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On the Weakly Nilpotent Graph of a Commutative Semiring *

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ABSTRACT: Let S be a commutative semiring with unity. The weakly nilpotent graph of S, denoted by $\Gamma_w(S)$ is defined as an undirected simple graph whose vertices are S^* and two distinct vertices x and y are adjacent if and only if $xy \in N(S)^*$, where $S^* = S\setminus \{0\}$ and $N(S)^*$ is the set of all non-zero nilpotent elements of S. In this paper, we determine the diameter of weakly nilpotent graph of an Artinian semiring. We prove that if $\Gamma_w(S)$ is a forest, then $\Gamma_w(S)$ is a union of a star and some isolated vertices. We determine the clique number, chromatic number and independence number of $\Gamma_w(S)$. Among other results, we show that for an Artinian semiring S, $\Gamma_w(S)$ is not a disjoint union of cycles or a unicyclic graph. For Artinian semirings, we determine $\operatorname{diam}(\overline{\Gamma_w(S)})$, where $\overline{\Gamma_w(S)}$ is the complement of the weakly nilpotent graph of S. Finally, we characterize all commutative semirings S for which $\overline{\Gamma_w(S)}$ is a cycle.

Key Words: Weakly nilpotent graph, diameter, girth, commutative semiring.

Contents

1. Introduction

The study of graphs associated to algebraic structures has become an exciting research topic in the last two decades, leading to many fascinating results and questions. Many fundamental papers devoted to graphs assigned to rings and modules have appeared recently, for instance see [1-5, 10-11, 13] etc. The study of graph-theoretic aspects of commutative semirings is also one of the fascinating research areas now a days. There are many research papers on assigning a graph to a commutative semiring, for instance, see [7, 8], etc. Recently, Khojasteh and Nikmehr [13] have introduced the weakly nilpotent graph of a commutative ring and studied its various properties as well as its complement. Being motivated by this work, in this paper we analogously introduce the weakly nilpotent graph of a commutative semiring and attempt to study some of its properties under semiring theoretic settings. Our main goal is to study the connection between the algebraic properties of a semiring and the graph-theoretic properties of the graph associated with it.

A semiring is a non-empty set S together with two binary operations addition and multiplication denoted by "+" and "." respectively, satisfying the following properties:

1. $(S, +)$ is a commutative monoid with identity element 0.
2. $(S, .)$ is a monoid with identity element $1 \neq 0$.
3. $a(b+c) = ab + ac$ and $(b+c)a = ba + ca$ for all $a, b, c \in S$.

We simply denote a semiring by S. The semiring S is said to be a commutative semiring if the semigroup $(S, .)$ is a commutative semigroup and monoid if $1 \in S$ such that $a.1 = 1.a = a$ for all $a \in S$, then we say the semiring is a semiring with unity. An element a of a semiring S is called nilpotent if $x^n = 0$ for some

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$n > 0$ and n is called index of nilpotency of x. S is called an Artinian semiring if and only if S satisfies the descending chain condition for ideals of S. We denote the set of unit elements of S, the set of zero divisors of S, the set of nilpotent elements of S by $U(S)$, $Z(S)$, and $N(S)$. If S has a unique maximal ideal M, then S is said to be local semiring and it is denoted by (S, M). A semiring S is said to be a reduced semiring if $N(S) = 0$. Any other undefined terminology can be found in [9].

Let G be a graph with vertex set $V(G)$. A path from x to y is a series of adjacent vertices $x—x_1—x_2—......—x_n—y$. For $x, y \in V(G)$ with $x \neq y$, $d(x, y)$ denotes the length of the shortage path from x to y; otherwise $d(x, y) = \infty$. The diameter of G is defined as $diam(G) = sup\{d(x, y) \mid x \text{ and } y \text{ are vertices of } G\}$. For any $x \in V(G)$, $d(x)$ denotes the number of edges incident with x, called the degree of x. A cycle is a path that begins and ends at the same vertex in which no edge is repeated and all vertices other than the starting and ending vertex are distinct. We use C_n to denote the cycle with n vertices, where $n \geq 3$. We denote the complete graph with n vertices by K_n. If a graph G has a cycle, then the length of the shortage cycle is called girth of G and it is denoted by $gr(G)$; otherwise $gr(G) = \infty$. A bipartite graph is a graph whose vertices can be partitioned into two disjoint sets U and V such that every edge connects a vertex in U to one in V. A complete bipartite graph is a bipartite graph in which every vertex of one part is joined to every vertex of the other part. We denote by $K_{m,n}$ the complete bipartite graph, with part size m and n. The star graph is denoted by $K_{1,n}$, for a positive integer n. We say that a graph G is totally disconnected if no two vertices of G are adjacent. The disjoint union of graphs G_1 and G_2, which is denoted by $G_1 \cup G_2$, where G_1 and G_2 are two vertex-disjoint graphs, is a graph with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. A unicyclic graph is a connected graph with a unique cycle, or we can regard a unicyclic graph as a cycle attached with each vertex a (rooted) tree. A clique of a graph is a complete subgraph and the number of vertices in the largest clique of graph G, denoted by $\omega(G)$, is called the clique number of G. The minimum number of colors that can be assigned to the vertices of G in such a way that every two adjacent vertices have different colors is called the chromatic number and it is denoted by $\chi(G)$. Any undefined terminology can be obtained in [6] and [12].

Throughout this paper, S is a commutative semiring with unity. The weakly nilpotent graph of S denoted by $\Gamma_w(S)$ is defined to be the undirected simple graph with the vertex set $S^* = S \setminus \{0\}$ and two distinct vertices x and y are adjacent if and only if $xy \in N(S)^*$, where $N(S)^*$ is the set of non-zero nilpotent elements of S. We think that the weakly nilpotent graph of a semiring helps us to study the algebraic properties of semiring using graph-theoretical tools. Now, we consider the complement of the weakly nilpotent graph of S, denoted by $\overline{\Gamma_w(S)}$. For any two distinct vertices $x, y \in S^*$, x is adjacent to y if and only if $xy \notin N(S)^*$. Obviously, the usual zero divisor graph is a subgraph of $\overline{\Gamma_w(S)}$.

In section 2, we determine the diameter of weakly nilpotent graph of an Artinian semiring. We prove that if $\Gamma_w(S)$ is forest, then $\Gamma_w(S)$ is a union of a star and isolated vertices. We study the clique number, the chromatic number of $\Gamma_w(S)$. Among other results, we show that for an Artinian semiring S, $\overline{\Gamma_w(S)}$ is not a disjoint union of cycles or a unicyclic graph. In section 3, we determine $diam(\overline{\Gamma_w(S)})$, where S is an Artinian semiring. We characterize all commutative semirings S for which $\overline{\Gamma_w(S)}$ is a cycle.

2. On the structure and properties of the weakly nilpotent graph of a commutative semiring

In this section, we will focus on the weakly nilpotent graph of an Artinian semiring. Here we prove the total disconnectedness and connectedness property of S. We also prove that if Γ_w(S) has no isolated vertex, then diam(Γ_w(S)) ≤ 4, where S is an Artinian semiring. It is shown that if Γ_w(S) is a forest, then Γ_w(S) is a union of a star and some isolated vertices.
As we mention in the introduction, the weakly nilpotent graph of a commutative semiring S denoted by $\Gamma_w(S)$ is defined to be the undirected simple graph with the vertex set S^* and two distinct vertices x and y are adjacent if and only if $xy \in N(S)^*$.

Example 2.1. *The commutative semirings $\mathbb{Z}_0 = \mathbb{Z}^+ \cup \{0\}$ and $\mathbb{Q}_0 = \mathbb{Q}^+ \cup \{0\}$ with unity, ordinary* *addition and multiplication have no nontrivial nilpotent and so $\Gamma_w(\mathbb{Z}_0)$ and $\Gamma_w(\mathbb{Q}_0)$ are empty graph. Also, for the commutative semirings $\mathbb{Z}_n$ (n being prime or square free integer) the graphs $\Gamma_w(\mathbb{Z}_n)$ are empty.*

Example 2.2. *In the commutative semiring $\mathbb{Z}_4$, the nontrivial nilpotent is $\{2\}$. So $\Gamma_w(\mathbb{Z}_4)$ is the following graph.*

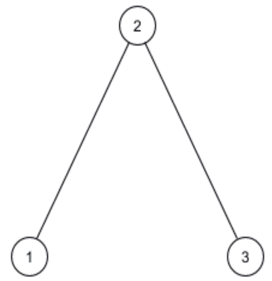

Figure 1: $\Gamma_w(\mathbb{Z}_4)$

Example 2.3. *In the commutative semiring $\mathbb{Z}_8$, the nontrivial nilpotents are $\{2, 4, 6\}$. So $\Gamma_w(\mathbb{Z}_8)$ is the following graph.*

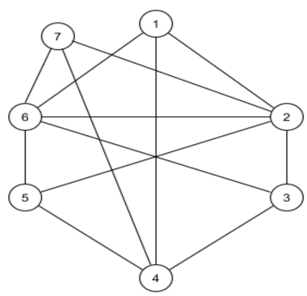

Figure 2: $\Gamma_w(\mathbb{Z}_8)$

Example 2.4. *In the commutative semiring $\mathbb{Z}_{12}$, the nontrivial nilpotent is $\{6\}$. So $\Gamma_w(\mathbb{Z}_{12})$ is the following graph.*

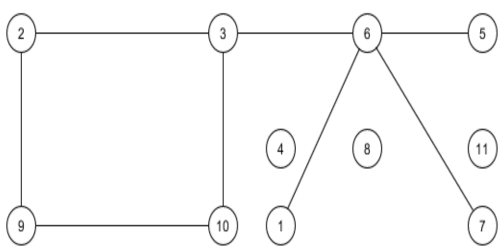

Figure 3: $\Gamma_w(\mathbb{Z}_8)$

Example 2.5. *In the commutative semiring $\mathbb{Z}_{16}$, the nontrivial nilpotent are $\{2, 4, 6, 8, 10, 12, 14\}$. So $\Gamma_w(\mathbb{Z}_{16})$ is the following graph.*

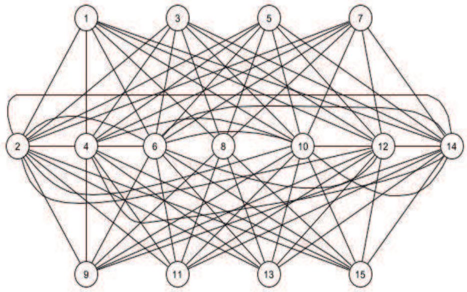

Figure 4: $\Gamma_w(\mathbb{Z}_{16})$

Proposition 2.1. *Let S be a commutative semiring with unity. Then the graph $\Gamma_w(S)$ is totally disconnected if and only if S is a reduced semiring.*

Proof. Let S be a commutative semiring with unity. If S is reduced semiring, then $N(S)^* = \phi$, which yields $\Gamma_w(S)$ is totally disconnected.
Conversely, assume that $\Gamma_w(S)$ is totally disconnected and $N(S)^* \neq \phi$, then every elements of $N(S)^*$ are adjacent to 1, a contradiction. □

Proposition 2.2. *Let S be a commutative Artinian semiring. Then $\Gamma_w(S)$ is a complete graph if and only if $S \cong \mathbb{Z}_2$.*

Proof. One side is clear. For other side, assume that $\Gamma_w(S)$ is a complete graph. Hence, $|U(S)| = 1$. Since, S is an Artinian semiring, so $S \cong \mathbb{Z}_2^r$, for some positive integer r. Let e_i be the $1 \times n$ vector whose i-th component is 1 and other components are 0. If $r \geq 2$ then e_1 is not adjacent to e_2 a contradiction. Therefore, $S \cong \mathbb{Z}_2$. □

Proposition 2.3. *Let (S, M) be a local semiring and $M \neq 0$. If $M = N(S)$, then $diam(\Gamma_w(S)) = 2$.*

Proof. Obviously, every element of $N(S)^*$ is adjacent to each element of $U(S)$. This shows that $\Gamma_w(S)$ is connected graph and $diam(\Gamma_w(S)) \leq 2$. If $U(S) = \{1\}$, then $M = 0$. Since, $1 + M \subseteq U(S)$, a contradiction. Therefore, $|U(S)| \geq 2$. Let $u, v \in U(S)$. So, u and v are adjacent to every element of $N(S)^*$. Since, u and v are not adjacent and $diam(\Gamma_w(S)) \leq 2$. Thus, $diam(\Gamma_w(S)) = 2$. □

Proposition 2.4. *Let S be a commutative Artinian semiring. If $\Gamma_w(S)$ has not any isolated vertex, then $diam(\Gamma_w(S)) \leq 4$.*

Proof. Let S be a commutative Artinian semiring. We know that $S \cong \Pi_{i=1}^{n} S_i$, where $n \geq 1$ and (S_i, M_i) is a local semiring for every i, $1 \leq i \leq n$. Let e_i be the $1 \times n$ vector whose i-th component is 1 and other components are zero. If $M_1 = 0$, then e_1 is an isolated vertex, a contradiction. Therefore $M_1 \neq 0$. Similarly, $M_i \neq 0$ for every i, $1 \leq i \leq n$. If $n = 1$ then by Proposition 2.3 $diam(\Gamma_w(S)) = 2$. Therefore we can assume that $n \geq 2$. Let $a = \Sigma_{i=1}^{n} a_i e_i$, $b = \Sigma_{i=1}^{n} b_i e_i \in S^*$ then we have the following three cases.

Case-1: Let $a, b \in U(S)$. Then we have $a — x — b$ where $x \in N(S)^*$. Hence, $d(a, b) = 2$.

Case-2: Let $a \in U(S)$ and $b \notin U(S)$. If $b_i \in M_i$ for every i, $1 \leqslant i \leqslant n$, then a is adjacent to b and $d(a, b) = 1$, otherwise suppose that $J = \{i \mid 1 \leqslant i \leqslant n,\ b_i \in \mathrm{U}(S_i)\}$. Let $x = \sum_{i=1}^{n} r_i e_i$ and $y = \sum_{i \notin J} e_i + \sum_{i \in J} r_i e_i$ for $r_i \in M_i^*$. Then $a — x — y — b$ is a path between a and b. Therefore, $d(a, b) = 3$.

Case-3: Let $a, b \notin U(S)$. If $a_i, b_i \in M_i$ for every i, $1 \leqslant i \leqslant n$. Then $\sum_{i=1}^{n} e_i$ is adjacent to a and b. Therefore $d(a, b) \leq 2$. Otherwise, let $I = \{i \mid 1 \leq i \leq n,\ a_i \in U(S_i)\}$ and $J = \{i \mid 1 \leq i \leq n, b_i \in U(S_i)\}$.

If $x = \sum_{i \notin I} e_i + \sum_{i \in I} r_i e_i$, $y = \sum_{i=1}^{n} r_i e_i$ and $w = \sum_{i \notin J} e_i + \sum_{i \in J} r_i e_i$, for $r_i \in m_i^*$, then $a — x — y — w — b$ is path between a and b. Therefore, $d(a, b) \leq 4$. Thus, $diam(\Gamma_w(S) \leq 4$. □

Proposition 2.5. *Let $n = p_1^{k_1} \times .... \times p_s^{k_s}$, where p_i is a prime number and k_i is a positive integer. Then $\Gamma_w(\mathbb{Z}_n)$ is a connected graph if and only if $k_i \geq 2$ for every i, $1 \leq i \leq s$. Moreover if $\Gamma_w(\mathbb{Z}_n)$ is a connected graph then $diam(\Gamma_w(\mathbb{Z}_n)) = 2$.*

Proof. Assume that $\Gamma_w(\mathbb{Z}_n)$ is a connected graph. If $k_1 = 1$, then by Proposition 2.1 $s \geq 2$. We show that $a = p_2^{k_2} \times .... \times p_s^{k}$ is an isolated vertex. To see this we note that if a is adjacent to b for some $b \in \mathbb{Z}_n^*$, then since $ab \neq 0$ and $a \notin p_1\mathbb{Z}$, we conclude that $b \notin p_1\mathbb{Z}$. On the other hand, since $ab \in N(S)^*$, $b \in p_1\mathbb{Z}$, a contradiction. Therefore $k_1 \geqslant 2$. Similarly, $k_i \geqslant 2$, for every i, $1 \leqslant i \leqslant s$. We show that $diam(\Gamma_w(\mathbb{Z}_n)) = 2$. Let $a, b \in \mathbb{Z}_n^*$. Then we have the following three cases.

Case-1: If $a, b \in U(\mathbb{Z}_n)$. Then a is not adjacent to b and we have $a — p_1 \times ..... \times p_s — b$. Therefore, $d(a, b) = 2$.

Case-2: If $a \in U(\mathbb{Z}_n)$ and $b \notin U(\mathbb{Z}_n)$. We can assume that $b = up_1^{t_1} \times .... \times p_r^{t_r}$, where $1 \leq r \leq s$ and $u \notin p_i\mathbb{Z}$, for every i, $1 \leq i \leq s$ and t_i is a positive integer for every i, $1 \leq i \leq r$. If $r = s$ then a is adjacent to b. If $r \neq s$, then since $k_s \geqslant 2$, $bp_1 \times .... \times p_s \neq 0$. Now, a and b are adjacent to $p_1 \times ..... \times p_s$. Thus, $d(a, b) \leq 2$.

Case-3: If $a, b \notin U(\mathbb{Z}_n)$. Then let $a = up_1^{t_1} \times .... \times p_r^{t_r}$ and $b = vp_1^{t_1} \times .... \times p_r^{t_j}$, where $u, v \notin p_i\mathbb{Z}$, for every i, $1 \leq i \leq s$, $1 \leq i \leq j \leq s$, t_i is a positive integer for every i, $1 \leq i \leq r$ and l_i is a positive integer for every i, $1 \leq i \leq j$. If $r = j < s$, then a and b are adjacent to $p_1 \times ..... \times p_s$. If $r = j = s$, then a and b are adjacent to 1. Now suppose that $r < j$. If $r < j < s$, then a and b are adjacent to $p_1 \times ..... \times p_s$. Therefore in this case $d(a, b) \leq 2$. This complete the proof. □

Proposition 2.6. *The graph $\Gamma_w(\mathbb{Z}_n)$ is a bipartite graph if $n = p^2$ (p being prime).*

Proof. Let us consider two partitions of the vertex set $V(\Gamma_w(\mathbb{Z}_{p^2})) = \mathbb{Z}_{p^2}^*$ namely, $V_1 : \mathbb{Z}_{p^2}^* \backslash N(\mathbb{Z}_{p^2})^*$ and $V_2 : N(\mathbb{Z}_{p^2})^*$. Then the following three cases arise,

Case-1: No two vertices of V_1 are adjacent.
For if $a, b \in V_1$ then $ab \notin N(\mathbb{Z}_{p^2})^*$
$\Rightarrow a \notin p\mathbb{Z}$ and $b \notin p\mathbb{Z}$
$\Rightarrow p \nmid a$ and $p \nmid b$
$\Rightarrow p \nmid ab$
$\Rightarrow ab \notin p\mathbb{Z} \Rightarrow ab \notin N(\mathbb{Z}_{p^2})^*$
And so, a and b are not adjacent in $\Gamma_w(\mathbb{Z}_{p^2})$.

Case-2: No two vertices of V_2 are adjacent.
For if $a, b \in V_2$ then a and b are zero divisors of $\mathbb{Z}_{p^2}^*$ which implies $ab = 0$ and so $ab \notin N(\mathbb{Z}_{p^2})^*$. Showing that a and b are not adjacent in $\Gamma_w(\mathbb{Z}_{p^2})$.

Case-3: Elements of V_1 are adjacent with elements of V_2.
Let $a \in V_1$ and $b \in V_2$ then $a \notin N(\mathbb{Z}_{p^2})^*$ and $b \in N(\mathbb{Z}_{p^2})^*$.
$\Rightarrow a \notin p\mathbb{Z}$ and $b \in p\mathbb{Z}$. But, $ab \in p\mathbb{Z}$
$\Rightarrow$ ab $\in$ N$(\mathbb{Z}_{p^2})^*$
So, a and b are adjacent in $\Gamma_w(\mathbb{Z}_{p^2})$.
Hence, $\Gamma_w(\mathbb{Z}_{p^2})$ is bipartite. □

Proposition 2.7. *Let S be a commutative semiring with unity and $|N(S)| \geq 3$. Then $gr(\Gamma_w(S)) \in \{3, 4\}$.*

Proof. Let S be a commutative semiring with unity and $|N(S)| \geq 3$. Assume that $x, y \in N(S)^*$ and $x^2 = 0$ then 1 is adjacent to x and y. Again we know that $1 + x$ is a unit element. Now $x(1 + x) = x + x^2 = x + 0 = x \in N(S)^*$ and $y(1+x) = y + yx = y + 0 = y \in N(S)^*$ [Since, x and y are zero divisor]. Therefore, $1 + x$ is adjacent to both x and y. So, 1 — x — $1 + x$ — y — 1 forms a 4-cycle in $\Gamma_w(S)$. Which yields $gr(\Gamma_w(S)) \in \{3, 4\}$. □

Proposition 2.8. *Let S be a commutative semiring with unity. If $\Gamma_w(S)$ is a forest, then the following holds:*
(i) $|N(S)| \leq 2$.
(ii) If $|N(S)| = 1$, then $\Gamma_w(S)$ is a totally disconnected.
(iii) If $|N(S)| = 2$, then $\Gamma_w(S)$ is a union of star and some isolated vertices.

Proof. Let S be a commutative semiring with unity.
(i) If $|N(S)| \geq 3$ then from above Proposition 2.7 $gr(\Gamma_w(S)) \in \{3, 4\}$, which is a contradiction. Therefore $|N(S)| \leq 2$.

(ii) If $|N(S)| = 1$, then S is a reduced semiring and by Proposition 2.1 Γ_w(S) is totally disconnected.

(iii) Let $|N(S)| = 2$, then $|N(S)^*| = 1$ and say $x \in N(S)^*$ hence $x^2 = 0$. We note that every element of $U(S)$ is adjacent to x. If x is adjacent to all the vertices, then $\Gamma_w(S)$ is a star graph. If every vertex that is not adjacent to x then x is an isolated vertex, then we are done. Therefore we can assume that there exist $y \in V(\Gamma_w(S))$ such that $d(y) = 1$ and y is not adjacent to x. Since x is not adjacent to y and $xy \in N(S)$. So, $xy = 0$. Assume that y is adjacent to a. Then, $ya = x$. Now, $y(a+x) = ya + yx = x + 0 = x \in N(S)^*$. Therefore, y is adjacent to $a + x$, a contradiction. This implies that if $d(y) = 1$ then x is adjacent to y. Since $\Gamma_w(S)$ is a forest. So, $\Gamma_w(S)$ is a union of a star graph (with centre x) and some isolated vertex. □

Proposition 2.9. *If S is an Artinian semiring, then the following holds:*
(i) If $\Gamma_w(S)$ is totally disconnected, then $S = \Pi_{i=1}^n F_i$, where F_i is a semifield, for $1 \leq i \leq n$.
(ii) $\Gamma_w(S)$ is a forest if and only if S is isomorphic to one of the semiring $\mathbb{Z}_4$, $\mathbb{Z}_2(x)/(x^2)$ and $\pi_{i=1}^n F_i$ is a semifield, for $i = 1, ..., n$.
(iii) $\Gamma_w(S)$ is not a disjoint union of cycles or a unicyclic graph.

Proof. Since, S is an Artinian semiring. So, $S \cong \Pi_{i=1}^n S_i$, where (S_i, M_i) is a local semiring and every M_i is nilpotent. Let e_i be the $1 \times n$ vector whose i-th component is 1 and other component are 0.
(i) Since $\sum_{i=1}^n e_i$ is adjacent to every non-zero element of $\Pi_{i=1}^n M_i$. But, $\Gamma_w(S)$ is a totally disconnected graph. So, $\Pi_{i=1}^n M_i = 0$. This implies that every S_i is a semifield and so $S = \Pi_{i=1}^n F_i$. This complete the proof.

(ii) Let us assume that $\Gamma_w(S)$ is a forest. Then we have the following two cases:

Case-1: $n \geq 2$. If $M_1, M_2 \neq 0$, then we have $e_1 + e_2$ — a_1e_1 — $e_1 + a_2e_2$ — $a_1e_1 + e_2$ — a_2e_2 — $e_1 + e_2$, for $a_i \in M_i^*$ and $i = 1, 2$, a contradiction. Therefore, we can assume that S_2 is a semifield. Similarly, we can assume that $S_2, ..., S_n$ are semifield. If $|M_1| \geq 3$, then let $\{a, b\} \subseteq M_1$. Since, $1 + a$ is a unit element of S_1, we conclude that $(1 + a)e_1$ — ae_1 — $e_1 + e_2$ — be_1 — $(1 + a)e_1$ is 4-cycle, a contradiction. Therefore, $|M_1| \leq 2$. If $|M| = 2$, then $S_1 \cong \mathbb{Z}_4$, $\mathbb{Z}_2(x)/(x^2)$. Therefore $S \cong \mathbb{Z}_4 \times \Pi_{i=1}^n F_i$ or $S \cong \mathbb{Z}_2(x)/(x^2) \times \Pi_{i=2}^n F_i$. If $S \cong \mathbb{Z}_4 \times \Pi_{i=1}^n F_i$, then e_1 — $2e_1$ — $3e_1$ — $2e_1 + e_2$ — e_1 is a 4-cycle, a contradiction. Thus, $|M| = 1$ and so S_1 is a semifield. Hence, $S \cong \pi_{i=1}^n F_i$.

Case-2: If $n = 1$. Since every element of $U(S)$ is adjacent to each element of $N(S)^*$, we conclude that $|N(S)^*| = 0, 1$ and $M = N(S)^*$. Therefore, $M = 0$ or $|M| = 2$. If $M = 0$, then S is a semifield and if $|M| = 2$ then $S \cong \mathbb{Z}_4$, $\mathbb{Z}_2(x)/(x^2)$. Conversely, assume that $S \cong \mathbb{Z}_4$, $\mathbb{Z}_2(x)/(x^2)$, then $\Gamma_w(S) = K_{1,2}$. If $S \cong \Pi_{i=1}^n F_i$, then by proposition 2.1 $\Gamma_w(S)$ is totally disconnected. This complete the proof.

(iii) To the contrary, let us assume that $\Gamma_w(S)$ is a disjoint union of cycle or a unicyclic graph. Then, we have the following two cases:

Case-1: If $n \geq 2$ and S_1 is a semifield then e_1 is an isolated vertex, a contradiction. Therefore, S_1 is not a semifield. Similarly, S_i is not a semifield for $i = 1, ..., n$. Since, $M_1 \neq 0$ and $|U(S)_1| \geq 2$. Let $\{1, u\} \subseteq U(S_1)$, $a \neq 0 \in M_1$ and $b \neq 0 \in M_2$. Then e_1 — $ae_1 + e_2$ — $ae_1 + be_2$ — e_1 and ue_1 — $ae_1 + e_2$ — $ae_1 + be_2$ — ue_1 are two cycles, a contradiction.

Case-2: If $n = 1$. Then (S, M) is an Artinian local semiring and $M = N(S)$. We note that every element of $U(S)$ is adjacent to each element of M^*. This implies that $|M^*| \leqslant 1$. Otherwise, if $a, b \in M^*$, then 1 — a — $1 + a$ — b —1 and 1 — a — $1 + b$ — b — 1 are two cycles of $\Gamma_w(S)$, a contradiction. Now, since $|M^*| \leqslant 1$. So, we can conclude that $S \cong \mathbb{Z}_2$, $\mathbb{Z}_4$, $\mathbb{Z}_2(x)/(x^2)$. It is easy to see that $\Gamma_w(\mathbb{Z}_2)$ is an isolated vertex and $\Gamma_w(\mathbb{Z}_4) = \Gamma_w(\mathbb{Z}_2(x)/(x^2)) = K_{1,2}$, a contradiction. □

Proposition 2.10. *Let S be a commutative semiring with unity and $S = \Pi_{i=1}^n S_i$, then the following holds:*
(i) $\omega(\Gamma_w(S)) \geq \Pi_{i=1}^n \omega(\Gamma_w(S_i))$.
(ii) Let $\chi(\Gamma_w(S)) = \chi$ and $\chi(\Gamma_w(S_i)) = \chi_i$ for every i, $1 \leqslant i \leqslant n$. If χ_i is finite for every i, $1 \leqslant i \leqslant n$ then $\chi \leqslant \sum_{J \in P} \Pi_{i \in J} \chi_i$, where P is the set of all subset of $\{1, ..., n\}$.

Proof. Let S be a commuative semiring with unity.
(i) Let C_i be the clique in $\Gamma_w(S_i)$ for $1 \leq i \leq n$. It is easy to see that $C = \{(a_1, ..., a_n) \mid a_i \in C_i, 1 \leq i \leq n\}$ is a clique in Γ_w(S). This complete the proof.

(ii) We first assume that $n = 2$ and $(x, y) \in S$. If $x, y \neq 0$, then we define $f((x, y)) = (\chi_1(x), \chi_2(y))$. If $x = 0$ and $y \neq 0$ then let $f((x, y)) = (0, \chi_2(y))$. Otherwise, since $(x, y) \neq 0$, we conclude that $x \neq 0$ and $y = 0$. In this case, suppose that $f((x, y)) = (\chi_1(x), 0)$. Obviously, f is a proper vertex coloring for S^*. Hence, $\chi \leq \chi_1 + \chi_2 + \chi_1\chi_2$. Now, assume that $n \geq 3$. By induction we can easily prove that $\chi \leq \sum_{J \in P} \Pi_{i \in J} \chi_i$, where P is the set of all subset of $\{1, ..., n\}$. □

Proposition 2.11. *Let (R, M) be a local semiring and $M \neq 0$. Then the following holds:*
(i) If $M^2 = 0$, then $\omega(\Gamma_w(S)) = 2$.
(ii) If S is a finite semiring, then $\chi(\Gamma_w(S)) \leq |M|$.

Proof. Let S be a commutative semiring with unity and M is a unique maximal ideal.
(i) Assume that $x \neq 0 \in M$. So, x is a nilpotent element of S and 1 is adjacent to x. Which yields $\omega(\Gamma_w(S)) \geq 2$ ....(1).
If C is a clique for $\Gamma_w(S)$ with maximal cardinal then C has at most one unit element. Since, each unit element is adjacent to every element of $N(S)^*$. Hence, $|C \cap U(S)| \leq 1$. On the other hand $M^* = 0$. So, $|C \cap M| \leq 1$. Thus, $|C| \leq 2$ ....(2)
Now (1) and (2) implies $\omega(\Gamma_w(S)) = 2$.

(ii) we assume that $M^* = \{x_1, ..., x_{|M|-1}\}$. We define $f(x_i) = i$, for every i, $1 \leq i \leq |M| - 1$ and $f(u) = |M|$ for every $u \in U(S)$. Clearly, f is a proper vertex coloring for $V(\Gamma_w(S))$. Therefore, $\chi(\Gamma_w(S)) \leq |M|$. □

3. The structure and properties of the complement of the weakly nilpotent graph of a commutative semiring

As we mention in the introduction, the complement of the weakly nilpotent graph $\overline{\Gamma_w(S)}$ is a graph with the vertex set S^* and two distinct vertices x and y in S^* are adjacent if and only if xy $\notin N(S)^*$.

Example 3.1. *In the commutative semiring $\mathbb{Z}_n$. If n being prime or square free integer, then the nontrivial nilpotent is ϕ, so $\overline{\Gamma_w(\mathbb{Z}_n)}$ is complete graph.*

Example 3.2. *In the commutative semiring $\mathbb{Z}_8$, $N(S)^* = \{2, 4, 6\}$. So, the graph $\overline{\Gamma_w(\mathbb{Z}_8)}$ is as follows:*

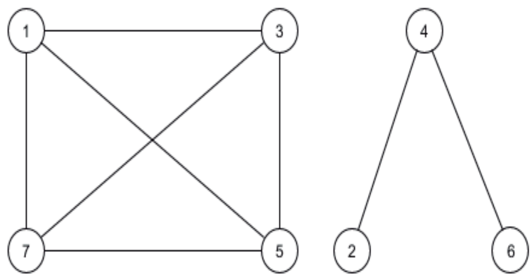

Figure 5: $\overline{\Gamma_w(\mathbb{Z}_8)}$

From the statement of Proposition 2.1, we have the following remark:

Remark 3.3. *For a commutative semiring S with unity, the graph $\overline{\Gamma_w(S)}$ is complete if and only if S is a reduced semiring.*

Proposition 3.1. *If S is a Artinian semiring. Then the following holds:*
(i) If S is a local semiring, then $\overline{\Gamma_w(S)}$ is connected if and only if S is a semifield.
(ii) If S is a non-local semiring, then $diam(\overline{\Gamma_w(S)}) \leq 4$.

Proof. (i) One side is clear. For other side assume that (S, M) is a local semiring and $\overline{\Gamma_w(S)}$ is connected. If $M \neq 0$, then there is not any path between a and u for every $a \in M^*$ and $u \in U(S)$. Since, $au \in N(S)^*$. This yield $\overline{\Gamma_w(S)}$ is disconnected, a contradiction. Therefore S is a semifield.

(ii) We know that $S \cong \Pi_{i=1}^n S_i$, where $n \geq 1$ and (S_i, M_i) is a local semiring for every i, $1 \leq i \leq n$. Let e_i be the $1 \times n$ vector whose i-th component is 1 and other component are 0. Let $a = \sum_{i=1}^n a_i e_i$, $b = \sum_{i=1}^n b_i e_i \in S^*$. Then, we have the following three cases:

Case-1: If $a, b \in U(S)$. Then a is adjacent to b. Since $ab \notin U(S)^*$. Therefore, $diam(\overline{\Gamma_w(S)}) = 1$.

Case-2: If $a \in U(S)$, $b \notin U(S)$ and $b_i \in M_i$ for every i, $1 \leq i \leq n$ then $b_j \neq 0$ for every j, $1 \leq j \leq n$. Suppose that r is the least positive integer such that $b_j^r = 0$. Hence a — $b_j^{(r-1)} a_j e_j$ — b is a path. If $b_j \in U(S)$, for some j, $1 \leq j \leq n$ then a is adjacent to b. Therefore $diam(\overline{\Gamma_w(S)}) \leq 2$.

Case-3: $a, b \notin U(S)$. Let $I = \{i \mid 1 \leq i \leq n, a_i \in U(S_i)\}$ and $J = \{i \mid 1 \leq i \leq n, b_i \in U(S_i)\}$. If $I \cap J \neq \phi$, then let $t \in I \cap J$. It is easy to see that a and b are adjacent to e_t. Now, assume that $I \cap J = \phi$, then we have the path a — $\sum_{i \in I} e_i$ — $\sum_{i=1}^n e_i$ — $\sum_{i \in J} e_i$ — b. Therefore, $diam(\overline{\Gamma_w(S)}) \leq 4$. □

Proposition 3.2. *Let S be a commutative semiring with unity. Then the following holds:*
(i) $\omega(\overline{\Gamma_w(S)}) \geq |U(S)|$.
(ii) $\omega(\overline{\Gamma_w(S)}) = |U(S)|$ if and only if S is a local semiring with maximal ideal $N(S)$.

Proof. (i) If $|U(S)| = 1$, then it is clear that $\omega(\overline{\Gamma_w(S)}) \geq |U(S)|$. Now let us assume that $u, v \in U(S)$ implies $uv \notin N(S)^*$, which yields u and v are adjacent in $\overline{\Gamma_w(S)}$ and so $\omega(\overline{\Gamma_w(S)}) \geq |U(S)|$.

(ii) First, suppose that S is a local semiring with maximal ideal $N(S)$. If C is a clique of $\overline{\Gamma_w(S)}$ and $|C| = \omega(\overline{\Gamma_w(S)})$, then $C \subseteq U(S)$ or $C \subseteq N(S)$, otherwise u is adjacent to x for some $u \in U(S)$ and $x \in N(S)$ which is impossible. Since, $ux \in N(S)^*$. Again we know that $1 + N(S) \subseteq U(S)$ implies $|N(S)| \leqslant |U(S)|$ and so $\omega(\overline{\Gamma_w(S)}) = |U(S)|$.

Conversely suppose that $\omega(\overline{\Gamma_w(S)}) = |U(S)|$. Since $U(S)$ is a clique of $\overline{\Gamma_w(S)}$ and $\omega(\overline{\Gamma_w(S)}) = |U(S)|$. u is not adjacent to x for every $u \in U(S)$ and $x \in S \setminus U(S)$. This yields that x is a nilpotent element. Therefore, $S \setminus U(S) = N(S)$. Thus, $N(S)$ is a maximal ideal. This show that S is a local semiring with maximal ideal $N(S)$. □

Proposition 3.3. *If S is a commutative semiring, then $gr(\overline{\Gamma_w(S)}) \in \{3, \infty\}$*

Proof. Assume that $x_1 — x_2 —$ .... $— x_n — x_1$ is a cycle and $n \geq 4$. If $|U(S)| \geq 3$ then by Proposition 3.2 $gr(\overline{\Gamma_w(S)}) = 3$. Therefore, we can assume that $|U(S)| \leq 2$. Then, there are two cases:

Case-1: If $|U(S)| = 2$. Let $1, u \in U(S)$. If $x \neq 0 \in S \setminus (N(S) \bigcup \{1, u\})$ then $1 — u — x — 1$ forms 3-cycles in $\overline{\Gamma_w(S)}$. Since, $1.u = u, u.x = ux, x.1 = x \in N(S)^*$ and so $gr(\overline{\Gamma_w(S)}) = 3$. If $S \setminus N(S) = \{1, u\}$ then $N(S)$ is a maximal ideal of S. This implies S is a local semiring with maximal ideal $N(S)$. Since $1 + N(S) \subseteq U(S)$. $|N(S)| \leqslant 2$. If $|N(S)| = 1$ then $S \cong \mathbb{Z}_2, \mathbb{Z}_3$ and $\overline{\Gamma_w(S)} = K_1$ and $\overline{K_2}$ respectively, a contradiction. If $|N(S)| = 2$ then $S \cong \mathbb{Z}_4, \mathbb{Z}_2(x)/(x^2)$ and $\overline{\Gamma_w(S)} = \overline{K_3}$, a contradiction.

Case-2: If $|U(S)| = 1$. We know that $1 + N(S) \subseteq U(S)$. So, S is a reduced semiring. This implies 1 is adjacent to x_1 and x_2. Now, $x_1 — x_2 — 1 — x_1$ is a 3-cycle. This completes the proof. □

Proposition 3.4. *Let S be a commutative semiring with unity. Then $\overline{\Gamma_w(S)}$ is a cycle if and only if S is a semifield of order 4 or $S \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof. Clearly, if S is a semifield of order 4 or $S \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then $\overline{\Gamma_w(S)}$ is a cycle.
Conversely, assume that $\overline{\Gamma_w(S)}$ is a cycle. Then by Proposition 3.3, $\overline{\Gamma_w(S)}$ is a 3-cycle. This implies that $|S| = 4$. If S is a local semiring, then S is a semifield or $S \cong \mathbb{Z}_4$, $\mathbb{Z}_2(x)/(x^2)$. Obviously, $gr(\overline{\Gamma_w(\mathbb{Z}_4)} = gr(\overline{\Gamma_w(\mathbb{Z}_2(x)/(x^2))} = \infty$. Therefore, S is a semifield of order 4. Now, suppose that S ia a non-local semiring. Then $\overline{\Gamma_w(S)}$ is a 3-cycle, and so $|S| = 4$. Hence $S \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, as desired. □

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