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Applying the (1,2)-Pitchfork Domination and Its Inverse on Some Special Graphs

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Abstract: Let G be a finite graph, simple, undirected and with no isolated vertex. For any non-negative integers j and k, a dominating set D of $V(G)$ is called a pitchfork dominating set of G if every vertex in it dominates j vertices (at least) and k vertices (at most) from $V - D$. A set D^{-1} of $V - D$ is an inverse pitchfork dominating set if it is pitchfork dominating set. In this paper, pitchfork domination and inverse pitchfork domination are applied when $j = 1$ and $k = 2$ on some special graphs such as: tadpole graph, lollipop graph, lollipop flower graph , daisy graph and Barbell graph.

Key Words: Dominating set, inverse dominating set, pitchfork domination, pnverse pitchfork domination.

Contents

1. Introduction

Let G be a graph with no isolated vertex has a vertex set V of order n and an edge set E of size m. The number of edges incident on vertex w is denoted by $deg(w)$ and represent the degree of w. A vertex of degree 0 is called isolated and a vertex of degree 1 is a leaf. The vertex that adjacent with a leaf is a support vertex. The minimum and maximum degrees of vertices in G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. For basic concepts and other graph theoretic terminologies not defined here, we refer to [11,12,22]. Also, we refer for basic concepts of domination to [13,14,15,21]. A set $D \subseteq V$ is a dominating set if every vertex in $V - D$ is adjacent to a vertex in D. A dominating set D is said to be a minimal if it has no proper dominating subset. The domination number $\gamma(G)$ is the cardinality of a minimum dominating set D of G. There are several models of domination, see for example [6,7,9,10,18,19,20,23,25]. Domination in graphs play a wide role in different kinds of fields in graph theory as labeled graph [8], topological graph [16], fuzzy graph [24] and other. The pitchfork domination and its inverse are introduced by Al-Harere and Abdlhusein [1,2,3,4]. They discuss several bounds and properties and gave an important information and applications of this model. A dominating set D of V is called a pitchfork dominating set if every vertex in it dominates j vertices (at least) and k vertices (at most) of $V - D$ for any non-negative integers j and k. A set D^{-1} of $V - D$ is an inverse pitchfork dominating set if it is pitchfork dominating set. The pitchfork domination number of G, denoted by $\gamma_{pf}(G)$ is the minimum cardinality over all pitchfork dominating sets in G. The inverse pitchfork domination number of G, denoted by $\gamma^{-1}_{pf}(G)$ is the minimum cardinality over all inverse pitchfork dominating sets in G. In this paper, pitchfork domination and its inverse are applied with their bounds and properties on some graphs.

Proposition 1.1. *[5]: Let G be any graph with $\Delta(G) \leq 2$. Then, $\gamma(G) = \gamma_{pf}(G)$.*

Theorem 1.2. *[2] The cycle graph C_n; $(n \geq 3)$ has an inverse pitchfork domination such that: $\gamma^{-1}_{pf}(C_n) = \gamma_{pf}(C_n) = \lceil \frac{n}{3} \rceil$.*

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Theorem 1.3. *[2] The path graph P_n; $(n \geq 2)$ has an inverse pitchfork domination such that:*

$$\gamma_{pf}^{-1}(P_n) = \begin{cases} \frac{n}{3}+1 & if\ n \equiv 0\,(mod\,3) \\ \lceil \frac{n}{3} \rceil & if\ n \equiv 1,2\,(mod\,3) \end{cases}$$

where $\gamma_{pf}^{-1}(P_2) = 1$.

Proposition 1.4. *[5] Let $G = K_n$ the complete graph with $n \geq 3$, then $\gamma_{pf}(K_n) = n-2$.*

Proposition 1.5. *[2] The complete graph K_n has an inverse pitchfork domination if and only if $n = 3, 4$ and $\gamma_{pf}^{-1}(K_n) = n-2$.*

2. Pitchfork Domination

In this section, pitchfork domination is applied to discuss minimum pitchfork dominating set and its order for some graphs such as: tadpole graph, lollipop graph, daisy graph and Barbell graph.

Tadpole graph $T_{m,n}$ is formed by joining a vertex of its cycle C_m to a path P_n by an edge as in Fig 1. (see [5,7,11,17]).

Theorem 2.1. *Let G be the tadpole graph $T_{m,n}$ where $(m \geq 3)$ and $(n \geq 2)$. Then:*

$$\gamma_{pf}(T_{m,n}) = \begin{cases} \lceil \frac{m}{3} \rceil + \lceil \frac{n}{3} \rceil, & if\ m \equiv 0,2\ (mod\,3)\ or\ (m \equiv 1\ \wedge\ n \equiv 0\ (mod\,3)) \\ \lceil \frac{m-1}{3} \rceil + \lceil \frac{n}{3} \rceil, & if\ m \equiv 1\ \wedge\ n \equiv 1,\,2\ (mod\,3) \end{cases}$$

Proof: Since $T_{m,n}$ contains a cycle C_m joined by a bridge to a path P_n, then $V(T_{m,n}) = E(T_{m,n}) = m+n$ where $E(C_m) = m$, $E(P_n) = n-1$ and the bridge u_1v_n. The vertices of P_n can be labeled as: $\{v_i; i = 1, 2, \ldots, n\}$. Also the vertices of C_m as: $\{u_j; j = 1, 2, \ldots, m\}$ such that the vertex $u_1 \in C_m$ adjacent with vertex $v_n \in P_n$ and $deg(u_1) = 3$, $deg(v_1) = 1$. Let the pitchfork dominating set $D = D_1 \cup D_2$ where D_1 is the pitchfork dominating set in C_m and D_2 is the pitchfork dominating set in P_n. According to m we have two cases:
Case 1: There are two parts:
part i: If $m \equiv 0,\,2\,(mod\,3)$, then let:

$$D_1 = \begin{cases} \{u_{3j}; j = 1, 2, \ldots, \lceil \frac{m}{3} \rceil\}\ if\ m \equiv 0. \\ \{u_{3j}; j = 1, 2, \ldots, \lceil \frac{m}{3} \rceil - 1\} \cup \{u_m\}\ if\ m \equiv 2. \end{cases}$$

$$D_2 = \begin{cases} \{v_{3i-1}; i = 1, 2, \ldots, \lceil \frac{n}{3} \rceil\}\ if\ n \equiv 0, 2. \\ \{v_{3i-1}; i = 1, 2, \ldots, \lceil \frac{n}{3} \rceil - 1\} \cup \{v_n\}\ if\ n \equiv 1. \end{cases}$$

in this case if $u_1 \in D_1$, then $u_2 \in D_1$ or $u_m \in D_1$ or $v_n \in D_2$ to avoid that u_1 dominates three vertices. Hence, let $u_1 \notin D_1$ and dominated only by D_1 when $n \equiv 0\,(mod\,3)$, but it is also dominated by D_2 when $n \equiv 1,\,2$. Hence, D is a pitchfork dominating set in $T_{m,n}$ and $\gamma_{pf}(T_{m,n}) = |D_1| + |D_2| = \lceil \frac{m}{3} \rceil + \lceil \frac{n}{3} \rceil$.
part ii: If $m \equiv 1\ \wedge\ n \equiv 0\ (mod\,3)$, then let: $D_1 = \{u_{3i-1}; i = 1, 2, \ldots, \lceil \frac{m}{3} \rceil\}$ and $D_2 = \{v_{3i-1}; i = 1, 2, \ldots, \lceil \frac{n}{3} \rceil\}$. Then, $\gamma_{pf}(T_{m,n}) = |D_1| + |D_2| = \lceil \frac{m}{3} \rceil + \lceil \frac{n}{3} \rceil$.
Case 2: If $m \equiv 1 \wedge n \equiv 1,\,2\,(mod\,3)$, then the vertex u_1 is not dominated by D_1 since it is dominated by the vertex $v_n \in D_2$, then: $D_1 = \{u_{3j}; j = 1, 2, \ldots, \lceil \frac{m}{3} \rceil - 1\}$ and $D_2 = \{v_{3i-1}; i = 1, 2, \ldots, \lceil \frac{n}{3} \rceil\}$. Thus, D is a minimum pitchfork dominating set in $T_{m,n}$ and $\gamma_{pf}(T_{m,n}) = |D_1| + |D_2| = \lceil \frac{m-1}{3} \rceil + \lceil \frac{n}{3} \rceil$. □

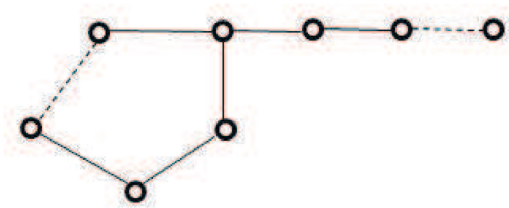

Figure 1: The tadpole graph

The lollipop graph $L_{m,n}$ is obtained by joining a vertex of K_m to P_n by edge as in Fig 2. (see [5,11,17]).

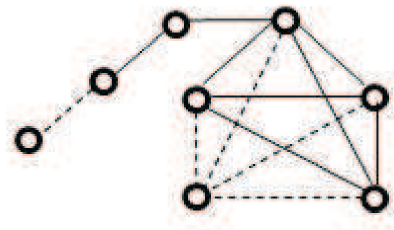

Figure 2: The lollipop graph

Proposition 2.2. *The lollipop graph $L_{m,n}$ has pitchfork domination for $m \geq 3$ and $n \geq 2$ such that* $\gamma_{pf}(L_{m,n}) = (m-2) + \lceil \frac{n}{3} \rceil$.

Proof: All vertices of K_m can be labeled as: $\{u_i; i = 1, 2, \ldots, m\}$, so that the vertices of P_n as: $\{v_j; j = 1, 2, \ldots, n\}$ where the vertex u_1 is adjacent with a vertex v_n. If $u_1 \in D$, then it will dominates two vertices of K_m and adjacent with v_n of P_n, then if $v_n \notin D$, the vertex u_1 dominates three vertices which is contradiction, and if $v_n \in D$, then $\gamma_{pf}(L_{m,n})$ may be increase. Therefore, let $u_1 \in V - D$. Since $\gamma_{pf}(K_m) = (m-2)$ by Proposition 1.4 and $\gamma_{pf}(P_n) = \lceil \frac{n}{3} \rceil$ by Observation 1.1. Thus, $\gamma_{pf}(L_{m,n}) = (m-2) + \lceil \frac{n}{3} \rceil$. □

The daisy graph D_{n_1,n_2} is formed by joined two cycles C_{n_1} and C_{n_2} by a common vertex as in Fig 3. (see [5,11,13]).

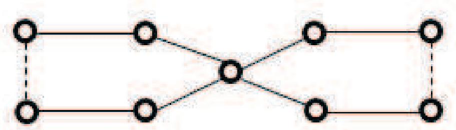

Figure 3: The daisy graph

Theorem 2.3. *Let G be the $(n_1, n_2)-$daisy graph D_{n_1,n_2}, then:*

$$\gamma_{pf}(D_{n_1,n_2}) = \begin{cases} \lceil \frac{n_1}{3} \rceil + \lceil \frac{n_2-1}{3} \rceil, & \text{if } \{n_1 \equiv 0, 2 \ (mod\, 3)\} \text{ or } \{n_1 \equiv 1 \wedge n_2 - 1 \equiv 0 \ (mod\, 3)\} \\ \lceil \frac{n_1-1}{3} \rceil + \lceil \frac{n_2-1}{3} \rceil, & \text{if } n_1 \equiv 1 \wedge n_2 - 1 \equiv 1, 2 \ (mod\, 3) \end{cases}$$

Proof: Suppose that D_{n_1,n_2} has two cycles C_{n_1} and C_{n_2} with common vertex. Let us label the vertices of C_{n_1} as: $\{v_i; i = 1, 2, \ldots, n_1\}$ so that the vertices of C_{n_2} as: $\{u_j; j = 1, 2, \ldots, n_2 - 1\}$ such that $|V(C_{n_1})| \geq |V(C_{n_2})|$ where this two cycles common by the vertex v_{n_1} of degree 4 which is adjacent with v_1, v_{n_1-1} from C_{n_1} and with u_1, u_{n_2-1} from C_{n_2}. Let the pitchfork dominating set of D_{n_1,n_2} is $D = D_1 \cup D_2$ where D_1 is pitchfork dominating set of C_{n_1} and D_2 is pitchfork dominating set of C_{n_2}. If we select $v_{n_1} \in D$, then it can be dominates at most two vertices and adjacent with two other vertices of D every one of them must be adjacent with one vertex from $V - D$. But this matter will increase $|D|$ (unless when $n_1 \equiv 1 \wedge n_2 - 1 \equiv 0$). Hence, to avoid this matter let us select $v_{n_1} \in V - D$. Now, to choose D:

Case 1: If $n_1 \equiv 0, 2 \ (mod\, 3)$. Let $D_1 = \{v_{3i-2}; i = 1, 2, \ldots, \lceil \frac{n_1}{3} \rceil\}$ and

$$D_2 = \begin{cases} \{u_{3i-1}; i = 1, 2, \ldots, \lceil \frac{n_2-1}{3} \rceil\} \ \text{if } n_2 - 1 \equiv 0, 2 \\ \{u_{3i-1}; i = 1, 2, \ldots, \lceil \frac{n_2-1}{3} \rceil - 1\} \cup \{u_{n_2-1}\} \ \text{if } n_2 - 1 \equiv 1. \end{cases}$$

Then, $\gamma_{pf}(D_{n_1,n_2}) = |D_1| + |D_2| = \lceil \frac{n_1}{3} \rceil + \lceil \frac{n_2-1}{3} \rceil$.
Case 2: If $n_1 \equiv 1\,(mod\,3)$:
Part i: If $n_1 \equiv 1 \wedge n_2 - 1 \equiv 0\,(mod\,3)$, then let: $D_1 = \{v_{3i-2}; i = 1, 2, \dots, \lceil \frac{n_1}{3} \rceil - 1\} \cup \{v_{n_1-1}\}$ and $D_2 = \{u_{3i-1}; i = 1, 2, \dots, \lceil \frac{n_2-1}{3} \rceil\}$. Hence, $\gamma_{pf}(D_{n_1,n_2}) = |D_1| + |D_2| = \lceil \frac{n_1}{3} \rceil + \lceil \frac{n_2-1}{3} \rceil$.
Part ii: If $n_1 \equiv 1 \wedge n_2 - 1 \equiv 1, 2\,(mod\,3)$, then in this case, the vertex v_{n_1} can be dominated by the vertex u_1 or u_{n_2-1} or together. Hence, the set D_1 will dominates only $n_1 - 1$ vertices of cycle C_{n_1}, therefore $|D_1|$ will be decreasing and we can choose D_1 and D_2 as follows: $D_1 = \{v_{3i-1}; i = 1, 2, \dots, \lceil \frac{n_1}{3} \rceil - 1\}$ and $D_2 = \{u_{3i-2}; i = 1, 2, \dots, \lceil \frac{n_2-1}{3} \rceil\}$. Hence, $\gamma_{pf}(D_{n_1,n_2}) = |D_1| + |D_2| = \lceil \frac{n_1-1}{3} \rceil + \lceil \frac{n_2-1}{3} \rceil$. □

The Barbell graph $B_{n,n}$, $(n \geq 3)$ contains two complete graphs K_n joined by edge as in Fig 4. (see [5,11,17]).

Figure 4: The Barbell graphs

Proposition 2.4. *The Barbell graph $B_{n,n}$, $(n \geq 3)$ has pitchfork domination such that $\gamma_{pf}(B_{n,n}) = 2n - 4$.*

Proof: Since $\gamma_{pf}(K_n) = n - 2$ by Proposition 1.4 such that the bridge is incident on two vertices of D or $V - D$ together. □

3. Inverse Pitchfork Domination

In this section, an inverse pitchfork domination is studied to discuss minimum inverse pitchfork dominating set and its order for the previous graphs.

Theorem 3.1. *For the tadpole graph $T_{m,n}$; $m \geq 3, n \geq 2$, we have:*

$$\gamma^{-1}_{pf}(T_{m,n}) = \begin{cases} \lceil \frac{m}{3} \rceil + \lceil \frac{n+1}{3} \rceil, & \text{if } n \equiv 0\,(mod\,3) \\ \lceil \frac{m}{3} \rceil + \lceil \frac{n}{3} \rceil, & \text{if } (m \equiv 0, 2 \wedge n \equiv 1, 2) \text{ or } (m \equiv 1 \wedge n \equiv 2) \\ \lceil \frac{m-1}{3} \rceil + \lceil \frac{n}{3} \rceil, & \text{if } m \equiv 1 \wedge n \equiv 1 \end{cases}$$

Proof: According to Theorem 2.1, $D = D_1 \cup D_2$ where D_1 is the pitchfork dominating set of C_m and D_2 is the pitchfork dominating set of P_n. Let $D^{-1} = D_1^{-1} \cup D_2^{-1}$ where D_1^{-1} is an inverse pitchfork dominating set of C_m and D_2^{-1} is an inverse pitchfork dominating set of P_n. Then, we choose D^{-1} as:
Case 1: If $m \equiv 0\ mod\,3$. Then, $D_1^{-1} = \{u_{3j-1}; j = 1, 2, \dots, \lceil \frac{m}{3} \rceil\}$ where $|D_1^{-1}| = \lceil \frac{m}{3} \rceil$ and

$$D_2^{-1} = \begin{cases} \{v_{3i-2}; i = 1, 2, \dots, \lceil \frac{n}{3} \rceil\} \cup \{v_n\}, & \text{if } n \equiv 0 \\ \{v_{3i-2}; i = 1, 2, \dots, \lceil \frac{n}{3} \rceil - 1\} \cup \{v_{n-1}\}, & \text{if } n \equiv 1 \\ \{v_{3i-2}; i = 1, 2, \dots, \lceil \frac{n}{3} \rceil\}, & \text{if } n \equiv 2 \end{cases}$$

Where

$$|D_2^{-1}| = \begin{cases} \lceil \frac{n+1}{3} \rceil, & \text{if } n \equiv 0 \\ \lceil \frac{n}{3} \rceil, & \text{if } n \equiv 1, 2 \end{cases}$$

Case 2: If $m \equiv 1\ mod\,3$, then

$$D_1^{-1} = \begin{cases} \{u_{3j-2}; j = 1, 2, \dots, \lceil \frac{m}{3} \rceil - 1\} \cup \{u_{m-2}\}, & \text{if } n \equiv 0 \\ \{u_{3j}; j = 1, 2, \dots, \lceil \frac{m}{3} \rceil - 1\}, & \text{if } n \equiv 1 \\ \{u_{3j-2}; j = 1, 2, \dots, \lceil \frac{m}{3} \rceil\}, & \text{if } n \equiv 2 \end{cases}$$

Where

$$|D_1^{-1}| = \begin{cases} \lceil \frac{m}{3} \rceil, & \text{if } n \equiv 0,2 \\ \lceil \frac{m-1}{3} \rceil, & \text{if } n \equiv 1 \end{cases}$$

And

$$D_2^{-1} = \begin{cases} \{v_{3i-2}; i = 1,2,\dots,\lceil \frac{n}{3} \rceil\} \cup \{v_n\}, & \text{if } n \equiv 0 \\ \{v_{3i-2}; i = 1,2,\dots,\lceil \frac{n}{3} \rceil\}, & \text{if } n \equiv 1,2 \end{cases}$$

Where

$$|D_2^{-1}| = \begin{cases} \lceil \frac{n+1}{3} \rceil, & \text{if } n \equiv 0 \\ \lceil \frac{n}{3} \rceil, & \text{if } n \equiv 1,2 \end{cases}$$

Case 3: If $m \equiv 2 \; mod\, 3$, then $D_1^{-1} = \{u_{3j-1}; j = 1,2,\dots,\lceil \frac{m}{3} \rceil - 1\} \cup \{u_{m-1}\}$ where $|D_1^{-1}| = \lceil \frac{m}{3} \rceil$ and

$$D_2^{-1} = \begin{cases} \{v_{3i-2}; i = 1,2,\dots,\lceil \frac{n}{3} \rceil\} \cup \{v_n\}, & \text{if } n \equiv 0 \\ \{v_{3i-2}; i = 1,2,\dots,\lceil \frac{n}{3} \rceil - 1\} \cup \{v_{n-1}\}, & \text{if } n \equiv 1 \\ \{v_{3i-2}; i = 1,2,\dots,\lceil \frac{n}{3} \rceil\}, & \text{if } n \equiv 2 \end{cases}$$

Where

$$|D_2^{-1}| = \begin{cases} \lceil \frac{n+1}{3} \rceil, & \text{if } n \equiv 0 \\ \lceil \frac{n}{3} \rceil, & \text{if } n \equiv 1,2 \end{cases}$$

□

Theorem 3.2. *The lollipop graph $L_{m,n}$; $m \geq 3, n \geq 2$, has an inverse pitchfork domination if and only if $m = 3,4$ such that:*

$$\gamma_{pf}^{-1}(L_{m,n}) = \begin{cases} \lceil \frac{n+1}{3} \rceil + (m-2), & \text{if } n \equiv 0\,(mod\, 3) \\ \lceil \frac{n}{3} \rceil + (m-2), & \text{if } n \equiv 1,2 \end{cases}$$

Proof: Let D is chosen as in Proposition 2.2 as $D = D_1 \cup D_2$, where D_1 is the pitchfork dominating set of K_m and D_2 is a pitchfork dominating set of P_n. Therefore, let $D^{-1} = D_1^{-1} \cup D_2^{-1}$ where D_1^{-1} is an inverse pitchfork dominating set in K_m and D_2^{-1} is an inverse pitchfork dominating set in P_n, then D^{-1} chosen according to D as the following cases:
Case 1: If $m = 3$, let $D_1 = \{u_2\}$, then $D_1^{-1} = \{u_3\}$. Also, if

$$D_2 = \begin{cases} \{v_{3i-1}; i = 1,2,\dots,\lceil \frac{n}{3} \rceil\}, & \text{if } n \equiv 0,2\,(mod\, 3) \\ \{v_{3i-1}; i = 1,2,\dots,\lceil \frac{n}{3} \rceil - 1\} \cup \{v_n\}, & \text{if } n \equiv 1\,(mod\, 3) \end{cases}$$

Then,

$$D_2^{-1} = \begin{cases} \{v_{3i-2}; i = 1,2,\dots,\lceil \frac{n}{3} \rceil\} \cup \{v_n\}, & \text{if } n \equiv 0\,(mod\, 3) \\ \{v_{3i-2}; i = 1,2,\dots,\lceil \frac{n}{3} \rceil - 1\} \cup \{v_{n-1}\}, & \text{if } n \equiv 1\,(mod\, 3) \\ \{v_{3i-2}; i = 1,2,\dots,\lceil \frac{n}{3} \rceil\}, & \text{if } n \equiv 2\,(mod\, 3) \end{cases}$$

Case 2: If $m = 4$, let $D_1 = \{u_2, u_3\}$, then $D_1^{-1} = \{u_1, u_4\}$. Also, if

$$D_2 = \begin{cases} \{v_{3i-1}; i = 1,2,\dots,\lceil \frac{n}{3} \rceil\}, & \text{if } n \equiv 0,2\,(mod\, 3) \\ \{v_{3i-1}; i = 1,2,\dots,\lceil \frac{n}{3} \rceil - 1\} \cup \{v_{n-1}\}, & \text{if } n \equiv 1\,(mod\, 3) \end{cases}$$

Then,

$$D_2^{-1} = \begin{cases} \{v_{3i-2}; i = 1,2,\dots,\lceil \frac{n}{3} \rceil\} \cup \{v_n\}, & \text{if } n \equiv 0\,(mod\, 3) \\ \{v_{3i-2}; i = 1,2,\dots,\lceil \frac{n}{3} \rceil\}, & \text{if } n \equiv 1,2\,(mod\, 3) \end{cases}$$

Therefore,

$$\gamma_{pf}^{-1}(L_{3,n}) = \begin{cases} \lceil \frac{n+1}{3} \rceil + 1, & \text{if } n \equiv 0\,(mod\, 3) \\ \lceil \frac{n}{3} \rceil + 1, & \text{if } n \equiv 1,2 \end{cases}$$

$$\gamma_{pf}^{-1}(L_{4,n}) = \begin{cases} \lceil \frac{n+1}{3} \rceil + 2, & \text{if } n \equiv 0\,(mod\, 3) \\ \lceil \frac{n}{3} \rceil + 2, & \text{if } n \equiv 1,2 \end{cases}$$

□

The lollipop flower $F_{m,n}$ is defined in [5] as a complete graph K_m, every vertex in which joins (by edge) with a path P_n, where $V(F_{m,n}) = m + mn$ so that $E(F_{m,n}) = \binom{m}{2} + mn$. See Fig 5.

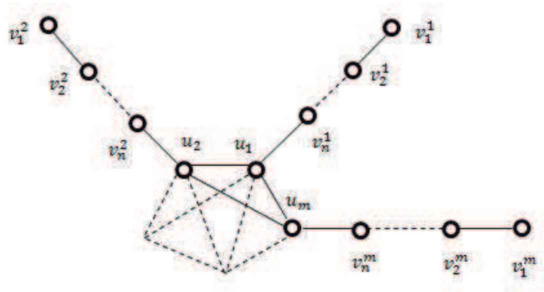

Figure 5: The lollipop flower graph

Theorem 3.3. *[5] For $m \geq 3$ and $n \geq 2$,*

$$\gamma_{pf}(F_{m,n}) = \begin{cases} m\lceil \frac{n}{3} \rceil, & \text{if } n \equiv 1,2 \ (mod\, 3) \\ m\lceil \frac{n}{3} \rceil + m - 1, & \text{if } n \equiv 0 \ (mod\, 3) \end{cases}$$

Theorem 3.4. *The lollipop flower graph $F_{m,n}$; $m \geq 3$, $n \geq 2$ has an inverse pitchfork domination if and only if $m = 3, 4$ such that:*

$$\gamma_{pf}^{-1}(F_{m,n}) = \begin{cases} \frac{mn}{3} + m, & \text{if } n \equiv 0 \ (mod\, 3) \\ m\lceil \frac{n}{3} \rceil + (m-2), & \text{if } n \equiv 1,2 \ (mod\, 3) \end{cases}$$

Proof: Let D is chosen as in Proposition 3.3 as $D = D_k \cup D_p$, where D_k is a pitchfork dominating set of K_m and D_p is a pitchfork dominating set of P_n^i where $D_p = \bigcup_{i=1}^{m} D_i$. Therefore, let $D^{-1} = D_k^{-1} \cup D_p^{-1}$ where D_k^{-1} is an inverse pitchfork dominating set in K_m and D_p^{-1} is an inverse pitchfork dominating set in P_n, then D^{-1} chosen according to D as:

$$D_k = \begin{cases} \{u_3\}, & \text{if } m = 3 \\ \{u_3, u_4\}, & \text{if } m = 4 \end{cases}$$

Hence,

$$D_k^{-1} = \begin{cases} \{u_1\}, & \text{if } m = 3 \\ \{u_1, u_2\}, & \text{if } m = 4 \end{cases}$$

Since

$$D_1 = D_2 = \begin{cases} \{v_{3i-1}; i = 1, 2, \ldots, \lceil \frac{n}{3} \rceil\}, & \text{if } n \equiv 0, 2 \, (mod\, 3) \\ \{v_{3i-1}; i = 1, 2, \ldots, \lceil \frac{n}{3} \rceil - 1\} \cup \{v_n\}, & \text{if } n \equiv 1 \, (mod\, 3) \end{cases}$$

Thus, D_1^{-1} and D_2^{-1} are formed as:

$$D_1^{-1} = D_2^{-1} = \begin{cases} \{v_{3i-2}; i = 1, 2, \ldots, \lceil \frac{n}{3} \rceil\} \cup \{v_n\}, & \text{if } n \equiv 0 \, (mod\, 3) \\ \{v_{3i-1}; i = 1, 2, \ldots, \lceil \frac{n}{3} \rceil - 1\} \cup \{v_{n-1}\}, & \text{if } n \equiv 1 \, (mod\, 3) \\ \{v_{3i-2}; i = 1, 2, \ldots, \lceil \frac{n}{3} \rceil\}, & \text{if } n \equiv 2 \, (mod\, 3) \end{cases}$$

Where

$$|D_1^{-1}| = |D_2^{-1}| = \begin{cases} \lceil \frac{n}{3} \rceil + 1, & \text{if } n \equiv 0 \\ \lceil \frac{n}{3} \rceil, & \text{if } n \equiv 1, 2 \end{cases}$$

Now, we choose D_3^{-1} and D_4^{-1} according to Theorem 1.1 such that $|D_3^{-1}| = |D_4^{-1}| = \lceil \frac{n}{3} \rceil$. Therefore,

$$\gamma_{pf}^{-1}(F_{m,n}) = \begin{cases} (m-2) + 2(\lceil \frac{n}{3} \rceil + 1) + (m-2)\lceil \frac{n}{3} \rceil, & \text{if } n \equiv 0 \ (mod\, 3) \\ (m-2) + 2\lceil \frac{n}{3} \rceil + (m-2)\lceil \frac{n}{3} \rceil, & \text{if } n \equiv 1,2 \ (mod\, 3) \end{cases}$$

which is the required identity after few simplification. □

Theorem 3.5. *Let G be the $(n_1, n_2)-$daisy graph D_{n_1,n_2}, then:*

$$\gamma_{pf}^{-1}(D_{n_1,n_2}) = \begin{cases} \lceil \frac{n_1}{3} \rceil + \lceil \frac{n_2}{3} \rceil, & \text{if } n_1 \equiv 0,2 \wedge n_2 - 1 \equiv 0 \,(mod\, 3) \\ \lceil \frac{n_1-1}{3} \rceil + \lceil \frac{n_2}{3} \rceil, & \text{if } n_1 \equiv 1 \wedge n_2 - 1 \equiv 0 \,(mod\, 3) \\ \lceil \frac{n_1}{3} \rceil + \lceil \frac{n_2-1}{3} \rceil, & \text{otherwise (i.e. } n_1 \equiv 0,1,2 \wedge n_2 - 1 \equiv 1,2 \,(mod\, 3)) \end{cases}$$

Proof: Suppose that D_{n_1,n_2} has two cycles C_{n_1} and C_{n_2} with a common vertex and let us label the vertices of D_{n_1,n_2} and the pitchfork dominating set according to Theorem 2.3. An inverse pitchfork dominating set of D_{n_1,n_2} is $D^{-1} = D_1^{-1} \cup D_2^{-1}$ where D_1^{-1} and D_2^{-1} is an inverse pitchfork dominating sets of C_{n_1} and C_{n_2} respectively, which are selecting as follows:

$$D_1^{-1} = \begin{cases} \{v_{3i-1}; i = 1,2,\ldots,\lceil \frac{n_1}{3} \rceil\}, & \text{if } n_1 \equiv 0 \,(mod\, 3) \\ \{v_{3i-1}; i = 1,2,\ldots,\lceil \frac{n_1}{3} \rceil - 1\}, & \text{if } n_1 \equiv 1 \wedge n_2 - 1 \equiv 0 \,(mod\, 3) \\ \{v_{3i-2}; i = 1,2,\ldots,\lceil \frac{n_1}{3} \rceil - 1\} \cup \{v_{n_1-1}\}, & \text{if } n_1 \equiv 1 \wedge n_2 - 1 \equiv 1,2 \,(mod\, 3) \\ \{v_{3i-1}; i = 1,2,\ldots,\lceil \frac{n_1}{3} \rceil - 1\} \cup \{v_{n_1}\}, & \text{if } n_1 \equiv 2 \wedge n_2 - 1 \equiv 0 \,(mod\, 3) \\ \{v_{3i-2}; i = 1,2,\ldots,\lceil \frac{n_1}{3} \rceil - 1\} \cup \{v_{n_1-2}\}, & \text{if } n_1 \equiv 2 \wedge n_2 - 1 \equiv 1,2 \,(mod\, 3) \end{cases}$$

And

$$D_2^{-1} = \begin{cases} \{u_{3i-2}; i = 1,2,\ldots,\lceil \frac{n_2-1}{3} \rceil\} \cup \{u_{n_2-1}\}, & \text{if } n_2 - 1 \equiv 0 \,(mod\, 3) \\ \{u_{3i-1}; i = 1,2,\ldots,\lceil \frac{n_2-1}{3} \rceil - 1\} \cup \{u_{n_2-1}\}, & \text{if } n_2 - 1 \equiv 1 \wedge n_1 \equiv 0 \,(mod\, 3) \\ \{u_{3i-1}; i = 1,2,\ldots,\lceil \frac{n_2-1}{3} \rceil - 1\} \cup \{u_{n_2-2}\}, & \text{if } n_2 - 1 \equiv 1 \wedge n_1 \equiv 1,2 \,(mod\, 3) \\ \{u_{3i-2}; i = 1,2,\ldots,\lceil \frac{n_2-1}{3} \rceil\}, & \text{if } n_2 - 1 \equiv 2 \,(mod\, 3) \end{cases}$$

Where

$$|D_1^{-1}| = \begin{cases} \lceil \frac{n_1-1}{3} \rceil, & \text{if } n_1 \equiv 1 \wedge n_2 - 1 \equiv 0 \\ \lceil \frac{n_1}{3} \rceil, & \text{otherwise} \end{cases}$$

And

$$|D_2^{-1}| = \begin{cases} \lceil \frac{n_2}{3} \rceil, & \text{if } n_2 - 1 \equiv 0 \\ \lceil \frac{n_2-1}{3} \rceil, & \text{if } n_2 - 1 \equiv 1,2 \end{cases}$$

Therefore, D^{-1} is a minimum inverse pitchfork dominating set. □

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