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Semi-Fredholm and Semi-Browder Spectra For C_0-quasi-semigroups

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ABSTRACT: In [5] D.Barcenas and H. Leiva are introduced the notion of C_0-quasi-semigroups of bounded linear operators, as a generalization of C_0-semigroups of operators. In this paper, we shall show the connections between a different spectra of the C_0-quasi-semigroups by the spectra of their generators, specially, ascent, descent essential ascent and essential descent, upper and lower semi-Fredholm and semi-Browder spectra.

Key Words: C_0-quasi-semigroup, C_0-semigroup, semi-Fredholm, ascent, descent spectrum, semi-Browder spectrum.

Contents

1. Introduction

Let X be a complex Banach space and $\mathcal{B}(X)$ the algebra of all bounded linear operators on X. The theory of quasi-semigroups of bounded linear operators, as a generalization of semigroups of operators, was introduced by Leiva and Barcenas [3], [4] ,[5]. Recently Sutrima, Ch. Rini Indrati and others [11] are show some relations between a C_0-quasi-semigroup and its generator related to the time-dependent evolution equation.

A two parameter commutative family $\{R(t,s)\}_{t,s\geq 0} \subseteq \mathcal{B}(X)$ is called a strongly continuous quasi-semigroup (or C_0-quasi-semigroup) of operators if for every $t,s,r \geq 0$ and $x \in X$, we have

1. $R(t,0) = I$, the identity operator on X,
2. $R(t,s+r) = R(t+r,s)R(t,r)$,
3. $\lim\limits_{s\longrightarrow 0^+} ||R(t,s)x - x|| = 0$,
4. there exists a continuous increasing mapping $M : [0,+\infty[\longrightarrow [0,+\infty[$ such that,
$$||R(t,s)|| \leq M(t+s).$$

For a C_0-quasi-semigroup $\{R(t,s)\}_{t,s\geq 0}$ on a Banach space X, let $\mathcal{D}$ be the set of all $x \in X$ for which the following limits exist,

$$\lim_{s\to 0^+} \frac{R(0,s)x - x}{s}, \ \lim_{s\to 0^+} \frac{R(t,s)x - x}{s} \text{ and } \lim_{s\to 0^+} \frac{R(t-s,s)x - x}{s}$$

and

$$\lim_{s\to 0^+} \frac{R(t,s)x - x}{s} = \lim_{s\to 0^+} \frac{R(t-s,s)x - x}{s}.$$

In this case, for $t \geq 0$, we define an operator $A(t)$ on $\mathcal{D}$ as

$$A(t)x = \lim_{s\to 0^+} \frac{R(t,s)x - x}{s}.$$

The family $\{A(t)\}_{t\geq 0}$ is called infinitesimal generator of the C_0-quasi-semigroups $\{R(t,s)\}_{t,s\geq 0}$ and $\mathcal{D}$ the domain for $A(t)$, $t \geq 0$.

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Theorem 1.1. *[11]*
Let $\{R(t,s)\}_{t,s\geq 0}$ be a C_0-quasi-semigroup on X with generator $\{A(t)\}_{t\geq 0}$ then,

1. *For each $t \geq 0$, $R(t,.)$ is strongly continuous on $[0;+\infty[$.*

2. *For each $t \geq 0$ and $x \in X$,*

$$\lim_{s\to 0^+} \frac{1}{s}\int_0^s R(t,h)xdh = x.$$

3. *If $x \in \mathcal{D}$, $t \geq 0$ and $t_0, s_0 \geq 0$ then, $R(t_0,s_0)x \in \mathcal{D}$ and*

$$R(t_0,s_0)A(t)x = A(t)R(t_0,s_0)x.$$

4. *For each $s > 0$, $\dfrac{\partial}{\partial s}R(t,s)x = A(t+s)R(t,s)x = R(t,s)A(t+s)x; \quad x \in \mathcal{D}$.*

5. *If $A(.)$ is locally integrable, then for every $x \in \mathcal{D}$ and $s \geq 0$,*

$$R(t,s)x = x + \int_0^s A(t+h)R(t,h)xdh.$$

6. *If $f : [0;+\infty[\longrightarrow X$ is a continuous function, then for every $t \in [0;+\infty[$*

$$\lim_{r\to 0^+} \frac{1}{r}\int_s^{s+r} R(t,h)f(h)dh = R(t,s)f(s).$$

Theorem 1.2. *[11] Let $A(t)$ be a closed and densely defined generator of a C_0-quasi-semigroup $\{R(t,s)\}_{t,s\geq 0}$ such that the resolvent $\mathcal{R}(\lambda, A(t)) = (\lambda I - A(t))^{-1}$ exists in $S = \{\lambda \in \mathbb{C} : \ -\theta \leq \arg(\lambda) \leq \theta$ with $\theta \in]\frac{\pi}{2}, \pi[\}$.*
If $\lambda \in \rho(A(t))$, then for all $s \geq 0$ we have

$$\mathcal{R}(\lambda, A(t))R(t,s) = R(t,s)\mathcal{R}(\lambda, A(t)).$$

In the semigroups theory, if A is an infinitesimal generator of a C_0-semigroup with domain $D(A)$, then A is a closed operator and $D(A)$ is dense in X. That is not always true for any C_0-quasi-semigroup, see [11].

In [8], [9], [10], [12], [13] the authors have studied the different spectra of the C_0-semigroup. In this paper, we will study C_0-quasi-semigroups, we will investigate the relationships between the different spectra of the C_0-quasi-semigroup and their generators, precisely for ascent, descent essential ascent and essential descent, upper and lower semi-Fredholm and semi-Browder spectra.

Throughout this work, we need the following definitions and notations:
Let T be a closed linear operator on X with domain $D(T)$ and $\mathcal{C}(X)$ the space of closed operators, we denote by $Rg(T)$, $N(T)$, $\rho(T)$ and $\sigma(T)$, respectively the range, the kernel, the resolvent and the spectrum of T, where $\sigma(T) = \mathbb{C}\backslash\rho(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not bijective}\}$. The function resolvent of T is defined for all $\lambda \in \rho(T)$ by $\mathcal{R}(\lambda, T) = (\lambda I - T)^{-1}$.
The ascent and descent of a closed operator T are defined respectively by,

$$asc(T) = \min\{k \in \mathbb{N} : N(T^k) = N(T^{k+1})\} ; \ des(T) = \min\{k \in \mathbb{N} : Rg(T^k) = Rg(T^{k+1})\}.$$

with the convention $inf(\varnothing) = \infty$.
The essential ascent and descent of a closed operator T are defined respectively by,

- $asc_e(T) = \min\{k \in \mathbb{N} : dim[N(T^{k+1})/N(T^k)] < \infty\}$

- $des_e(T) := \inf\{k \in \mathbb{N} : \dim Rg(T^k)/Rg(T^{k+1}) < \infty\}$.

The ascent, descent, essential ascent and essential descent spectra are defined by,

- $\sigma_{asc}(T) = \{\lambda \in \mathbb{C} \, : \, asc(\lambda I - T) = \infty\}$; $\sigma_{des}(T) = \{\lambda \in \mathbb{C} \, : \, des(\lambda I - T) = \infty\}$.
- $\sigma_{asc_e}(T) = \{\lambda \in \mathbb{C} \, : \, asc_e(\lambda I - T) = \infty\}$; $\sigma_{des_e}(T) = \{\lambda \in \mathbb{C} \, : \, des_e(\lambda I - T) = \infty\}$.

The sets of upper and lower semi-Fredholm and their spectra are defined respectively by,

- $\Phi_+(X) = \{T \in \mathcal{C}(X) \, : \, \alpha(T) := dim(N(T)) < \infty$ and $Rg(T)$ is closed$\}$,
  $\sigma_{e_+}(T) = \{\lambda \in \mathbb{C} \, : \, \lambda I - T \notin \Phi_+(X)\}$.
- $\Phi_-(X) = \{T \in \mathcal{C}(X) \, : \, \beta(T) := codim(Rg(T)) = dim(X/Rg(T)) < +\infty\}$,
  $\sigma_{e_-}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin \Phi_-(X)\}$.

An operator $T \in \mathcal{C}(X)$ is called semi-Fredholm, in symbol $T \in \Phi_\pm(X)$, if $T \in \Phi_+(X) \cup \Phi_-(X)$.
An operator $T \in \mathcal{C}(X)$ is called Fredholm, in symbol $T \in \Phi(X)$, if $T \in \Phi_+(X) \cap \Phi_-(X)$.
The essential and semi-Fredholm spectra are defined by,

- $\sigma_e(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin \Phi(X)\}$
- $\sigma_{e_\pm}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin \Phi_\pm(X)\}$

The sets of upper and lower semi-Browder and their spectra are defined respectively by,

- $Br_+(X) = \{T \in \Phi_+(X) \, : \, asc(T) < +\infty\}$; $\sigma_{Br_+}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin Br_+(X)\}$
- $Br_-(X) = \{T \in \Phi_-(X) \, : \, des(T) < +\infty\}$; $\sigma_{Br_-}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin Br_-(X)\}$

An operator $T \in \mathcal{C}(X)$ is called semi-Browder, in symbol $T \in Br_\pm(X)$, if $T \in Br_+(X) \cup Br_-(X)$.
An operator $T \in \mathcal{C}(X)$ is called Browder, in symbol $T \in Br(X)$, if $T \in Br_+(X) \cap Br_-(X)$.
The semi-Browder and Browder spectra are defined by,

- $\sigma_{Br_\pm}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin Br_\pm(X)\}$.
- $\sigma_{Br}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin Br(X)\}$.

2. Main results

For later use, we introduce the following bounded linear operator acting on X and depending on the parameters $\lambda \in \mathbb{C}$ and $t, s > 0$:

$$D_\lambda(t,s)x = \int_0^s e^{\lambda(s-h)} R(t-h,h)xdh \quad \text{for all } x \in X.$$

We start by the following theorem,

Theorem 2.1. *Let $A(t)$ be a generator of a C_0-quasi-semigroup $\{R(t,s)\}_{t,s\geq 0}$ such that $A(t)$ is closed and densely defined. Then for all $t \geq s > 0$ and all $\lambda \in \mathbb{C}$, we have*

1. *For all $x \in \mathcal{D}$,*
$$D_\lambda(t,s)(\lambda I - A(t))x = [e^{\lambda s} - R(t-s,s)]x,$$
2. *For all $x \in X$, we have $D_\lambda(t,s)x \in \mathcal{D}$ and*
$$(\lambda I - A(t))D_\lambda(t,s)x = [e^{\lambda s} - R(t-s,s)]x.$$

Proof. 1. By Theorem 1.1, we know that for all $h > 0$ and for all $x \in \mathcal{D}$,

$$\frac{\partial R(t-h,h)}{\partial h}x = A(t)R(t-h,h)x = R(t-h,h)A(t)x.$$

Therefore, we conclude that

$$\begin{aligned} D_\lambda(t,s)[A(t)x] &= \int_0^s e^{\lambda(s-h)}R(t-h,h)[A(t)x]dh \\ &= \int_0^s e^{\lambda(s-h)}\frac{\partial R(t-h,h)}{\partial h}xdh \\ &= \Big[e^{\lambda(s-h)}R(t-h,h)x\Big]_0^s + \lambda\int_0^s e^{\lambda(s-h)}R(t-h,h)xdh \\ &= R(t-s,s)x - e^{\lambda s}x + \lambda D_\lambda(t,s)x. \qquad (*) \end{aligned}$$

Finally, we obtain for all $x \in \mathcal{D}$

$$D_\lambda(t,s)(\lambda I - A(t))x = [e^{\lambda s} - R(t-s,s)]x.$$

2. Let $\mu \in \rho(A(t))$. From Theorem 1.2, we have for all $x \in X$

$$\mathcal{R}(\mu,A(t))R(t,s)x = R(t,s)\mathcal{R}(\mu,A(t))x.$$

Hence, for all $x \in X$ we conclude

$$\begin{aligned} \mathcal{R}(\mu,A(t))D_\lambda(t,s)x &= \mathcal{R}(\mu,A(t))\int_0^s e^{\lambda(s-h)}R(t-h,h)xdh \\ &= \int_0^s e^{\lambda(s-h)}\mathcal{R}(\mu,A(t))R(t-h,h)xdh \\ &= \int_0^s e^{\lambda(s-h)}R(t-h,h)\mathcal{R}(\mu,A(t))xdh \\ &= D_\lambda(t,s)\mathcal{R}(\mu,A(t))x. \end{aligned}$$

Therefore, we obtain for all $x \in X$

$$\begin{aligned} D_\lambda(t,s)x &= \int_0^s e^{\lambda(s-h)}R(t-h,h)xdh \\ &= \int_0^s e^{\lambda(s-h)}R(t-h,h)(\mu - A(t))\mathcal{R}(\mu,A(t))xdh \\ &= \mu\int_0^s e^{\lambda(s-h)}R(t-h,h)\mathcal{R}(\mu,A(t))xdh - \int_0^s e^{\lambda(s-h)}R(t-h,h)A(t)\mathcal{R}(\mu,A(t))xdh \\ &= \mu\int_0^s e^{\lambda(s-h)}\mathcal{R}(\mu,A(t))R(t-h,h)xdh - \int_0^s e^{\lambda(s-h)}R(t-h,h)A(t)\mathcal{R}(\mu,A(t))xdh \\ &= \mu\mathcal{R}(\mu,A(t))\int_0^s e^{\lambda(s-h)}R(t-h,h)xdh - \int_0^s e^{\lambda(s-h)}R(t-h,h)[A(t)\mathcal{R}(\mu,A(t))x]dh \\ &= \mu\mathcal{R}(\mu,A(t))D_\lambda(t,s)x - D_\lambda(t,s)[A(t)\mathcal{R}(\mu,A(t))x] \\ &\overset{(*)}{=} \mu\mathcal{R}(\mu,A(t))D_\lambda(t,s)x - \Big[R(t-s,s)\mathcal{R}(\mu,A(t))x - e^{\lambda s}\mathcal{R}(\mu,A(t))x + \lambda D_\lambda(t,s)\mathcal{R}(\mu,A(t))x\Big] \\ &= \mu\mathcal{R}(\mu,A(t))D_\lambda(t,s)x - \mathcal{R}(\mu,A(t))R(t-s,s)x + e^{\lambda s}\mathcal{R}(\mu,A(t))x - \lambda\mathcal{R}(\mu,A(t))D_\lambda(t,s)x \\ &= \mathcal{R}(\mu,A(t))\Big[\mu D_\lambda(t,s)x - R(t-s,s)x + e^{\lambda s}x - \lambda D_\lambda(t,s)x\Big]. \end{aligned}$$

Therefore, for all $x \in X$ we deduce $D_\lambda(t,s)x \in \mathcal{D}$ and we have

$$(\mu - A(t))D_\lambda(t,s)x = \mu D_\lambda(t,s)x - R(t-s,s)x + e^{\lambda s}x - \lambda D_\lambda(t,s)x.$$

Finally, if $\mu \longrightarrow \lambda$ we obtain for all $x \in X$,

$$(\lambda I - A(t))D_\lambda(t,s)x = [e^{\lambda s} - R(t-s,s)]x.$$

□

For $t \geq 0$, we fix $\mathcal{D}^0 = \mathcal{D}(A(t)^0) = X$, $A(t)^0 = I$, and for $n \in \mathbb{N}$ we define by recurrence:

$$\mathcal{D}^n = \mathcal{D}(A(t)^n) := \{x \in \mathcal{D}(A(t)^{n-1}) : A(t)^{n-1}x \in \mathcal{D}(A(t))\},$$

$$A(t)^n x = A(t)A(t)^{n-1}x \text{ pour } x\ in \mathcal{D}(A(t)^n),$$

We introduce :

$$X = D(A(t)^0) \supseteq D(A(t)) \supseteq D(A(t)^2) \supseteq ... \supseteq D(A(t)^n).$$

Corollary 2.2. *Let $A(t)$ be a generator of the C_0-quasi-semigroup $\{R(t,s)\}_{t,s\geq 0}$ such that $A(t)$ is closed and densely defined. Then for all $t \geq s > 0$ and all $\lambda \in \mathbb{C}$, we obtain*

1. *For all $x \in X$,*

$$(\lambda I - A(t))^n[D_\lambda(t,s)]^n x = [e^{\lambda s} - R(t-s,s)]^n x.$$

2. *For all $x \in \mathcal{D}^n$,*

$$[D_\lambda(t,s)]^n(\lambda I - [A(t)])^n x = [e^{\lambda s} - R(t-s,s)]^n x.$$

3. $N[\lambda I - A(t)] \subseteq N[e^{\lambda s} - R(t-s,s)]$.
4. $Rg[e^{\lambda s} - R(t-s,s)] \subseteq Rg[\lambda I - A(t)]$.
5. $N[\lambda I - A(t)]^n \subseteq N[e^{\lambda s} - R(t-s,s)]^n$.
6. $Rg[e^{\lambda s} - R(t-s,s)]^n \subseteq Rg[\lambda I - A(t)]^n$.
7. $Rg^\infty[e^{\lambda s} - R(t-s,s)] \subseteq Rg^\infty[\lambda I - A(t)]$.

Proof. It's automatic by Theorem 2.1. □

To obtain the results concerning the semi-Fredholm and semi-Browder spectra we need the following theorem.

Theorem 2.3. *Let $A(t)$ be a generator of the C_0-quasi-semigroup $\{R(t,s)\}_{t,s\geq 0}$ such that $A(t)$ is closed and densely defined. Then for all $t \geq s > 0$ and all $\lambda \in \mathbb{C}$, we have*

1. *$(\lambda I - A(t))L_\lambda(t,s) + \varphi_\lambda(s)D_\lambda(t,s) = I$, where I is identity operator, $L_\lambda(t,s) = \dfrac{1}{s}\displaystyle\int_0^s e^{-\lambda h}D_\lambda(t,h)dh$ and $\varphi_\lambda(s) = \frac{1}{s}e^{-\lambda s}$.*

   Moreover, the operators $L_\lambda(t,s)$, $D_\lambda(t,s)$ and $(\lambda I - A(t))$ are mutually commuting.
2. *For all $n \in \mathbb{N}^*$, there exists an operator $F_{\lambda,n}(t,s) \in \mathcal{B}(X)$ such that,*

$$(\lambda I - A(t))^n[L_\lambda(t,s)]^n + F_{\lambda,n}(t,s)D_\lambda(t,s) = I.$$

   Moreover, the operator $F_{\lambda,n}(t,s)$ is commute with each one of $D_\lambda(t,s)$ and $L_\lambda(t,s)$.
3. *For all $n \in \mathbb{N}^*$, there exists an operator $B_{\lambda,n}(t,s) \in \mathcal{B}(X)$ such that,*

$$(\lambda I - A(t))^n B_{\lambda,n}(t,s) + [F_{\lambda,n}(t,s)]^n[D_\lambda(t,s)]^n = I.$$

   Moreover, the operator $B_{\lambda,n}(t,s)$ is commute with each one of $D_\lambda(t,s)$ and $F_{\lambda,n}(t,s)$.

Proof. 1. Let $\mu \in \rho(A(t))$. By theorem 2.1, for all $x \in X$ we have $D_\lambda(t,h)x \in \mathcal{D}$ and hence, for all $t, s \geq 0$,

$$\begin{aligned} sL_\lambda(t,s)x &= \int_0^s e^{-\lambda h} D_\lambda(t,h)xdh \\ &= \int_0^s e^{-\lambda h} \mathcal{R}(\mu, A(t))(\mu - A(t))D_\lambda(t,h)xdh, \\ &= \mathcal{R}(\mu, A(t))[\mu \int_0^s e^{-\lambda h} D_\lambda(t,h)xdh - \int_0^s e^{-\lambda h} A(t)D_\lambda(t,h)xdh] \\ &= \mathcal{R}(\mu, A(t))[\mu s L_\lambda(t,s)x - \int_0^s e^{-\lambda h} A(t)D_\lambda(t,h)xdh] \end{aligned}$$

Therefore for all $x \in X$, we have $L_\lambda(t,s)x \in \mathcal{D}$ and

$$s(\mu - A(t))L_\lambda(t,s)x = \mu s L_\lambda(t,s)x - \int_0^s e^{-\lambda h} A(t)D_\lambda(t,h)xdh.$$

Thus

$$A(t)(sL_\lambda(t,s)x) = \int_0^s e^{-\lambda h} A(t)D_\lambda(t,h)xdh.$$

Hence, we conclude that

$$\begin{aligned} (\lambda I - A(t))(sL_\lambda(t,s)x) &= \lambda s L_\lambda(t,s)x - \int_0^s e^{-\lambda h} A(t)D_\lambda(t,h)xdh \\ &= \lambda s L_\lambda(t,s)x - \int_0^s e^{-\lambda h} \big[\lambda D_\lambda(t,h)x - e^{\lambda h}x + R(t-h,h)x\big]dh \ \ (\mathit{Theorem}\, 2.1) \\ &= \lambda s L_\lambda(t,s)x - \lambda \int_0^s e^{-\lambda h} D_\lambda(t,h)xdh + \int_0^s xdh - \int_0^s e^{-\lambda h} R(t-h,h)xdh \\ &= \lambda s L_\lambda(t,s)x - \lambda s L_\lambda(t,s)x + sx - e^{-\lambda s} \int_0^s e^{\lambda(s-h)} R(t-h,h)xdh \\ &= sx - e^{-\lambda s} D_\lambda(t,s)x \\ &= \big[s - s\varphi_\lambda(s)D_\lambda(t,s)\big]x, \end{aligned}$$

Therefore, we obtain $(\lambda I - A(t))L_\lambda(t,s) + \varphi_\lambda(s)D_\lambda(t,s) = I$.
And since the family $\{R(t,s)\}_{t,s\geq 0}$ is commutative, then for all $t > s > h \geq 0$, we have $D_\lambda(t,h)R(t-s,s) = R(t-s,s)D_\lambda(t,h)$.
Hence, for all $s, r, t \geq h \geq 0$, we have $D_\lambda(t,s)D_\lambda(t,r) = D_\lambda(t,r)D_\lambda(t,s)$.

Thus,

$$D_\lambda(t,s)L_\lambda(t,s) = L_\lambda(t,s)D_\lambda(t,s).$$

Since for all $x \in X$, $A(t)L_\lambda(t,s)x = \int_0^s e^{-\lambda h} A(t)D_\lambda(t,h)xdh$ and for all $x \in \mathcal{D}$, $A(t)D_\lambda(t,h)x = D_\lambda(t,h)A(t)x$, then we obtain for all $x \in \mathcal{D}$,

$$\begin{aligned} (\lambda I - A(t))L_\lambda(t,s)x &= \lambda L_\lambda(t,s)x - A(t)L_\lambda(t)x \\ &= \lambda L_\lambda(t,s)x - \int_0^s e^{-\lambda h} A(t)D_\lambda(t,h)xdh \\ &= \lambda L_\lambda(t,s)x - \int_0^s e^{-\lambda h} D_\lambda(t,h)A(t)xdh \\ &= \lambda L_\lambda(t,s)x - L_\lambda(t,s)A(t)x \\ &= L_\lambda(t,s)(\lambda I - A(t))x. \end{aligned}$$

2. For all $n \in \mathbb{N}^*$, we obtain

$$\begin{aligned}[(\lambda I - A(t))L_\lambda(t,s)]^n &= [I - \varphi_\lambda(s)D_\lambda(t,s)]^n \\ &= \sum_{i=0}^{n} C_n^i[-\varphi_\lambda(s)D_\lambda(t,s)]^i \\ &= I + \sum_{i=1}^{n} C_n^i[-\varphi_\lambda(s)D_\lambda(t,s)]^i \\ &= I - D_\lambda(t,s)\sum_{i=1}^{n} C_n^i[\varphi_\lambda(s)]^i[-D_\lambda(t,s)]^{i-1} \\ &= I - D_\lambda(t,s)F_{\lambda,n}(t,s),\end{aligned}$$

Where

$$F_{\lambda,n}(t,s) = \sum_{i=1}^{n} C_n^i[\varphi_\lambda(s)]^i[-D_\lambda(t,s)]^{i-1}.$$

Therefore, we have

$$(\lambda I - A(t))^n[L_\lambda(t,s)]^n + D_\lambda(t,s)F_{\lambda,n}(t,s) = I.$$

Finally, for commutativity, it's clear that $F_{\lambda,n}(t,s)$ commute with each one of $D_\lambda(t,s)$ and $L_\lambda(t,s)$ since the operators $L_\lambda(t,s)$, $D_\lambda(t,s)$ and $(\lambda I - A(t))$ are mutually commuting.

3. We have $D_\lambda(t,s)F_{\lambda,n}(t,s) = I - (\lambda I - A(t))^n[L_\lambda(t,s)]^n$, then for all $n \in \mathbb{N}^*$

$$\begin{aligned}[D_\lambda(t,s)F_{\lambda,n}(t,s)]^n &= \left[I - (\lambda I - A(t))^n[L_\lambda(t,s)]^n\right]^n \\ &= I - \sum_{i=1}^{n} C_n^i\left[(\lambda I - A(t))^n[L_\lambda(t,s)]^n\right]^i \\ &= I - (\lambda I - A(t))^n\sum_{i=1}^{n} C_n^i\left[(\lambda I - A(t))^{n(i-1)}[L_\lambda(t,s)]^{ni}\right. \\ &= I - (\lambda I - A(t))^n B_{\lambda,n}(t,s),\end{aligned}$$

Where $B_{\lambda,n}(t,s) = \sum_{i=1}^{n} C_n^i(\lambda I - A(t))^{n(i-1)}[L_\lambda(t,s)]^{ni}$. Hence, we obtain

$$[D_\lambda(t,s)]^n[F_{\lambda,n}(t,s)]^n + (\lambda I - A(t))^n B_{\lambda,n}(t,s) = I.$$

Finally, the commutativity is clear.

□

We start by this result.

Proposition 2.4. *Let $A(t)$ be a closed and densely defined generator of a C_0-quasi-semigroup $\{R(t,s)\}_{t,s\geq 0}$ on a Banach space X. If $Rg[e^{\lambda s} - R(t-s,s)]^p$ is closed, then $Rg[\lambda I - A(t)]^p$ is also closed.*

Proof. Let $(y_n)_{n\in\mathbb{N}} \subseteq X$ such that $y_n \to y \in X$ and there exists $(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}$ satisfying

$$(\lambda I - A(t))^p x_n = y_n.$$

By (3) of theorem 2.3, for all $n \in \mathbb{N}^*$, there exists $F_{\lambda,p}(t,s)$, $B_{\lambda,p}(t,s) \in \mathcal{B}(X)$ such that,

$$(\lambda I - A(t))^p B_{\lambda,p}(t,s) + [F_{\lambda,p}(t,s)]^p[D_\lambda(t,s)]^p = I.$$

Hence, we conclude that

$$\begin{aligned}[e^{\lambda s}-R(t-s,s)]^p[F_{\lambda,p}(t,s)]^p x_n &= D_\lambda(t,s)^p(\lambda I-A(t))^p[F_{\lambda,p}(t,s)]^p x_n\ , \quad (\textit{by theorem } 2.1)\\ &= [F_{\lambda,p}(t,s)]^p D_\lambda(t,s)^p(\lambda I-A(t))^p x_n\\ &= [F_{\lambda,p}(t,s)]^p D_\lambda(t,s)^p y_n\\ &= y_n-(\lambda I-A(t))^p B_{\lambda,p}(t,s)y_n.\end{aligned}$$

Thus,

$$y_n-(\lambda I-A(t))^p B_{\lambda,p}(t,s)y_n\in Rg[e^{\lambda s}-R(t-s,s)]^p.$$

Therefore, $Rg[e^{\lambda s}-R(t-s,s)]^p$ is closed, $B_{\lambda,p}(t,s)$ is bounded linear and $y_n-(\lambda I-A(t))^p B_{\lambda,p}(t,s)y_n$ converges to $y-(\lambda I-A(t))^p B_{\lambda,p}(t,s)y$, we conclude that

$$y-(\lambda I-A(t))^p B_{\lambda,p}(t,s)y\in Rg[e^{\lambda s}-R(t-s,s)]^p.$$

Then there exists $z\in X$ such that

$$[e^{\lambda s}-R(t-s,s)]^p z=y-(\lambda I-A(t))^p B_{\lambda,p}(t,s)y.$$

Hence, we have

$$\begin{aligned}y &= [e^{\lambda s}-R(t-s,s)]^p z+(\lambda I-A(t))B_{\lambda,p}(t,s))y;\\ &= (\lambda I-A(t))^p D_\lambda(t,s)^p z+(\lambda I-A(t))^p B_{\lambda,p}(t,s)y;\\ &= (\lambda I-A(t))^p[D_\lambda(t,s)^p z+B_{\lambda,p}(t,s)y].\end{aligned}$$

Finally, we obtain

$$y\in Rg(\lambda I-A(t))^p.$$

□

Theorem 2.5. *Let $A(t)$ be a closed and densely defined generator of a C_0-quasi-semigroup $\{R(t,s)\}_{t,s\geq 0}$ on a Banach space X. For all $t\geq s>0$, we have*

1. $e^{\sigma_{e_+}(A(t))s}\subseteq\sigma_{e_+}(R(t-s,s))\backslash\{0\}$.
2. $e^{\sigma_{e_-}(A(t))s}\subseteq\sigma_{e_-}(R(t-s,s))\backslash\{0\}$.
3. $e^{\sigma_{e_\pm}(A(t))s}\subseteq\sigma_{e_\pm}(R(t-s,s))\backslash\{0\}$.

Proof. 1. Suppose that $e^{\lambda s}\notin\sigma_{e_+}(R(t-s,s))$, then there exists $n\in\mathbb{N}$ such that $\alpha[e^{\lambda s}-R(t-s,s)]=n$ and $Rg[e^{\lambda s}-R(t-s,s)]$ is closed.
By corollary 2.2, we have

$$N(\lambda I-A(t))\subset N[e^{\lambda s}-R(t-s,s)],$$

then

$$\alpha(\lambda I-A(t))\leq n.$$

On the other hand, from proposition 2.4, we deduce that $Rg(\lambda I-A(t))$ is closed.
Therefore $\lambda I-A(t)\in\Phi_+(\mathcal{D})$, Then$\lambda\notin\sigma_{e_+}(A(t))$.

2. Suppose that $e^{\lambda s}\notin\sigma_{e_-}(R(t-s,s))$, then there exist $n\in\mathbb{N}$ such that $\beta[e^{\lambda s}-R(t-s,s)]=n$.
By corollary 2.2, we obtain

$$Rg[e^{\lambda s}-R(t-s,s)]\subseteq Rg(\lambda I-A(t)),$$

then $\beta(\lambda I-A(t))\leq n$ and hence, $\lambda\notin\sigma_{e_-}(A(t))$

3. It is automatic by the previous assertions of this theorem.

□

Remark 2.6. *Note that the inclusion* $\{e^{\lambda s}, \lambda \in \sigma_*(A(t))\} \subseteq \sigma_*(R(t-s,s))\backslash\{0\}$ *, where* $\sigma_* \in \{\sigma_{e_+}, \sigma_{e_-}, \sigma_{e_\pm}\}$ *is strict as shown in the following example.*

Example 2.7. *Let* $R(t,s) = T(s)$ *where* $\{T(s)\}_{s\geq 0}$ *is the translation group on the space* $C_{2\pi}(\mathbb{R})$ *of all* 2π *periodic continuous functions on* $\mathbb{R}$ *and denote its generator by* A *(see* [10, Paragraph I.4.15]). From [10, Examples 2.6.iv] we have, $\sigma(A(t)) = \sigma(A) = i\mathbb{Z}$, then $e^{\sigma(A(t))s}$ is at most countable, therefore $e^{\sigma_*(A(t))s}$ are also.
The spectra of the operators $T(s)$ are always contained in $\Gamma = \{z \in \mathbb{C} : |z| = 1\}$ and contain the eigenvalues e^{iks} for $k \in \mathbb{Z}$. Since $\sigma(T(s))$ is closed, it follows from [10, Theorem IV.3.16] that $\sigma(T(s)) = \Gamma$ whenever $s/2\pi \notin \mathbb{Q}$, then $\sigma(T(s))$ is not countable, so $\sigma_*(R(t-s,s))\backslash\{0\}$ are also.

To obtain a results for ascent and descent spectra we need the following proposition.

Proposition 2.8. *Let* $A(t)$ *be a closed and densely defined generator of a* C_0-*quasi-semigroup* $\{R(t,s)\}_{t,s\geq 0}$ *on a Banach space* X*. For all* $t \geq s > 0$*, we have*

1. *If* $des[e^{\lambda s} - R(t-s,s)] = n$*, then* $des[\lambda I - A(t)] \leq n$*.*
2. *If* $asc[e^{\lambda s} - R(t-s,s)] = n$*, then* $asc[\lambda I - A(t)] \leq n$*.*

Proof.

1. Let $y \in Rg[\lambda I - A(t)]^n$, then there exists $x \in \mathcal{D}^n$ (domain of $A(t)^n$) satisfying,

$$(\lambda I - A(t))^n x = y.$$

Since $des[e^{\lambda s} - R(t-s,s)] = n$, therefore $Rg[e^{\lambda s} - R(t-s,s)]^n = Rg[e^{\lambda s} - R(t-s,s)]^{n+1}$. Hence, there exists $z \in X$ such that

$$[e^{\lambda s} - R(t-s,s)]^n x = [e^{\lambda s} - R(t-s,s)]^{n+1} z.$$

On the other hand, by theorem 2.3, we have,

$$(\lambda I - A(t))^n B_{\lambda,n}(t,s) + [F_{\lambda,n}(t,s)]^n [D_\lambda(t,s)]^n = I,$$

Thus we have,

$$\begin{aligned}
y &= (\lambda I - A(t))^n x \\
&= (\lambda I - A(t))^n [(\lambda I - A(t))^n B_{\lambda,n}(t,s) + [F_{\lambda,n}(t,s)]^n [D_\lambda(t,s)]^n] x \\
&= (\lambda I - A(t))^n (\lambda I - A(t))^n B_{\lambda,n}(t,s)x + [F_{\lambda,n}(t,s)]^n (\lambda I - A(t))^n [D_\lambda(t,s)^n] x \\
&= (\lambda I - A(t))^{2n} B_{\lambda,n}(t,s)x + [F_{\lambda,n}(t,s)]^n [e^{\lambda s} - R(t-s,s)]^n x \\
&= (\lambda I - A(t))^{2n} B_{\lambda,n}(t,s)x + [F_{\lambda,n}(t,s)]^n [[e^{\lambda s} - R(t-s,s)]^{n+1} z] \\
&= (\lambda I - A(t))^{2n} B_{\lambda,n}(t,s)x + [F_{\lambda,n}(t,s)]^n [(\lambda I - A(t))^{n+1} [D_\lambda(t,s)]^{n+1} z] \\
&= (\lambda I - A(t))^{n+1} [(\lambda I - A(t))^{n-1} B_{\lambda,n}(t,s)x + [F_{\lambda,n}(t,s)]^n [D_\lambda(t,s)]^{n+1} z].
\end{aligned}$$

Therefore, we conclude that $y \in Rg[\lambda I - A(t)]^{n+1}$ and hence,

$$Rg[\lambda I - A(t)]^n = Rg[\lambda I - A(t)]^{n+1}.$$

Finally, we conclude that

$$des(\lambda I - A(t)) \leq n.$$

2. Let $x \in N(\lambda I - A(t))^{n+1}$ and we suppose that $asc[e^{\lambda s} - R(t-s,s)] = n$, then we obtain

$$N[e^{\lambda s} - R(t-s,s)]^n = N[e^{\lambda s} - R(t-s,s)]^{n+1}.$$

From corollary 2.2, we have

$$N(\lambda I - A(t))^{n+1} \subseteq N[e^{\lambda s} - R(t-s,s)]^{n+1},$$

hence

$$x \in N[e^{\lambda s} - R(t-s,s)]^n.$$

Thus we have,

$$\begin{aligned}(\lambda I - A(t))^n x &= (\lambda I - A(t))^n[(\lambda I - A(t))^n B_{\lambda,n}(t,s) + [F_{\lambda,n}(t,s)]^n[D_\lambda(t,s)]^n]x\\ &= (\lambda I - A(t))^{2n} B_{\lambda,n}(t,s)x + [F_{\lambda,n}(t,s)]^n(\lambda I - A(t))^n[D_\lambda(t,s)]^n x\\ &= B_{\lambda,n}(t,s)(\lambda I - A(t))^{n-1}(\lambda I - A(t))^{n+1}x + [F_{\lambda,n}(t,s)]^n[e^{\lambda s} - R(t-s,s)]^n x\\ &= 0.\end{aligned}$$

Therefore, we obtain $x \in N(\lambda I - A(t))^n$ and hence

$$asc(\lambda I - A(t)) \leq n.$$

□

Theorem 2.9. *Let $A(t)$ be a closed and densely defined generator of a C_0-quasi-semigroup $\{R(t,s)\}_{t,s\geq 0}$ on a Banach space X. $t \geq s > 0$, we have*

1. $e^{\sigma_{asc}(A(t))s} \subseteq \sigma_{asc}(R(t-s,s))\backslash\{0\}$.
2. $e^{\sigma_{des}(A(t))s} \subseteq \sigma_{des}(R(t-s,s))\backslash\{0\}$.

Proof. Immediately comes from proposition 2.8. □

The following theorem examines the semi-Browder spectrum.

Theorem 2.10. *Let $A(t)$ be a closed and densely defined generator of a C_0-quasi-semigroup $\{R(t,s)\}_{t,s\geq 0}$ on a Banach space X. For all $\lambda \in \mathbb{C}$ and all $t \geq s > 0$, we have*

1. $e^{\sigma_{Br_+}(A(t))s} \subseteq \sigma_{Br_+}(R(t-s,s))\backslash\{0\}$.
2. $e^{\sigma_{Br_-}(A(t))s} \subseteq \sigma_{Br_-}(R(t-s,s))\backslash\{0\}$.
3. $e^{\sigma_{Br_\pm}(A(t))s} \subseteq \sigma_{Br_\pm}(R(t-s,s))\backslash\{0\}$.

Proof. 1. Suppose that $e^{\lambda s} \notin \sigma_{Br_+}(R(t-s,s))$, then there exist $n, m \in \mathbb{N}$ such that $\alpha[e^{\lambda s} - R(t-s,s)] = m$, $Rg[e^{\lambda s} - R(t-s,s)]$ is closed and $asc[e^{\lambda s} - R(t-s,s)] = n$. From corollary 2.2 and Propositions 2.4 and 2.8, we obtain

$$\alpha(\lambda I - A(t)) \leq m,\ Rg(\lambda I - A(t)) \text{ is closed and } asc(\lambda I - A(t)) \leq n.$$

Therefore $\lambda I - A(t) \in \Phi_+(\mathcal{D})$ and $asc(\lambda I - A(t)) < +\infty$ and hence, $\lambda \notin \sigma_{Br_+}(A(t))$.

2. Suppose that $e^{\lambda s} \notin \sigma_{Br_-}(R(t-s,s))$, then there exist $n, m \in \mathbb{N}$ such that $\beta[e^{\lambda s} - R(t-s,s)] = m$ and $des[e^{\lambda s} - R(t-s,s)] = n$. By corollary 2.2 and Proposition 2.8, we obtain

$$\beta(\lambda I - A(t)) \leq m \text{ and } des(\lambda I - A(t)) \leq n.$$

Therefore $\lambda I - A(t) \in \Phi_-(\mathcal{D})$ and $des(\lambda I - A(t)) < +\infty$ and hence, $\lambda \notin \sigma_{Br_-}(A(t))$.

3. It is automatic by the previous assertions of this theorem and proposition 2.4.

□

Proposition 2.11. *Let $A(t)$ be a closed and densely defined generator of a C_0-quasi-semigroup $\{R(t,s)\}_{t,s\geq 0}$ on a Banach space X. For all $t \geq s > 0$, we have*

1. *If $des_e[e^{\lambda s} - R(t-s,s)] = n$, then $des_e[\lambda I - A(t)] \leq n$.*
2. *If $asc_e[e^{\lambda s} - R(t-s,s)] = n$, then $asc_e[\lambda I - A(t)] \leq n$.*

Proof. 1. Suppose that $des_e[e^{\lambda s} - R(t-s,s)] = n$, Since $Rg[e^{\lambda s} - R(t-s,s)]^n \subseteq Rg(\lambda I - A(t))^n$ we define the linear surjective application ϕ by

$$\begin{aligned} \phi : Rg(\lambda I - A(t))^n &\rightarrow Rg[e^{\lambda s} - R(t-s,s)]^n / Rg[e^{\lambda s} - R(t-s,s)]^{n+1}, \\ y = (\lambda I - A(t))^n x &\mapsto [e^{\lambda s} - R(t-s,s)]^n x + Rg[e^{\lambda s} - R(t-s,s)]^{n+1}. \end{aligned}$$

Thus, by isomorphism Theorem, we obtain

$$Rg(\lambda I - A(t))^n / N(\phi) \simeq Rg[e^{\lambda s} - R(t-s,s)]^n / Rg[e^{\lambda s} - R(t-s,s)]^{n+1}.$$

Therefore

$$dim(Rg(\lambda I - A(t))^n / N(\phi)) = des_e[e^{\lambda s} - R(t-s,s)] = n.$$

And since $N(\phi) \subseteq Rg[e^{\lambda s} - R(t-s,s)]^{n+1} \subseteq Rg(\lambda I - A(t))^{n+1}$, then

$$Rg(\lambda I - A(t))^n / Rg(\lambda I - A(t))^{n+1} \subseteq Rg(\lambda I - A(t))^n / N(\phi).$$

Then, $dim(Rg(\lambda I - A(t))^n / R(\lambda I - A(t))^{n+1}) \leq dim(Rg(\lambda I - A(t))^n / N(\phi)) = n.$
Finally, we obtain that $des_e(\lambda I - A(t)) \leq n$

2. Suppose that

$$asc_e[e^{\lambda s} - R(t-s,s)] = n.$$

And since $N(\lambda I - A)^{n+1} \subseteq N[e^{\lambda s} - R(t-s,s)]^{n+1}$, we define the linear application ψ by

$$\begin{aligned} \psi : N(\lambda I - A(t))^{n+1} &\rightarrow N[e^{\lambda s} - R(t-s,s)]^{n+1} / N[e^{\lambda s} - R(t-s,s)]^n, \\ x &\mapsto x + N[e^{\lambda s} - R(t-s,s)]^n. \end{aligned}$$

Thus, by isomorphism Theorem, we obtain

$$N(\lambda I - A(t))^{n+1} / N(\psi) \simeq Rg(\psi) \subseteq N[e^{\lambda s} - R(t-s,s)]^{n+1} / N[e^{\lambda s} - R(t-s,s)]^n.$$

Therefore

$$dim(N(\lambda I - A)^{n+1} / N(\psi)) \leq asc_e[e^{\lambda s} - R(t-s,s)] = n.$$

And since $N(\psi) \subseteq N[e^{\lambda s} - R(t-s,s)]^n \subseteq Rg(\lambda I - A(t))^n$, then

$$N(\lambda I - A(t))^{n+1} / N(\lambda I - A(t))^n \subseteq N(\lambda I - A(t))^{n+1} / N(\psi).$$

Finally, we obtain $asc_e(\lambda I - A)) \leq n.$

□

We will discuss in the following result the essential ascent and descent spectrum.

Theorem 2.12. *Let $A(t)$ be a closed and densely defined generator of a C_0-quasi-semigroup $\{R(t,s)\}_{t,s\geq 0}$ on a Banach space X. For all $t \geq s > 0$, we have*

1. $e^{\sigma_{asc_e}(A(t))s} \subseteq \sigma_{asc_e}(R(t-s,s))\backslash\{0\}$.

2. $e^{\sigma_{dsc_e}(A(t))s} \subseteq \sigma_{dsc_e}(R(t-s,s))\backslash\{0\}$.

Proof. 1. Suppose that , $e^{\lambda s} \notin \sigma_{asc_e}(R(t-s,s))$. Then there exists $n \in \mathbb{N}$ satisfying

$$asc_e[e^{\lambda s} - R(t-s,s)] = n.$$

Therefore, by Proposition 2.11, we obtain $asc_e[\lambda I - A(t)] \leq n$ and hence

$$\lambda \notin \sigma_{asc_e}(A(t)).$$

2. Suppose that

$$e^{\lambda s} \notin \sigma_{des_e}(R(t-s,s)).$$

Then there exists $n \in \mathbb{N}$ satisfying

$$des_e[e^{\lambda s} - R(t-s,s)] = n.$$

Therefore, by Proposition 2.11, we obtain $des_e[\lambda I - A(t)] \leq n$ and hence $\lambda \notin \sigma_{des_e}(A(t))$.

□

Remark 2.13. *The inclusions of the previous theorem 2.4 and 2.6 is strict. Because according to example 2.7, the spectra $e^{\sigma_{asc}(A(t))s}$, $e^{\sigma_{des}(A(t))s}$, $e^{\sigma_{asc_e}(A(t))s}$ and $e^{\sigma_{des_e}(A(t))s}$ are at most countable, but according to corollary 2.10, 1.8 in preprints [6],[7] the spectra $\sigma_{asc}(R(t-s,s))$, $\sigma_{des}(R(t-s,s))$, $\sigma_{asc_e}(R(t-s,s))$, and $\sigma_{des_e}(R(t-s,s))$ are not countable.*

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