cap S^{-1}P_n$. Hence $J = S^{-1}P_1 \cap ... \cap S^{-1}P_n$. As P_l are primary ideals, then so are $S^{-1}P_l$. This completes the proof of the proposition.

□

By this proposition, we study the transfer of the above-mentioned classes to homomorphic image. We consider I an ideal of R and $f : R \to R/I$ the canonical surjection, we have then the next results:

Proposition 2.6. *Suppose that the following two conditions hold:*

1. *Each regular principal ideal of R contains I.*

2. *$f^{-1}\{\bar{r}\}$ contains a regular element where $\bar{r} \in Reg(R/I)$.*

Then:

(a) *If R is a weak π-ring, then so is R/I.*

(b) *If R is an almost weak π-ring, then so is R/I.*

(c) *If R is a $(*)$-ring, then so is R/I.*

Proof:

(a) Let $J =< \bar{r} >$ be a regular principal ideal of R/I. Along with the hypothesis that $f^{-1}\{\bar{r}\}$ contains a regular element, we may assume, without loss of generality, r is regular. We get then $K =< r >$ a regular principal ideal of R. As R is a weak π-ring, it follows that $K = P_1...P_n$ and hence $J = K/I = (P_1...P_n)/I = P_1/I...P_n/I$. Therefore J is a finite product of prime ideals.

(b) Let $J =< \bar{r} >$ be a regular principal ideal such that $f^{-1}\{\bar{r}\}$ contains a regular element. Thus, we may assume, without loss of generality, $K =< r >$ a regular principal ideal. Since R is an almost weak π-ring, we obtain $K_M = P_{1M}...P_{nM}$ for all $M \in Max(R, K)$, in particular for all $N \in Max(R/I, J)$ with $N = M/I$ for some $M \in Max(R, K)$, and hence $J_N = (K/I)_N \cong K_M/I_M = P_{1M}...P_{nM}/I_M = P_{1M}/I_M...P_{nM}/I_M$, which completes the proof.

(c) Let $J =< \bar{r} >$ be a regular principal ideal of R/I. Along with the hypothesis that $f^{-1}\{\bar{r}\}$ contains a regular element, without loss of generality, we may assume $K =< r >$ a regular principal ideal. As R is a $(*)$-ring, then $K = P_1 \cap ... \cap P_n$ and hence $J = K/I = (P_1 \cap ... \cap P_n)/I = P_1/I \cap ... \cap P_n/I$, as desired.

□

3. The transfer to the trivial ring extension

In the sequel, we study the possible transfer of the properties of being a weak π-ring, an almost weak π-ring and a $(*)$-ring between a commutative ring A and $A \propto E$.

A homogeneous ideal of $A \propto E$ is an ideal with the form $I \propto F$ where I is an ideal of A, F is a submodule of E, and $IE \subseteq F$, also we have $I = \{a \in A \mid (a, e) \in J \text{ } for \text{ } some \text{ } e \in E\}$ and $F = \{e \in E \mid (a, e) \in J \text{ } for \text{ } some \text{ } a \in A\}$.

Theorem 3.1. *Let A be a ring, E an A-module and $R = A \propto E$ be the trivial ring extension of A by E. Then:*

1. *R is a weak π-ring if and only if every principal ideal not disjoint of S is a finite product of prime ideals and $sE = E$ for all $s \in S$ where $S = A - (Z(A) \cup Z(E))$.*

2. *If R is an almost weak π-ring, then every principal ideal not disjoint of S, I_M is a finite product of prime ideals for all $M \in Max(A, I)$.*

3. *Suppose that $sE_M = E_M$ for all $M \in Max(A)$ and all $s \in S$. If A is an almost weak π-ring, then so is R.*

4. *Suppose that $E = aE$ for all $a \in S$. R is a $(*)$-ring if and only if every regular principal ideal not disjoint of S has a primary decomposition.*

Proof:

1. As R is a weak π-ring and by [3, Theorem 3.3] a product of homogeneous ideals is homogeneous, it follows that every regular principal ideal of R is homogeneous, so by [3, Theorem 3.9] $sE = E$ for all $s \in S$. Let $I =< a >$ be a principal ideal of A with $I \cap S \neq \emptyset$. Thus, $J =< (a, 0) >= I \propto E$ is a regular ideal of R, hence $I \propto E = (P_1 \propto E)...(P_n \propto E)$ where P_i's are prime ideals of A. We conclude that $I = P_1...P_n$.
   Conversely, let J be a regular principal ideal of R. By hypothesis and [3, Theorem 3.9], $J = I \propto E$ for some ideal I of A with $I \cap S \neq \emptyset$. Again by hypothesis, I is a finite product of prime ideals of A, set $I = P_1...P_n$. Since $sE = E$ for all $s \in S$, we get $J = I \propto E = (P_1 \propto E)...(P_n \propto E)$, therefore R is a weak π-ring.

   (2) and (3) are similar to the first statement.

(4) Suppose that R is a $(*)$-ring. Along with the hypothesis $E = aE$ for all $a \in S$, for every regular principal ideal I of A such that $I \cap S \neq \emptyset$, $I \propto E$ is a regular principal ideal of R. Then $I \propto E = (P_1 \propto E) \cap ... \cap (P_n \propto E) = (P_1 \cap ... \cap P_n) \propto E$ where P_k's are primary ideals of A, and hence $I = P_1 \cap ... \cap P_n$, as desired.
Conversely, let J be a regular principal ideal of R. By hypothesis, $J = I \propto E$ where I is a regular principal ideal of R such that $I \cap S \neq \emptyset$ and $I = P_1 \cap ... \cap P_n$ where P_k's are primary ideals of A. So, $J = (P_1 \cap ... \cap P_n) \propto E = (P_1 \propto E) \cap ... \cap (P_n \propto E)$. Therefore, R is a $(*)$-ring.

□

As a particular case of the previous theorem, we get:

Corollary 3.2. *Let A be a ring, E an A-module and $R = A \propto E$ be the trivial ring extension such that $Z(E) \subseteq Z(A)$. Then:*

1. *R is a weak π-ring if and only if A is a weak π-ring and $sE = E$ for all $s \in S$.*

2. *If R is an almost weak π-ring, then so is A.*

3. *Suppose that $sE_M = E_M$ for all $M \in Max(A)$ and all $s \in S$. If A is an almost weak π-ring, then so is R.*

4. *Suppose that $E = aE$ for all $a \in S$. Then R is a $(*)$-ring if and only if so is A.*

Emmy Noether proved that a Noetherian ring R is a Laskerian ring [6], hence a $(*)$-ring. The next corollary provide a non-Noetherian example of a $(*)$-ring.

Corollary 3.3. *Let D be a domain and E a divisible R-module. Then:*

1. *D is a weak π-ring if and only if so is $D \propto E$.*

2. *D is an almost weak π-ring if and only if so is $D \propto E$.*

3. *D is a $(*)$-ring if and only if so is $D \propto E$.*

4. *If E is a non-finitely generated D-module, then $D \propto E$ is a non-Noetherian ring and we have D is a $(*)$-ring if and only if so is $D \propto E$.*

Corollary 3.4. *Let (A, M) be a local ring and E an A/M-vector space. Then:*

1. *$A \propto E$ is a weak π-ring.*

2. *$A \propto E$ is a $(*)$-ring.*

As an application of our result, we construct the following example of a weak π-ring ($(*)$-ring) which is neither a Dedekind ring nor a π-ring.

Example 3.5. *Let $R = \mathbb{Z} \propto \mathbb{Z}/4\mathbb{Z}$, it's clear that $\mathbb{Z}/4\mathbb{Z} = a\mathbb{Z}/4\mathbb{Z}$ for all $a \in S = \mathbb{Z} - Z(\mathbb{Z}/4\mathbb{Z})$, then:*

- *R is a weak π-ring.*
- *R is a $(*)$-ring.*
- *R not a Dedekind ring (by [10, Theorem 1]).*
- *R not a π-ring (by [3, Theorem 4.10]).*

By the next example, we show that $(*)$-rings are not necessary weak π-rings.

Example 3.6. *Let $R = \mathbb{Z}$ and $E = 4\mathbb{Z}$, we prove that:*

1. *$R \propto E$ is not a weak π-ring (by theorem 3.1 (1)).*

2. *$R \propto E$ is a $(*)$-ring.*

The condition "$f^{-1}(r)$ contains a regular element for every regular element $\overline{r} \in R/I$" is necessary in Proposition 2.6 (1). To see this, consider the following example:

Example 3.7. *Let $A = \mathbb{Z}_2 \propto \mathbb{Z}_2$, $M = 2\mathbb{Z}_2 \propto \mathbb{Z}_2$, $E = A/M$ and $R = A \propto E$. Easily, we can see that $E = rE$ for all $r \in S = A - (Z(A) \cup Z(E))$. Thus every regular principal ideal J of R has the form $I \propto E$ where I is a regular principal ideal not disjoint with S. Hence the first condition in Proposition 2.6 is satisfied. Now, let $(a, 0)$ be a regular element of A. For all $e \in E$ we have $((a, 0), e)((0, 0), e) = ((0, 0), 0)$. Clearly R is a weak π-ring, however A is not by corollary 3.4.*

Acknowledgments

This work is dedicated to our beloved Professor Abdelmoujib Benkirane.

References

1. D.D. Anderson and C. Jayaram, *Principal element lattices*, Czech. Math. J., (1996) 46:99-109.
2. D.D. Anderson and L.A. Mahaney, (1988). *On primary factorizations.* J. Pure Appl. Algebra 54:141-154.
3. D.D. Anderson and M. Winders, *Idealization of a module.* J. Comm. Algebra 1 (2009), 3-56.
4. C. Bakkari, S. Kabbaj and N. Mahdou, *Trivial extension definided by Prûfer conditions*, J. Pure App. Algebra 214 (2010), 53-60.
5. R.W. Gilmer, *Multiplicative Ideal Theory.* New York: Marcel Dekker, Inc., (1972).
6. R.W. Gilmer, Noether's work in commutative ring theory, in "Emmy Noether" (J. Brewer and M. Smith, Eds.), Dekker, New York, in press.
7. S. Glaz, *Commutative Coherent Rings*, Springer-Verlag, Lecture Notes in Mathematics 1371 (1989).
8. J.A. Huckaba, *Commutative Rings with Zero Divisors.* New York: Marcel Dekker, Inc., (1988).
9. C. Jayaram, *Almost Q-rings.* Arch. Math. (Brno)(2004), 40:249-257.
10. C. Jayaram, *Some characterizations of Dedekind rings.* Commun. Algebra (2012), 40:206-212.

11. C. Jayaram, *Weak -rings*. Comm. Algebra, 45:6, (2017) 2394-2400.

12. S. Kabbaj and N. Mahdou, *Trivial extensions defined by coherent-like conditions*, Comm. Algebra 32 (2004), no. 10, 3937-3953.

13. P.J. McCarthy, *Principal elements of lattices of ideals*. Proc. Amer. Math. Soc., (1971) 30:43-45.

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