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Global Existence and General Decay of Moore–Gibson–Thompson Equation with not Necessarily Decreasing Kernel

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ABSTRACT: In this paper, we consider the Moore–Gibson–Thompson equations. By using the potential well theory we obtain the existence of a global solution. Then, we prove the general decay result of solutions under weaker assumptions than the ones frequently used in the literature. In particular, the kernels we are considering are not necessarily exponentially decaying to zero as was assumed before. The present results improve also a previous work of the authors.

Key Words: Viscoelastic equations, global existence, general decay, Moore–Gibson–Thompson (MGT).

Contents

1. Introduction

In this works, we study the global existence and general decay of the following for the Moore-Gibson-Thompson equation with term viscoelastic memory

$$\tau u_{ttt} + \alpha u_{tt} - c^2 \Delta u - b\Delta u_t + \int_0^t g(t-s)\Delta u(s)\,ds = 0, \quad (x,t) \in \Omega \times \mathbb{R}_+, \tag{1.1}$$

with initial data

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ u_{tt}(x,0) = u_2(x), \quad x \in \Omega, \tag{1.2}$$

and boundary conditions

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}_+, \tag{1.3}$$

where $\Omega \in \mathbb{R}^n,\ g(.) : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are given functions which will be spaced later, and $u_0(x),\ u_1(x)$ and $u_2(x)$ are given functions. All the parameters τ and b are assumed to be positive constants. In a physical context of the acoustic waves, the variable u denotes a scalar acoustic velocity potential $v = -\nabla u$ with v denoting the acoustic particle velocity, c^2 denotes the speed of sound, α denotes thermal relaxation resulting from replacing Fourier law by the Maxwell-Cattaneo law, the coefficient $b \approxeq \delta + ac^2$ where δ is the diffusibility of the sound and the coefficient $\alpha > 0$ describes natural damping effects associated with an acoustic environment, see Lebon and Cloot [18]. The convolution term $\int_0^t g(t-s)\Delta u(s)\,ds$ reflects the memory effects of materials due to viscoelasticity. Here the convolution kernel g satisfies proper conditions exhibiting "memory character" which will be explained later. This model of (1.1) arises in high-frequency ultrasound applications accounting for thermal flux and molecular relaxation times. According to revisited extended irreversible thermodynamics, thermal flux relaxation leads to the third-order derivative in time while molecular relaxation leads to non-local effects governed by memory terms.

The presence of the third time derivative is typical in extended irreversible thermodynamics (EIT) a theory originally proposed to remove the unpleasant property of propagation of heat and velocity signals

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with an infinite velocity when Fourier–Navier–Stokes equations are used [20]. The guiding idea behind is that physical quantities such as thermodynamic fluxes typically given by constitutive relations, in EIT theory, are governed by evolution equations with a suitable relaxation time τ. In addition, more recently the EIT theory has been revisited by adding non-local effects with an eye on reaching agreement between theory and experiment particularly in systems with long relaxation times (viscoelastic fluids) and phenomena involving high frequencies. The latter leads to a presence of additional integral terms in the equation [20]. Moore–Gibson–Thompson (MGT) equation arises from modeling high-frequency ultrasound waves.Without memory, the linearized MGT equation reads

$$\tau u_{ttt} + \alpha u_{tt} + c^2 Au + bA\Delta u_t = 0. \tag{1.4}$$

Certainly this equation is in abstract form, and it has a simple prototype where $A = -\Delta$ with Dirichlet boundary conditions. In [15], the well posedness of (1.4) and the uniform decay of its energy are studied under proper functional setting and initial boundary conditions. Spectral analysis for this model has been carried out in [22], which confirms the validity and sharpness of the results in [21]. A linear MGT equation is the prelude to nonlinear ones. The classical nonlinear acoustics models include the Kuznetsov equation, the Westervelt equation and the KZK equation. This research field is highly active due to a wide range of applications such as the medical and industrial use of high intensity ultrasound in lithotripsy, thermotherapy, ultrasound cleaning, etc. There have been quite a few works in this aspect, more from engineering viewpoint. The motivation of our work is due to some results regarding the following research papers: Lasiecka, I. and Wang, X. [16] studied the Moore–Gibson–Thompson equation with memory, part I: exponential decay of energy

$$\begin{cases} \tau u_{ttt} + \alpha u_{tt} + c^2 Au + bA\Delta u_t - \int_0^t g(t-s) Aw(s)\, ds = 0, \quad (x,t) \in \Omega \times \mathbb{R}_+, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ u_{tt}(x,0) = u_2(x), \quad x \in \Omega, \\ u(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}_+, \end{cases}$$

where $\Omega \in \mathbb{R}^n$, $h(.) : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are given functions (See [16]), $u_0(x)$, $u_1(x)$ and $u_2(x)$ are given functions, and τ, α, c^2 and b parameters in MGT equation. A is a positive self-adjoint operator on a Hilbert space H.

Medjden, M. Tatar, N. [21] studied the asymptotic behavior for a viscoelastic problem with not necessarily decreasing kernel. Mesloub, F. Boulaaras, S. [22] studied the general decay for a viscoelastic problem with not necessarily decreasing kernel. Boumaza, N. Boulaaras, S. [2] studied the general decay for Kirchhoff type in viscoelasticity with not necessarily decreasing kernel. Boulaaras, S. Draifia, A. Alnegga, M. [3] studied the polynomial decay rate for kirchhoff type in viscoelasticity with logarithmic nonlinearity and not necessarily decreasing kernel.

However, Lasiecka, I. and Wang, X. [17] did not study the general decay of problem $(1.1) - (1.3)$ with not necessarily decreasing kernel. Motivated by the above research, we will consider the general decay with not necessarily decreasing kernel of the model $(1.1) - (1.3)$ in this paper.

The outline of the paper is as follows. In the second section we define the classical energy $E(t)$ associated to $(1.1) - (1.3)$ and define the modified energy $e(t)$ associated to $(1.1) - (1.3)$ and show that it is a non-increasing function of t. In section 3, we prove global existence of solution of $(1.1) - (1.3)$. Finally, in section 4, we prove general decay of solution of the posed problem.

2. Assumptions and main results

In this section, we define the classical energy $E(t)$ associated to $(1.1) - (1.3)$ and define the modified energy $e(t)$ associated to $(1.1) - (1.3)$ and show that it is a non-increasing function of t. In order to state our main results we make further assumptions on g :
(**A1**) We suppose that the kernel $g(t)$ is a $C^1(\mathbb{R}_+, \mathbb{R}_+)$ and $\int_0^\infty g(s)\, ds < c^2$.
(**A2**) There exists a positive differentiable function $\psi(t)$ such that
$g'(t) + \psi(t) g(t) \geq 0$ and $e^{\alpha t}[g'(t) + \psi(t) g(t)] \in L^1(\mathbb{R}_+)$ for $\alpha > 0$, and $\psi(t)$ satisfis, for somme positive

constant L,

$$\left|\frac{\psi'(t)}{\psi(t)}\right| \leq L, \quad \psi'(t) \leq 0, \quad \int_0^\infty \psi(s)\,ds = \infty, \quad t \geq 0. \tag{2.1}$$

(**A3**) $g'(t) \leq 0$ and $g''(t) \geq 0$ for all $t \geq 0$.

We recall the binary notation

$$(g\Box w)(t) := \int_0^t g(t-s)\,\|w(x,s)-w(x,t)\|^2_{L^2(\Omega)}\,ds. \tag{2.2}$$

Lemma 2.1. *Assume that* (**A1**) − (**A3**) *holds. Then the classical energy associated to* (1.1) − (1.3) *is defined by*

$$\begin{aligned} E(t) \quad : \quad &= \frac{k\tau}{2}\|u_{tt}\|^2_{L^2(\Omega)} + kc^2(\nabla u, \nabla u_t)_{L^2(\Omega)} + \frac{kb}{2}\|\nabla u_t\|^2_{L^2(\Omega)} \\ &+\tau(u_{tt},u_t)_{L^2(\Omega)} + \frac{\alpha}{2}\|u_t\|^2_{L^2(\Omega)} + \frac{c^2}{2}\|\nabla u\|^2_{L^2(\Omega)}, \end{aligned} \tag{2.3}$$

and its derivative is

$$\begin{aligned} \frac{d}{dt}\{E(t)\} \quad &= \quad -k\alpha\|u_{tt}\|^2_{L^2(\Omega)} + kc^2\|\nabla u_t\|^2_{L^2(\Omega)} + \tau\|u_{tt}\|^2_{L^2(\Omega)} \\ &\quad -b\|\nabla u_t\|^2_{L^2(\Omega)} + k\int_0^t g(t-s)\,(\nabla u(s), \nabla u_{tt}(t))_{L^2(\Omega)}\,ds \\ &\quad +\int_0^t g(t-s)\,(\nabla u(s), \nabla u_t(t))_{L^2(\Omega)}\,ds, \end{aligned} \tag{2.4}$$

where

$$\frac{\tau}{\alpha} < 1 < k < \frac{b}{2c^2} < \frac{b}{c^2}. \tag{2.5}$$

Proof. **Step .1** Multiplying (1.1) by u_{tt} and integrating over Ω, we have

$$\begin{aligned} &\tau(u_{ttt},u_{tt})_{L^2(\Omega)} + \alpha(u_{tt},u_{tt})_{L^2(\Omega)} - c^2(\Delta u, u_{tt})_{L^2(\Omega)} \\ &-b(\Delta u_t, u_{tt})_{L^2(\Omega)} + \left(\int_0^t g(t-s)\Delta u(s)ds, u_{tt}(t)\right)_{L^2(\Omega)} \\ = \quad & 0. \end{aligned} \tag{2.6}$$

By direct calculations, we get

$$\tau(u_{ttt},u_{tt})_{L^2(\Omega)} = \frac{\tau}{2}\frac{d}{dt}\left\{\|u_{tt}\|^2_{L^2(\Omega)}\right\}, \tag{2.7}$$

and

$$\alpha(u_{tt},u_{tt})_{L^2(\Omega)} = \alpha\|u_{tt}\|^2_{L^2(\Omega)}. \tag{2.8}$$

And using integration by parts, we have

$$\begin{aligned} &-c^2(\Delta u, u_{tt})_{L^2(\Omega)} \\ = \quad & c^2\frac{d}{dt}\left\{(\nabla u, \nabla u_t)_{L^2(\Omega)}\right\} - c^2\|\nabla u_t\|^2_{L^2(\Omega)}, \end{aligned} \tag{2.9}$$

$$-b(\Delta u_t, u_{tt})_{L^2(\Omega)} = \frac{b}{2}\frac{d}{dt}\left\{\|\nabla u_t\|^2_{L^2(\Omega)}\right\}, \tag{2.10}$$

$$\begin{aligned} &\left(\int_0^t g(t-s)\Delta u(s)ds, u_{tt}(t)\right)_{L^2(\Omega)} \\ = \quad & -\int_0^t g(t-s)\,(\nabla u(s), \nabla u_{tt}(t))_{L^2(\Omega)}\,ds. \end{aligned} \tag{2.11}$$

By replacement of (2.7) – (2.11) into (2.6), we get

$$\begin{aligned}
&\frac{d}{dt}\left\{\frac{\tau}{2}\left\|u_{tt}\right\|^2_{L^2(\Omega)}+c^2\left(\nabla u,\nabla u_t\right)_{L^2(\Omega)}+\frac{b}{2}\left\|\nabla u_t\right\|^2_{L^2(\Omega)}\right\}\\
=\;&-\alpha\left\|u_{tt}\right\|^2_{L^2(\Omega)}+c^2\left\|\nabla u_t\right\|^2_{L^2(\Omega)}\\
&+\int_0^t g(t-s)\left(\nabla u(s),\nabla u_{tt}(t)\right)_{L^2(\Omega)}ds. \qquad (2.12)
\end{aligned}$$

Step .2 Multiplying (1.1) by u_t and integrating over Ω over

$$\begin{aligned}
&\tau\left(u_{ttt},u_t\right)_{L^2(\Omega)}+\alpha\left(u_{tt},u_t\right)_{L^2(\Omega)}-c^2\left(\Delta u,u_t\right)_{L^2(\Omega)}\\
&-b\left(\Delta u_t,u_t\right)_{L^2(\Omega)}+\left(\int_0^t g(t-s)\Delta u(s)ds,u_t(t)\right)_{L^2(\Omega)}\\
=\;&0. \qquad (2.13)
\end{aligned}$$

By direct calculations, we get

$$\tau\left(u_{ttt},u_t\right)_{L^2(\Omega)}=\tau\frac{d}{dt}\left\{\left(u_{tt},u_t\right)_{L^2(\Omega)}\right\}-\tau\left\|u_{tt}\right\|^2_{L^2(\Omega)}, \qquad (2.14)$$

$$\alpha\left(u_{tt},u_t\right)_{L^2(\Omega)}=\frac{\alpha}{2}\frac{d}{dt}\left\{\left\|u_t\right\|^2_{L^2(\Omega)}\right\}. \qquad (2.15)$$

Using integration by parts, we have

$$-c^2\left(\Delta u,u_t\right)_{L^2(\Omega)}=\frac{c^2}{2}\frac{d}{dt}\left\{\left\|\nabla u\right\|^2_{L^2(\Omega)}\right\}, \qquad (2.16)$$

$$-b\left(\Delta u_t,u_t\right)_{L^2(\Omega)}=b\left\|\nabla u_t\right\|^2_{L^2(\Omega)}, \qquad (2.17)$$

$$\begin{aligned}
&\left(\int_0^t g(t-s)\Delta u(s)ds,u_t(t)\right)_{L^2(\Omega)}\\
=\;&-\int_0^t g(t-s)\left(\nabla u(s),\nabla u_t(t)\right)_{L^2(\Omega)}ds. \qquad (2.18)
\end{aligned}$$

By replacement of (2.14) – (2.18) into (2.13), we get

$$\begin{aligned}
&\frac{d}{dt}\left\{\tau\left(u_{tt},u_t\right)_{L^2(\Omega)}+\frac{\alpha}{2}\left\|u_t\right\|^2_{L^2(\Omega)}+\frac{c^2}{2}\left\|\nabla u\right\|^2_{L^2(\Omega)}\right\}\\
=\;&\tau\left\|u_{tt}\right\|^2_{L^2(\Omega)}-b\left\|\nabla u_t\right\|^2_{L^2(\Omega)}+\int_0^t g(t-s)\left(\nabla u(s),\nabla u_t(t)\right)_{L^2(\Omega)}ds. \qquad (2.19)
\end{aligned}$$

On multiplying (2.12) by k and summing by (2.19), we get

$$\begin{aligned}
&\frac{d}{dt}\left\{\frac{k\tau}{2}\left\|u_{tt}\right\|^2_{L^2(\Omega)}+kc^2\left(\nabla u,\nabla u_t\right)_{L^2(\Omega)}+\frac{kb}{2}\left\|\nabla u_t\right\|^2_{L^2(\Omega)}\right.\\
&\left.+\tau\left(u_{tt},u_t\right)_{L^2(\Omega)}+\frac{\alpha}{2}\left\|u_t\right\|^2_{L^2(\Omega)}+\frac{c^2}{2}\left\|\nabla u\right\|^2_{L^2(\Omega)}\right\}\\
=\;&-k\alpha\left\|u_{tt}\right\|^2_{L^2(\Omega)}+kc^2\left\|\nabla u_t\right\|^2_{L^2(\Omega)}+\tau\left\|u_{tt}\right\|^2_{L^2(\Omega)}\\
&-b\left\|\nabla u_t\right\|^2_{L^2(\Omega)}+k\int_0^t g(t-s)\left(\nabla u(s),\nabla u_{tt}(t)\right)_{L^2(\Omega)}ds\\
&+\int_0^t g(t-s)\left(\nabla u(s),\nabla u_t(t)\right)_{L^2(\Omega)}ds, \qquad (2.20)
\end{aligned}$$

using (2.3) into (2.20), we get (2.4).
This completes the proof. □

Lemma 2.2. *Assume that* $(\mathbf{A1})-(\mathbf{A3})$ *holds. Then the modified energy to* $(1.1)-(1.3)$ *is defined by*

$$\begin{aligned}
e(t) \quad : \quad &= \frac{k\tau}{2}\|u_{tt}\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u_t\|^2_{L^2(\Omega)} + \tau(u_{tt},u_t)_{L^2(\Omega)} + kc^2(\nabla u,\nabla u_t)_{L^2(\Omega)} \\
&+\frac{kb}{2}\|\nabla u_t\|^2_{L^2(\Omega)} + \frac{k}{2}(-g'\square\nabla u)(t) + \frac{k}{2}g(t)\|\nabla u\|^2_{L^2(\Omega)} \\
&+\frac{1}{2}\left(c^2-\int_0^t g(s)\,ds\right)\|\nabla u\|^2_{L^2(\Omega)} - k\int_0^t g(t-s)(\nabla u(s),\nabla u_t(t))_{L^2(\Omega)}\,ds \\
&+\frac{1}{2}(g\square\nabla u)(t), \qquad (2.21)
\end{aligned}$$

and its derivative satisfies the following

$$\begin{aligned}
\frac{d}{dt}\{e(t)\} &= \tau\|u_{tt}\|^2_{L^2(\Omega)} - k\alpha\|u_{tt}\|^2_{L^2(\Omega)} + kc^2\|\nabla u_t\|^2_{L^2(\Omega)} \\
&\quad -b\|\nabla u_t\|^2_{L^2(\Omega)} - \frac{k}{2}(g''\square\nabla u)(t) + \frac{1}{2}(g'\square\nabla u)(t) \\
&\quad +\frac{k}{2}g'(t)\|\nabla u\|^2_{L^2(\Omega)} - \frac{1}{2}g(t)\|\nabla u\|^2_{L^2(\Omega)} \\
&\leq 0, \qquad (2.22)
\end{aligned}$$

where k *is definite in* (2.5).

Proof. By direct calculations, we get

$$\begin{aligned}
&-\int_0^t g(t-s)(\nabla u(s),\nabla u_t(t))_{L^2(\Omega)}\,ds \\
=\; &\frac{1}{2}\frac{d}{dt}\left\{(g\square\nabla u)(t) - \left(\int_0^t g(s)\,ds\right)\|\nabla u\|^2_{L^2(\Omega)}\right\} \\
&-\frac{1}{2}(g'\square\nabla u)(t) + \frac{1}{2}g(t)\|\nabla u\|^2_{L^2(\Omega)}, \qquad (2.23)
\end{aligned}$$

$$\begin{aligned}
&-k\int_0^t g(t-s)(\nabla u(s),\nabla u_{tt}(t))_{L^2(\Omega)}\,ds \\
=\; &\frac{d}{dt}\left\{\frac{k}{2}(-g'\square\nabla u)(t) + \frac{k}{2}g(t)\|\nabla u\|^2_{L^2(\Omega)}\right. \\
&\left.-k\int_0^t g(t-s)(\nabla u(s),\nabla u_t(t))_{L^2(\Omega)}\,ds\right\} \\
&+\frac{k}{2}(g''\square\nabla u)(t) - \frac{k}{2}g'(t)\|\nabla u\|^2_{L^2(\Omega)}. \qquad (2.24)
\end{aligned}$$

By replacement (2.23) and (2.24) into (2.4), we get

$$\begin{aligned}
&\frac{d}{dt}\left\{\frac{k\tau}{2}\|u_{tt}\|^2_{L^2(\Omega)} + kc^2(\nabla u,\nabla u_t)_{L^2(\Omega)} + \frac{kb}{2}\|\nabla u_t\|^2_{L^2(\Omega)}\right. \\
&+\tau(u_{tt},u_t)_{L^2(\Omega)} + \frac{\alpha}{2}\|u_t\|^2_{L^2(\Omega)} + \frac{1}{2}(g\square\nabla u)(t) \\
&+\frac{1}{2}\left(c^2-\int_0^t g(s)\,ds\right)\|\nabla u\|^2_{L^2(\Omega)} + \frac{k}{2}(-g'\square\nabla u)(t) + \frac{k}{2}g(t)\|\nabla u\|^2_{L^2(\Omega)} \\
&\left.-k\int_0^t g(t-s)(\nabla u(s),\nabla u_t(t))_{L^2(\Omega)}\,ds\right\} \\
=\; &-k\alpha\|u_{tt}\|^2_{L^2(\Omega)} + kc^2\|\nabla u_t\|^2_{L^2(\Omega)} + \tau\|u_{tt}\|^2_{L^2(\Omega)} - b\|\nabla u_t\|^2_{L^2(\Omega)} \\
&-\frac{k}{2}(g''\square\nabla u)(t) + \frac{k}{2}g'(t)\|\nabla u\|^2_{L^2(\Omega)} + \frac{1}{2}(g'\square\nabla u)(t) - \frac{1}{2}g(t)\|\nabla u\|^2_{L^2(\Omega)}, \qquad (2.25)
\end{aligned}$$

using (2.21) into (2.25), we get (2.22).
This completes the proof. □

3. Global existence

In this section we show that any solution of the system (1.1) − (1.3) is global and decays uniformly provided that $e(0)$ is positive and small enough.

Theorem 3.1. *Assume that* (**A1**) − (**A3**) *holds. Then the solution to problem* (1.1) − (1.3) *is bounded and global.*

Proof. It suffices to show that $\|u_t\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}$ is bounded independently of t.
Using (2.5) and (**A1**) into (2.22), we get

$$\omega_1 \|u_t\|^2_{L^2(\Omega)} + \omega_2 \|\nabla u\|^2_{L^2(\Omega)} \leq e(t) \leq e(0),$$

where $\omega_1 > 0$ and $\omega_2 > 0$, then

$$\|u_t\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} \leq \omega_3 e(0).$$

Then the solution to problem (1.1) − (1.3) is bounded and global.
This completes the proof. □

4. General decay

In this section we state and prove our result.
Notation We denote by $\theta,\ \overline{\theta},\ \theta_\alpha,\ \overline{\theta}_\alpha$ and $\bar{g}$ the following expressions

$$\begin{cases} \psi(t)\,\theta(t) := g'(t) + \psi(t)\,g(t), \\ \bar{\theta} := \int_0^\infty \theta(s)ds, \quad \theta_\alpha := e^{\alpha t}\theta(t), \\ \overline{\theta}_\alpha := \int_0^\infty \theta_\alpha(s)ds, \quad \overline{g} := \int_0^\infty g(s)\,ds. \end{cases} \tag{4.1}$$

In the previous work it supposed that $g^{'}(t) \leq 0$. Therefore from (2.22) we see that $e^{'}(t) \leq 0$. This implies that $e(t) \leq e(0)$, for all $t \geq 0$. In our case we are not assuming that $g'(t) \leq 0$. In fact, we are allowing the function $g(t)$ to oscillate.
To prove our result we need to introduce the following auxiliary functional

$$\Gamma(t) := (u_{tt}, u)_{L^2(\Omega)}, \tag{4.2}$$

and

$$\begin{aligned} \Theta(t) \quad : &= \int_\Omega \int_0^t k_\alpha(t-s)\,|\nabla u(t) - \nabla u(s)|^2\,dsdx \\ : &= (k_\alpha \Box \nabla u)(t), \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} k_\alpha(t) \quad : &= e^{-\alpha t} \int_t^{+\infty} \theta_\alpha(s)ds \\ : &= e^{-\alpha t} \int_t^{+\infty} \theta(s)e^{\alpha s}ds, \end{aligned} \tag{4.4}$$

and $\theta(t)$ is defined in (4.1). Further, we consider the functional

$$\begin{aligned} M(t) \quad : \quad &= e(t) + \varepsilon\psi(t)\Gamma(t) + \frac{\varepsilon\alpha}{\tau}\psi(t)(u_t,u)_{L^2(\Omega)} + \frac{\varepsilon b}{2\tau}\psi(t)\|\nabla u\|^2_{L^2(\Omega)} \\ &+\chi\psi(t)\Theta(t) - \chi\psi(t)\left(\int_0^t k_\alpha(s)ds\right)\|\nabla u\|^2_{L^2(\Omega)} \\ &+2\chi\psi(t)\left\{\int_\Omega \nabla u(t).\int_0^t k_\alpha(t-s)\nabla u(s)dsdx\right\} \\ &-2\beta_1\psi(0)\left(\int_0^t \|u_s\|^2_{L^2(\Omega)}ds\right) - 2\beta_2\psi(0)\left(\int_0^t \|u_{ss}\|^2_{L^2(\Omega)}ds\right) \\ &-2\beta_3\psi(0)\left(\int_0^t \|\nabla u\|^2_{L^2(\Omega)}ds\right) - 2\beta_4\psi(0)\left(\int_0^t \|\nabla u_s\|^2_{L^2(\Omega)}ds\right), \end{aligned} \tag{4.5}$$

for some positive constant $\varepsilon,\ k,\ \chi,\ \beta_1,\ \beta_2,\ \beta_3$ and β_4 to be determined.

Remark 4.1. *Let*

$$\begin{aligned} Q_i(t) \quad : \quad &= M(t) + i\beta_1\psi(0)\left(\int_0^t \|u_s\|^2_{L^2(\Omega)}ds\right) + i\beta_2\psi(0)\left(\int_0^t \|u_{ss}\|^2_{L^2(\Omega)}ds\right) \\ &+i\beta_3\psi(0)\left(\int_0^t \|\nabla u\|^2_{L^2(\Omega)}ds\right) + i\beta_4\psi(0)\left(\int_0^t \|\nabla u_s\|^2_{L^2(\Omega)}ds\right), \quad for \quad i=1,2, \end{aligned}$$

then

$$\begin{aligned} Q_2(t) \quad : \quad &= e(t) + \varepsilon\psi(t)\Gamma(t) + \frac{\varepsilon\alpha}{\tau}\psi(t)(u_t,u)_{L^2(\Omega)} + \frac{\varepsilon b}{2\tau}\psi(t)\|\nabla u\|^2_{L^2(\Omega)} \\ &+\chi\psi(t)\Theta(t) - \chi\psi(t)\left(\int_0^t k_\alpha(s)ds\right)\|\nabla u\|^2_{L^2(\Omega)} \\ &+2\chi\psi(t)\left\{\int_\Omega \nabla u(t).\int_0^t k_\alpha(t-s)\nabla u(s)dsdx\right\}. \end{aligned} \tag{4.6}$$

Proposition 4.2. *Assume that* $(\mathbf{A1})-(\mathbf{A3})$ *holds. Then there exists positive constants* $C_2>0$ *and* $C_5>0$ *such that*

$$C_2\{E(t) + (g\Box\nabla u)(t)\} \le Q_2(t) \le C_5\{E(t) + (g\Box\nabla u)(t) + (-g'\Box\nabla u)(t) + \Theta(t)\}, \tag{4.7}$$

where C_2 *definite in* (4.22) *and* C_5 *definite in* (4.25).

Proof. For the function $\Gamma(t)$ definite in (4.2).
Using Young's inequality (for $\varepsilon=\varepsilon_1$), Poincaré inequality and $-1 \le \dfrac{-\psi(t)}{\psi(0)} < 0$, we get

$$\varepsilon\psi(t)\Gamma(t) \ge -\frac{\varepsilon\psi(0)\varepsilon_1}{2}\|u_{tt}\|^2_{L^2(\Omega)} - \frac{\varepsilon\psi(0)C_p}{2\varepsilon_1}\|\nabla u\|^2_{L^2(\Omega)}, \tag{4.8}$$

where C_p is the Poincaré constant.
Similarly, by using Young's inequality (for $\varepsilon=\varepsilon_2$), we get

$$\frac{\varepsilon\alpha}{\tau}\psi(t)(u_t,u)_{L^2(\Omega)} \ge -\frac{\varepsilon\alpha\psi(0)\varepsilon_2}{2\tau}\|u_t\|^2_{L^2(\Omega)} - \frac{\varepsilon\alpha\psi(0)C_p}{2\tau\varepsilon_2}\|\nabla u\|^2_{L^2(\Omega)}. \tag{4.9}$$

Note that from (4.1), (4.4) and using $-1 \le \dfrac{-\psi(t)}{\psi(0)} < 0$, we get

$$-\chi\psi(t)\left(\int_0^t k_\alpha(s)ds\right)\|\nabla u\|^2_{L^2(\Omega)} \ge -\frac{\chi\psi(0)\bar{\theta}_\alpha}{\alpha}\|\nabla u\|^2_{L^2(\Omega)}. \tag{4.10}$$

Using Young's inequality $\left(\text{for } \varepsilon = \frac{\varepsilon_3}{2}\right)$, (4.3) and $-1 \leq \frac{-\psi(t)}{\psi(0)} < 0$, we get

$$\begin{aligned} &2\chi\psi(t)\left\{\int_\Omega \nabla u(t) \,.\, \int_0^t k_\alpha(t-s)\nabla u(s)dsdx\right\} \\ \geq\ & -\frac{\chi}{2\varepsilon_3}\psi(t)\Theta(t) - \frac{2\chi(1+\varepsilon_3)\psi(0)\bar{\theta}_\alpha}{\alpha}\|\nabla u\|^2_{L^2(\Omega)}. \end{aligned} \tag{4.11}$$

Using Young's inequality (for $\varepsilon = \varepsilon_4$), we get

$$\tau(u_{tt}(t), u_t(t))_{L^2(\Omega)} \geq -\frac{\tau\varepsilon_4}{2}\|u_{tt}(t)\|^2_{L^2(\Omega)} - \frac{\tau}{2\varepsilon_4}\|u_t(t)\|^2_{L^2(\Omega)}. \tag{4.12}$$

Using Young's inequality (for $\varepsilon = \varepsilon_5$), we get

$$kc^2(\nabla u, \nabla u_t)_{L^2(\Omega)} \geq -\frac{kc^2\varepsilon_5}{2}\|\nabla u\|^2_{L^2(\Omega)} - \frac{kc^2}{2\varepsilon_5}\|\nabla u_t\|^2_{L^2(\Omega)}. \tag{4.13}$$

Using Young's inequality (for $\varepsilon = \varepsilon_6$ and $\varepsilon = \varepsilon_7$) and using $\int_0^t g(s)\,ds \leq \bar{g}$, we get

$$\begin{aligned} &-k\int_0^t g(t-s)(\nabla u(s), \nabla u_t(t))_{L^2(\Omega)}\,ds \\ \geq\ & -\frac{k\varepsilon_6}{2}(g\Box\nabla u)(t) - \left\{\frac{k}{2\varepsilon_6} + \frac{k}{2\varepsilon_7}\right\}\bar{g}\|\nabla u_t\|^2_{L^2(\Omega)} - \frac{k\varepsilon_7\bar{g}}{2}\|\nabla u\|^2_{L^2(\Omega)}. \end{aligned} \tag{4.14}$$

By replacement (4.8) – (4.14) into (4.6), we get

$$\begin{aligned} Q_2(t) \geq\ & \left\{\frac{\alpha}{2} - \frac{\tau}{2\varepsilon_4} - \frac{\varepsilon\alpha\psi(0)\varepsilon_2}{2\tau}\right\}\|u_t\|^2_{L^2(\Omega)} \\ &+ \left\{\frac{k\tau}{2} - \frac{\tau\varepsilon_4}{2} - \frac{\varepsilon\psi(0)\varepsilon_1}{2}\right\}\|u_{tt}\|^2_{L^2(\Omega)} \\ &+ \left\{\frac{(c^2-\bar{g})}{2} - \frac{kc^2\varepsilon_5}{2} - \frac{k\bar{g}\varepsilon_7}{2} - \frac{\varepsilon\alpha\psi(0)C_p}{2\tau\varepsilon_2}\right. \\ &\left. -\frac{\varepsilon\psi(0)C_p}{2\varepsilon_1} - \frac{\chi\psi(0)\bar{\theta}_\alpha}{\alpha} - \frac{2\chi(1+\varepsilon_3)\psi(0)\bar{\theta}_\alpha}{\alpha}\right\}\|\nabla u\|^2_{L^2(\Omega)} \\ &+ \left\{\frac{kb}{2} - \frac{kc^2}{2\varepsilon_5} - \frac{k\bar{g}}{2\varepsilon_6} - \frac{k\bar{g}}{2\varepsilon_7}\right\}\|\nabla u_t\|^2_{L^2(\Omega)} \\ &+ \frac{(1-k\varepsilon_6)}{2}(g\Box\nabla u)(t) + \chi\left\{1 - \frac{1}{2\varepsilon_3}\right\}\psi(t)\Theta(t). \end{aligned} \tag{4.15}$$

Clearly, choosing $\varepsilon_1 := \frac{k\tau}{4\varepsilon\psi(0)}$, $\varepsilon_2 := \frac{(\alpha k - \tau)\tau}{2k\varepsilon\alpha\psi(0)}$, $\varepsilon_3 := 1$, $\varepsilon_4 := \frac{k}{2}$, $\frac{4c^2}{b} < \varepsilon_5 < \frac{c^2-\bar{g}}{5kc^2}$, $\frac{4\bar{g}}{b} < \varepsilon_6 < \frac{1}{2k}$, $\frac{4\bar{g}}{b} < \varepsilon_7 < \frac{c^2-\bar{g}}{5k\bar{g}}$, $\chi := \frac{(c^2-\bar{g})\alpha}{50\psi(0)\bar{\theta}_\alpha}$, and $\varepsilon < \frac{\tau\sqrt{(c^2-\bar{g})(\alpha k-\tau)k}}{\psi(0)\sqrt{10C_p[\alpha k + 2(\alpha k - \tau)\tau]}}$, into (4.15), we get

$$\begin{aligned} Q_2(t) \geq\ & \frac{(\alpha k - \tau)}{4k}\|u_t\|^2_{L^2(\Omega)} + \frac{k\tau}{8}\|u_{tt}\|^2_{L^2(\Omega)} + \frac{(c^2-\bar{g})}{10}\|\nabla u\|^2_{L^2(\Omega)} \\ &+ \frac{kb}{8}\|\nabla u_t\|^2_{L^2(\Omega)} + \frac{1}{4}(g\Box\nabla u)(t) + \frac{\chi}{2}\psi(t)\Theta(t), \end{aligned} \tag{4.16}$$

using

$$\begin{aligned} &\alpha_1\left\{\|u_t\|^2_{L^2(\Omega)} + \|u_{tt}\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} + \|\nabla u_t\|^2_{L^2(\Omega)}\right\} \\ \leq\ & E(t) \\ \leq\ & \alpha_2\left\{\|u_t\|^2_{L^2(\Omega)} + \|u_{tt}\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} + \|\nabla u_t\|^2_{L^2(\Omega)}\right\}, \end{aligned} \tag{4.17}$$

where

$$\begin{cases} \alpha_1 := \dfrac{\min\left\{(\alpha-\tau),\tau(k-1),\dfrac{c^2}{2},k\left(b-2kc^2\right)\right\}}{2} > 0, \\ \text{and} \\ \alpha_2 := \dfrac{\max\left\{(\alpha+\tau),\tau(k+1),c^2(k+1),k\left(c^2+b\right)\right\}}{2} > 0, \end{cases}$$

into (4.16), we get

$$Q_2(t) \geq C_2\left\{E(t) + (g\Box\nabla u)(t)\right\}, \tag{4.18}$$

where

$$\begin{cases} C_2 := C_1 \min\left\{\dfrac{1}{\alpha_2},1\right\} > 0, \\ \text{and} \\ C_1 := \dfrac{\min\left\{\dfrac{(\alpha k-\tau)}{2k},\dfrac{k\tau}{4},\dfrac{\left(c^2-\bar{g}\right)}{5},\dfrac{kb}{4},\dfrac{1}{2}\right\}}{2} > 0. \end{cases} \tag{4.19}$$

On the other hand, by replacement $(4.8)-(4.14)$ into (4.6) and taking $\varepsilon_i = 1$ for $i = 1,...,7$, we get

$$\begin{aligned} Q_2(t) \leq{} & \left(\frac{\tau+\alpha}{2} + \frac{\varepsilon\alpha\psi(0)}{2\tau}\right)\|u_t\|^2_{L^2(\Omega)} + \frac{\{\tau+\varepsilon\psi(0)+k\tau\}}{2}\|u_{tt}\|^2_{L^2(\Omega)} \\ & + C_3\|\nabla u\|^2_{L^2(\Omega)} + \frac{k\left\{c^2+b+2\bar{g}\right\}}{2}\|\nabla u_t\|^2_{L^2(\Omega)} \\ & + \frac{(k+1)}{2}(g\Box\nabla u)(t) + \frac{k}{2}(-g'\Box\nabla u)(t) + \frac{3\chi\psi(0)}{2}\Theta(t), \end{aligned} \tag{4.20}$$

where

$$\begin{aligned} C_3 \ :& = \left\{\frac{kc^2 + k\sup(g(t)) + k\bar{g} + \varepsilon\psi(0)C_p + \left(c^2+\bar{g}\right)}{2}\right. \\ & \left. + \frac{\psi(0)(\varepsilon\alpha C_p + \varepsilon b + 10\tau\chi\theta_\alpha)}{2\tau}\right\} > 0. \end{aligned}$$

Using (4.17) into (4.20), we get

$$Q_2(t) \leq C_5\left\{E(t) + (g\Box\nabla u)(t) + (-g'\Box\nabla u)(t) + \Theta(t)\right\}, \tag{4.21}$$

where

$$\begin{cases} C_5 := C_4 \max\left\{\dfrac{1}{\alpha_1},1\right\} > 0, \\ C_4 := \max\left\{\left(\dfrac{\tau+\alpha}{2} + \dfrac{\varepsilon\alpha\psi(0)}{2\tau}\right),\dfrac{\{\tau+\varepsilon\psi(0)+k\tau\}}{2},C_3,\right. \\ \left.\dfrac{k\left\{c^2+b+2\bar{g}\right\}}{2},\dfrac{(k+1)}{2},\dfrac{3\chi\psi(0)}{2}\right\} > 0, \end{cases} \tag{4.22}$$

using (4.18) and (4.21), we get (4.7).
This completes the proof. $\square$

Let

$$\begin{cases} \varepsilon := \min\left\{ \dfrac{\tau\sqrt{(c^2-\bar{g})(\alpha k-\tau)k}}{\psi(0)\sqrt{10C_p[\alpha k+2(\alpha k-\tau)\tau]}}, \dfrac{\beta_1\tau}{4\alpha}, \dfrac{\sqrt{\beta_1\beta_2}}{\sqrt{2}}, \right. \\ \left. \dfrac{4\beta_1\tau(k\alpha-\tau)(c^2-\bar{g})}{12(k\alpha-\tau)[\alpha^2L^2C_p+2\beta_1\bar{g}]+3\beta_1\tau^2L^2C_p\psi(0)} \right\} > 0, \\ \text{and} \\ \bar{l}_\alpha < \dfrac{\alpha^2(c^2-\bar{g})}{3\tau(L+2\alpha)(2L+3\alpha)}\varepsilon > 0. \end{cases}$$

Now we are in position to state and prove our first result.

Theorem 4.3. *Assume that the hypotheses* $(\mathbf{A1})-(\mathbf{A3})$ *holds, the initial data* (u_0,u_1) *satisfy* $E(0)>0$ *and that* $\overline{\theta}_\alpha$ *is as above. Then the classical energy* $E(t)$ *of* $(1.1)-(1.3)$ *decays to zero exponentially. That is, there exist positive constants* $C_2>0,\ C_5>0$ *and* $C_7>0$ *such that*

$$E(t) \le \frac{Q_1(0)}{C_2} e^{-\dfrac{C_7}{C_5}\left(\int_0^t \psi(s)ds\right)}, \quad t \ge 0,$$

where C_2 *is definite in* (4.19), C_5 *is definite in* (4.22) *and* C_7 *is definite in* (4.41).

Proof. Now a differentiation of $M(t)$ definite in (4.5) with respect to time gives

$$\begin{aligned} &\frac{d}{dt}\{Q_1(t)\} \\ =\ &\frac{d}{dt}\{e(t)\} + \varepsilon\frac{d}{dt}\left\{\psi(t)\Gamma(t) + \frac{\alpha}{\tau}\psi(t)(u_t,u)_{L^2(\Omega)} + \frac{b}{2\tau}\psi(t)\|\nabla u\|^2_{L^2(\Omega)}\right\} \\ &+\chi\frac{d}{dt}\left\{\psi(t)\psi(t) - \psi(t)\left(\int_0^t k_\alpha(s)ds\right)\|\nabla u\|^2_{L^2(\Omega)}\right. \\ &\left.+2\psi(t)\int_\Omega \nabla u(t).\int_0^t k_\alpha(t-s)\nabla u(s)dsdx\right\} \\ &-\beta_1\psi(0)\|u_t\|^2_{L^2(\Omega)} - \beta_2\psi(0)\|u_{tt}\|^2_{L^2(\Omega)} \\ &-\beta_3\psi(0)\|\nabla u\|^2_{L^2(\Omega)} - \beta_4\psi(0)\|\nabla u_t\|^2_{L^2(\Omega)}. \end{aligned} \tag{4.23}$$

A differentiation of (4.2) along the solution of $(1.1)-(1.3)$ yilds

$$\begin{aligned} &\frac{d}{dt}\{\psi(t)\Gamma(t)\} \\ =\ &\psi'(t)\int_\Omega u_{tt}udx + \psi(t)\int_\Omega u_{tt}u_tdx - \frac{\alpha}{\tau}\psi(t)\int_\Omega u_{tt}udx \\ &+\frac{c^2}{\tau}\psi(t)\int_\Omega \Delta u u dx + \frac{b}{\tau}\psi(t)\int_\Omega \Delta u_t u dx \\ &-\frac{1}{\tau}\psi(t)\int_\Omega u(t)\left[\int_0^t g(t-s)\Delta u(s)ds\right]dx. \end{aligned} \tag{4.24}$$

By direct calculations, we get

$$\begin{aligned} &-\frac{\alpha}{\tau}\psi(t)\int_\Omega u_{tt}udx \\ =\ &-\frac{\alpha}{\tau}\frac{d}{dt}\left\{\psi(t)(u_t,u)_{L^2(\Omega)}\right\} + \frac{\alpha}{\tau}\psi'(t)(u_t,u)_{L^2(\Omega)} \\ &+\frac{\alpha}{\tau}\psi(t)\|u_t\|^2_{L^2(\Omega)}. \end{aligned} \tag{4.25}$$

Using integration by parts, we get

$$\frac{c^2}{\tau}\psi(t)\int_\Omega \Delta u(t)\, u(t)\, dx = -\frac{c^2}{\tau}\psi(t)\left\|\nabla u(t)\right\|^2_{L^2(\Omega)}, \tag{4.26}$$

$$\begin{aligned} & \frac{b}{\tau}\psi(t)\int_\Omega \Delta u_t u dx \\ = & -\frac{b}{2\tau}\frac{d}{dt}\left\{\psi(t)\left\|\nabla u\right\|^2_{L^2(\Omega)}\right\} + \frac{b}{2\tau}\psi'(t)\left\|\nabla u\right\|^2_{L^2(\Omega)}, \end{aligned} \tag{4.27}$$

$$\begin{aligned} & -\frac{1}{\tau}\psi(t)\int_\Omega u(t)\left[\int_0^t g(t-s)\,\Delta u(s)\, ds\right] dx \\ = & \frac{1}{\tau}\psi(t)\left\{\int_0^t g(t-s)\int_\Omega \nabla u(t).\nabla u(s)\, dxds\right\}. \end{aligned} \tag{4.28}$$

Using $(4.25)-(4.28)$ into (4.24), we get

$$\begin{aligned} & \frac{d}{dt}\left\{\psi(t)\Gamma(t) + \frac{\alpha}{\tau}\psi(t)(u_t,u)_{L^2(\Omega)} + \frac{b}{2\tau}\psi(t)\left\|\nabla u\right\|^2_{L^2(\Omega)}\right\} \\ = & \psi'(t)(u_{tt},u)_{L^2(\Omega)} + \psi(t)(u_{tt},u_t)_{L^2(\Omega)} + \frac{\alpha}{\tau}\psi'(t)(u_t,u)_{L^2(\Omega)} \\ & + \frac{\alpha}{\tau}\psi(t)\left\|u_t\right\|^2_{L^2(\Omega)} - \frac{c^2}{\tau}\psi(t)\left\|\nabla u\right\|^2_{L^2(\Omega)} + \frac{b}{2\tau}\psi'(t)\left\|\nabla u\right\|^2_{L^2(\Omega)} \\ & + \frac{1}{\tau}\psi(t)\left\{\int_0^t g(t-s)\int_\Omega \nabla u(t).\nabla u(s)\, dxds\right\}. \end{aligned} \tag{4.29}$$

By direct calculations, we get

$$\begin{aligned} & \frac{d}{dt}\left\{\psi(t)\Theta(t) - \left(\int_0^t k_\alpha(s)ds\right)\psi(t)\left\|\nabla u\right\|^2_{L^2(\Omega)}\right. \\ & \left. + 2\psi(t)\left(\int_\Omega \nabla u(t).\int_0^t k_\alpha(t-s)\nabla u(s)dsdx\right)\right\} \\ = & \left\{\psi'(t) - \alpha\psi(t)\right\}\Theta(t) - \psi(t)(\theta\Box\nabla u)(t) \\ & - \left\{\psi(t)k_\alpha(t) + \left(\int_0^t k_\alpha(s)ds\right)\psi'(t) - 2k_\alpha(0)\psi(t)\right\}\left\|\nabla u\right\|^2_{L^2(\Omega)} \\ & + 2\left(\psi'(t) - \alpha\psi(t)\right)\left\{\int_\Omega \nabla u(t).\int_0^t k_\alpha(t-s)\nabla u(s)dsdx\right\} \\ & - 2\psi(t)\left\{\int_\Omega \nabla u(t).\int_0^t \theta(t-s)\nabla u(s)dsdx\right\}. \end{aligned} \tag{4.30}$$

Taking into account (2.19), (4.29) and (4.30) into (4.23), we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left\{Q_1(t)\right\} \\
=\;& \tau\|u_{tt}\|^2_{L^2(\Omega)} - k\alpha\|u_{tt}\|^2_{L^2(\Omega)} + kc^2\|\nabla u_t\|^2_{L^2(\Omega)} - b\|\nabla u_t\|^2_{L^2(\Omega)} \\
&-\frac{k}{2}\left(g''\Box\nabla u\right)(t) + \frac{1}{2}\left(g'\Box\nabla u\right)(t) + \frac{k}{2}g'(t)\|\nabla u\|^2_{L^2(\Omega)} \\
&-\frac{1}{2}g(t)\|\nabla u\|^2_{L^2(\Omega)} + \varepsilon\psi'(t)\left(u_{tt},u\right)_{L^2(\Omega)} + \varepsilon\psi(t)\left(u_{tt},u_t\right)_{L^2(\Omega)} \\
&+\frac{\varepsilon\alpha}{\tau}\psi'(t)\left(u_t,u\right)_{L^2(\Omega)} + \frac{\varepsilon\alpha}{\tau}\psi(t)\|u_t\|^2_{L^2(\Omega)} \\
&-\frac{\varepsilon c^2}{\tau}\psi(t)\|\nabla u\|^2_{L^2(\Omega)} + \frac{\varepsilon b}{2\tau}\psi'(t)\|\nabla u\|^2_{L^2(\Omega)} \\
&+\frac{\varepsilon}{\tau}\psi(t)\left\{\int_0^t g(t-s)\int_\Omega \nabla u(t).\nabla u(s)\,dxds\right\} \\
&+\chi\left\{\psi'(t)-\alpha\psi(t)\right\}\Theta(t) - \chi\psi(t)\left(\theta\Box\nabla u\right)(t) \\
&-\chi\left\{\psi(t)k_\alpha(t) + \left(\int_0^t k_\alpha(s)ds\right)\psi'(t) - 2k_\alpha(0)\psi(t)\right\}\|\nabla u\|^2_{L^2(\Omega)} \\
&+2\chi\left(\psi'(t)-\alpha\psi(t)\right)\left\{\int_\Omega \nabla u(t).\int_0^t k_\alpha(t-s)\nabla u(s)dsdx\right\} \\
&-2\chi\psi(t)\left\{\int_\Omega \nabla u(t).\int_0^t \theta(t-s)\nabla u(s)dsdx\right\} \\
&-\beta_1\psi(0)\|u_t\|^2_{L^2(\Omega)} - \beta_2\psi(0)\|u_{tt}\|^2_{L^2(\Omega)} \\
&-\beta_3\psi(0)\|\nabla u\|^2_{L^2(\Omega)} - \beta_4\psi(0)\|\nabla u_t\|^2_{L^2(\Omega)}. \qquad (4.31)
\end{aligned}
$$

Next, we use the estimate (4.31).

By using Young's inequality $\left(\text{for } \varepsilon = \frac{\delta_1}{2}\right)$, $\left|\frac{\psi'(t)}{\psi(t)}\right| \leq L$ and Poincaré inequality, we get

$$
\begin{aligned}
&\varepsilon\psi'(t)\left(u_{tt},u\right)_{L^2(\Omega)} \\
\leq\;& \varepsilon L\psi(t)\left\{\frac{1}{4\delta_1}\|u_{tt}\|^2_{L^2(\Omega)} + C_p\delta_1\|\nabla u\|^2_{L^2(\Omega)}\right\}. \qquad (4.32)
\end{aligned}
$$

Using Young's inequality (for $\varepsilon = \delta_2$), we get

$$
\begin{aligned}
&\varepsilon\psi(t)\left(u_{tt},u_t\right)_{L^2(\Omega)} \\
\leq\;& \varepsilon\psi(t)\left\{\frac{\delta_2}{2}\|u_{tt}\|^2_{L^2(\Omega)} + \frac{1}{2\delta_2}\|u_t\|^2_{L^2(\Omega)}\right\}. \qquad (4.33)
\end{aligned}
$$

Using Young's inequality $\left(\text{for } \varepsilon = \frac{\delta_3}{2}\right)$, $\left|\frac{\psi'(t)}{\psi(t)}\right| \leq L$ and Poincaré inequality, we get

$$
\begin{aligned}
&\frac{\varepsilon\alpha}{\tau}\psi'(t)\left(u_t,u\right)_{L^2(\Omega)} \\
\leq\;& \frac{\varepsilon\alpha L}{\tau}\psi(t)\left\{C_p\delta_3\|\nabla u\|^2_{L^2(\Omega)} + \frac{1}{4\delta_3}\|u_t\|^2_{L^2(\Omega)}\right\}. \qquad (4.34)
\end{aligned}
$$

Using Young's inequality (for $\varepsilon = \delta_4$) and using $\int_0^t g(s)\,ds \leq \bar{g}$, we get

$$
\begin{aligned}
&\frac{\varepsilon}{\tau}\psi(t)\left\{\int_0^t g(t-s)\int_\Omega \nabla u(t).\nabla u(s)\,dxds\right\} \\
\leq\;& \frac{\delta_4\varepsilon}{2\tau}\psi(t)\left(g\Box\nabla u\right)(t) + \frac{\varepsilon\bar{g}}{\tau}\left\{\frac{1}{2\delta_4}+1\right\}\psi(t)\|\nabla u\|^2_{L^2(\Omega)}. \qquad (4.35)
\end{aligned}
$$

Using Young's inequality (for $\varepsilon = \delta_5$) $\left|\frac{\psi'(t)}{\psi(t)}\right| \leq L$ and $\int_0^t k_\alpha(s)ds \leq \frac{\bar{\theta}_\alpha}{\alpha}$, we get

$$\begin{aligned}
&2\chi\left(\psi'(t) - \alpha\psi(t)\right)\left\{\int_\Omega \nabla u(t) \,.\int_0^t k_\alpha(t-s)\nabla u(s)dsdx\right\} \\
\leq\ & \frac{\chi(L+\alpha)}{2\delta_5}\psi(t)\Theta(t) + \frac{2\chi(L+\alpha)(1+\delta_5)\bar{\theta}_\alpha}{\alpha}\psi(t)\|\nabla u\|^2_{L^2(\Omega)}.
\end{aligned} \tag{4.36}$$

Similarly, by using Young's inequality (for $\varepsilon = \delta_6$), we get

$$\begin{aligned}
&-2\chi\psi(t)\left\{\int_\Omega \nabla u(t) \,.\int_0^t \theta(t-s)\nabla u(s)dsdx\right\} \\
\leq\ & \frac{\chi}{2\delta_6}\psi(t)(\theta\Box\nabla u)(t) + 2\chi(1+\delta_6)\bar{\theta}\psi(t)\|\nabla u(t)\|^2_{L^2(\Omega)}.
\end{aligned} \tag{4.37}$$

Making use of (4.32) – (4.37) into (4.31) and

$$\begin{aligned}
k_\alpha(0) &= \int_0^\infty \bar{\theta}_\alpha(s)ds \\
&= \bar{\bar{\theta}}_\alpha,
\end{aligned}$$

we get

$$\begin{aligned}
&\frac{d}{dt}\{Q_1(t)\} \\
\leq\ & -\left(\beta_1 - \frac{\varepsilon}{2\delta_2} - \frac{\varepsilon\alpha L}{4\tau\delta_3} - \frac{\varepsilon\alpha}{\tau}\right)\psi(t)\|u_t\|^2_{L^2(\Omega)} \\
&-\left(\beta_2 + \frac{(k\alpha-\tau)}{\psi(0)} - \frac{\varepsilon L}{4\delta_1} - \frac{\varepsilon\delta_2}{2}\right)\psi(t)\|u_{tt}\|^2_{L^2(\Omega)} \\
&-\left\{\beta_3 + \frac{\varepsilon}{\tau}\left(c^2 - \bar{g} - \alpha L C_p\delta_3 - \tau L C_p\delta_1 - \frac{\bar{g}}{2\delta_4}\right) - \frac{\chi\bar{\theta}_\alpha L}{\alpha}\right. \\
&\left.-2\chi\bar{\theta}_\alpha - \frac{2\chi(L+\alpha)(1+\delta_5)\bar{\theta}_\alpha}{\alpha} - 2\chi(1+\delta_6)\bar{\theta}\right\}\psi(t)\|\nabla u\|^2_{L^2(\Omega)} \\
&-\left(\beta_4 + \frac{(b-kc^2)}{\psi(0)}\right)\psi(t)\|\nabla u_t\|^2_{L^2(\Omega)} \\
&-\left(\frac{1}{4} - \frac{\delta_4\varepsilon}{2\tau}\right)\psi(t)(g\Box\nabla u)(t) - \frac{1}{4\psi(0)}\psi(t)(-g'\Box\nabla u)(t) \\
&-\chi\left(\alpha - \frac{(L+\alpha)}{2\delta_5}\right)\psi(t)\Theta(t) - \left(\chi\left\{1 - \frac{1}{2\delta_6}\right\} - \frac{1}{4}\right)\psi(t)(\theta\Box\nabla u)(t).
\end{aligned} \tag{4.38}$$

Finally, we choose $\delta_1 := \frac{\psi(0)\varepsilon L}{4(k\alpha-\tau)}$, $\delta_2 := \frac{2\varepsilon}{\beta_1}$, $\delta_3 := \frac{\varepsilon\alpha L}{\beta_1\tau}$, $\delta_4 := \frac{\tau}{4\varepsilon}$, $\delta_5 := \frac{L+\alpha}{\alpha}$, $\delta_6 := \frac{2}{3}$, $\chi := 1$, $\beta_3 := \frac{10\bar{\theta}}{3}$, and

$$\varepsilon < \min\left\{\frac{\beta_1\tau}{4\alpha}, \frac{\sqrt{\beta_1\beta_2}}{\sqrt{2}}, \frac{4\beta_1\tau(k\alpha-\tau)\left(c^2-\bar{g}\right)}{12(k\alpha-\tau)\left[\alpha^2L^2C_p + 2\beta_1\bar{g}\right] + 3\beta_1\tau^2L^2C_p\psi(0)}\right\}.$$

Then if $\bar{\theta}_\alpha < \dfrac{\alpha^2\left(c^2-\bar{g}\right)}{3\tau\left(L+2\alpha\right)\left(2L+3\alpha\right)}\varepsilon$, we entail from (4.38) that

$$
\begin{aligned}
&\frac{d}{dt}\left\{Q_1\left(t\right)\right\} \\
\leq\ & -\frac{\beta_1}{4}\psi\left(t\right)\left\|u_t\right\|^2_{L^2(\Omega)}-\frac{\beta_2}{2}\psi\left(t\right)\left\|u_{tt}\right\|^2_{L^2(\Omega)}-\frac{\varepsilon\left(c^2-\bar{g}\right)}{3\tau}\psi\left(t\right)\left\|\nabla u\right\|^2_{L^2(\Omega)} \\
& -\left(\beta_4+\frac{\left(b-kc^2\right)}{\psi\left(0\right)}\right)\psi\left(t\right)\left\|\nabla u_t\right\|^2_{L^2(\Omega)}-\frac{1}{8}\psi\left(t\right)\left(g\square\nabla u\right)\left(t\right) \\
& -\frac{1}{4\psi\left(0\right)}\psi\left(t\right)\left(-g'\square\nabla u\right)\left(t\right)-\frac{\alpha}{2}\psi\left(t\right)\Theta\left(t\right).
\end{aligned}
\tag{4.39}
$$

Using (4.21) into (4.39) , we get

$$
\frac{d}{dt}\left\{Q_1\left(t\right)\right\}\leq -C_7\psi\left(t\right)\left\{E\left(t\right)+\left(g\square\nabla u\right)\left(t\right)+\left(-g'\square\nabla u\right)\left(t\right)+\psi\left(t\right)\right\}, \tag{4.40}
$$

where

$$
\left\{
\begin{array}{c}
C_7:=C_6\min\left\{\dfrac{1}{\alpha_2},1\right\}>0, \\
\\
\text{and} \\
\\
C_6:=\min\left\{\dfrac{\beta_1}{4},\dfrac{\beta_2}{2},\dfrac{\varepsilon\left(c^2-\bar{g}\right)}{3\tau},\left(\beta_4+\dfrac{\left(b-kc^2\right)}{\psi\left(0\right)}\right),\dfrac{1}{8},\dfrac{1}{4\psi\left(0\right)},\dfrac{\alpha}{2}\right\}>0.
\end{array}
\right.
\tag{4.41}
$$

In virtue of **Proposition 1** (the right hand side inequality) into (4.40) and using $Q_1\left(t\right)\leq Q_2\left(t\right)\leq 2Q_1\left(t\right)$,, we find for all $t\geq 0$

$$
\frac{d}{dt}\left\{Q_1\left(t\right)\right\}\leq -\frac{C_7}{C_5}\psi\left(t\right)Q_1\left(t\right). \tag{4.42}
$$

Using Gromwell's Inequality in (4.42) , we find

$$
Q_1\left(t\right)\leq Q_1\left(0\right)e^{-\dfrac{C_7}{C_5}\left(\int_0^t\psi(s)ds\right)},\quad t\geq 0.
$$

Notice that by our assumption $E\left(0\right)>0$ in the theorem we have $Q_1\left(0\right)>0$. Again by **Proposition 1** (the left hand side inequality), we conclude the assertion of our theorem

$$
E\left(t\right)\leq\frac{Q_1\left(0\right)}{C_2}e^{-\dfrac{C_7}{C_5}\left(\int_0^t\psi(s)ds\right)},\quad t\geq 0.
$$

This completes the proof. □

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