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A New Method for Designing Involute Trajectory Timelike Ruled Surfaces in Minkowski 3-space

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ABSTRACT: In this study, we present the new concept of involute trajectory ruled surface in Minkowski 3-space. The involute trajectory timelike ruled surface is a surface generated by the motion of a timelike oriented line $\vec{X}$ along the spacelike involute curve $\vec{\gamma}(s)$ of a given timelike base curve $\vec{r}(s)$. The main purpose of this article is to present a new perspective on the generation of developable trajectory ruled surfaces in Minkowski 3-space. These surfaces are formed depending on the angle θ between the Darboux vector $\vec{D}$ and the binormal vector $\vec{b}$ of the evolute curve $\vec{r}(s)$. Also, some new results and theorems related to the developability of the involute trajectory timelike ruled surfaces are obtained. Finally, we illustrate these surfaces by presenting one example.

Key Words: Trajectory ruled surface, involute-evolute, developable surface, Frenet frame, Minkowski 3-space.

Contents

1. Introduction

Involute of a given space curve is a well-known concept in classical differential geometry. An involute of a given curve is some other curve that always remains perpendicular to the tangent lines to that given curve. This can also be thought as the process of winding or unwinding a string tautly around a curve. The original curve is called an evolute. In the literature, the relationship between the Frenet frames of the involute of a curve for the first time was expressed by using the angle between the Darboux vector and binormal vector of the evolute curve in the Euclidean 3-spaces. (see [1]). Then, Bilici expressed the transformation matrix between the Frenet frames of the involute of a curve by using the Lorentzian timelike (spacelike) angle θ between the Darboux vector $\vec{D}$ and binormal vector $\vec{b}$ of the evolute curve $\vec{r}(s)$ according to causal characteristic of the curve couple in the Minkowski 3-spaces, [2]. Later Bilici, in his work with Bayram, consructed a surface family possessing an involute of a given curve as an asymptotic curve [3].

Ruled surfaces and their special condition, developability, are another well known concepts of classical differential geometry. Ruled surfaces can be described as the movement of a line segment along a curve based on a curve is the resulting surface. For example, the cylinder and cone are the well-known ruled surfaces.

A developable ruled surface can be defined as isometrically mapped (i.e., developed) into the plane. Among the ruled surfaces, the developable play a distinguished role since they have broad applications in many areas from engineering to manufacturing. For instance, an aircraft designer uses them to design airplane wings, and a tinman uses them to connect two differently shaped tubes with planar segments of metal sheets.

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In the past, offsets of ruled surfaces have been the subject of many studies. Ravani and Ku [4] generalized the theory of Bertrand curves for Bertrand ruled surface-offsets, using line geometry. Some of the studies in the Computer Aided Geometric Design (CAGD) literature dealing with offsets of surfaces have been given by many authors [5-7]. The corresponding characterizations of timelike and spacelike ruled surfaces in Minkowski 3-space have been given by Turgut and Hacısalihoğlu [8] and Yaylı [9]. Woestijne [10] and Kim and Yoon [11] classified the Lorentz surfaces. These studies are important for space kinematics and mechanisms. Furthermore, the geometry of trajectory ruled surfaces is widely applied to the study of design problems in spatial mechanisms or space kinematics [12-14]. In addition, Orbay and Aydemir gave the spacelike surface with spacelike directional vector and the ruled surfaces generated by Frenet vectors of the base curve of this surface in the Minkowski 3-space [15]. In addition, there are a lot of remarkable studies on special curves and geometry of surfaces in different spaces [20-23].

The involute trajectory timelike ruled surface is a surface generated by the motion of a timelike oriented line $\overrightarrow{X}$ along the spacelike involute curve $\overrightarrow{\gamma}(s)$ of a given timelike base curve $\overrightarrow{r}(s)$.

In this paper, the production of trajectory ruled surfaces in Minkowski 3-space by the Frenet trihedron moving along the space like involute $\overrightarrow{\gamma}(s)$ of a given timelike space curve $\overrightarrow{r}(s)$ is stated according to the Lorentzian timelike angle θ between the binormal vector $\overrightarrow{b}$ and the unit vector $\overrightarrow{D_0}$ of direction of Darboux vector $\overrightarrow{D}$ of the evolute curve $\overrightarrow{r}(s)$. In addition, some special cases were investigated depending on the selection of oriented line. Further, some new results and theorems related to the developability of involute trajectory ruled surfaces are obtained. Finally, these surfaces $\Psi_{\overrightarrow{t^*}}$, $\Psi_{\overrightarrow{n^*}}$, $\Psi_{\overrightarrow{b^*}}$ are illustrated with an example.

2. Preliminaries

Let Minkowski 3-space $\mathbb{R}^3_1$ be the vector space $\mathbb{R}^3$ and provide with the Lorentzian inner product , $\left\langle \overrightarrow{u}, \overrightarrow{v} \right\rangle = -u_1v_1 + u_2v_2 + u_3v_3$, where $\overrightarrow{u} = (u_1, u_2, u_3)$, $\overrightarrow{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$.

A vector $\overrightarrow{u} \in \mathbb{R}^3$ is said to be timelike if $\left\langle \overrightarrow{u}, \overrightarrow{u} \right\rangle < 0$ spacelike if $\left\langle \overrightarrow{u}, \overrightarrow{u} \right\rangle > 0$ or $\overrightarrow{u} = 0$, and lightlike (or null) if $\left\langle \overrightarrow{u}, \overrightarrow{u} \right\rangle = 0$ and $\overrightarrow{u} \neq 0$ for all. Similarly, a smooth curve $\overrightarrow{r} : I \longrightarrow \mathbb{R}^3_1$, $(I \subset \mathbb{R})$ in $\mathbb{R}^3_1$ can be timelike, spacelike, or null (lightlike) if all of its velocity $\dot{\overrightarrow{r}}(s)$ vectors are timelike, spacelike, or null for every $s \in I \subset \mathbb{R}$, respectively [16]. If $\left\langle \overrightarrow{u}, \overrightarrow{v} \right\rangle = 0$ for all $\overrightarrow{u}$ and $\overrightarrow{v}$, the vectors $\overrightarrow{u}$ and $\overrightarrow{v}$ are called perpendicular in the sense of Lorentz. The norm of $\overrightarrow{u} \in \mathbb{R}^3_1$ is defined as $\left\| \overrightarrow{u} \right\| = \sqrt{\left| \left\langle \overrightarrow{u}, \overrightarrow{u} \right\rangle \right|}$. A timelike vector is said to be positive (resp. negative) if and only if $u_1 > 0$ $(resp. u_1 < 0)$. The vector product of $\overrightarrow{u}$ and $\overrightarrow{v}$ is given by the equality:

$$\overrightarrow{u} \times \overrightarrow{v} = (u_3v_2 - u_2v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1) . \tag{2.1}$$

A surface in $\mathbb{R}^3_1$ is called a timelike surface if the induced metric on the surface is the Lorentz metric, that is, the normal on the surface is a spacelike vector. Let $\overrightarrow{r}(s)$ be a unit speed timelike space curve with curvature κ and torsion τ . We denote by $\left\{ \overrightarrow{t}(s), \overrightarrow{n}(s), \overrightarrow{b}(s) \right\}$ the moving Frenet frame along the curve $\overrightarrow{r}(s)$. Then, $\overrightarrow{t}$, $\overrightarrow{n}$, $\overrightarrow{b}$ and are the tangent, the principal normal, and the binormal vector of the curve $\overrightarrow{r}(s)$, respectively. In this trihedron $\overline{t}$, is the timelike vector and $\overrightarrow{n}$ and $\overrightarrow{b}$ are spacelike vectors. Thus, the scalar product and cross product of these vectors is given by

$$\left\langle \overrightarrow{t}, \overrightarrow{t} \right\rangle = \left\langle \overrightarrow{n}, \overrightarrow{n} \right\rangle = \left\langle \overrightarrow{b}, \overrightarrow{b} \right\rangle = 1, \ \left\langle \overrightarrow{t}, \overrightarrow{n} \right\rangle = \left\langle \overrightarrow{n}, \overrightarrow{b} \right\rangle = \left\langle \overrightarrow{b}, \overrightarrow{t} \right\rangle = 0, \tag{2.2}$$

$$\overrightarrow{t} \times \overrightarrow{n} = \overrightarrow{b}, \ \overrightarrow{n} \times \overrightarrow{b} = -\overrightarrow{t}, \ \overrightarrow{b} \times \overrightarrow{t} = \overrightarrow{n} \tag{2.3}$$

In this situation, the Frenet formulas are given by the formulas:

$$\dot{\vec{t}} = \kappa \vec{n}, \ \dot{\vec{n}} = \kappa \vec{t} - \tau \vec{b}, \ \dot{\vec{b}} = \tau \vec{n} \tag{2.4}$$

We also define the rotation vector of this trihedron, the Darboux vector of $\vec{r}(s)$, which provides a useful way of interpreting κ and τ geometrically. We can express this in terms of the Frenet-Serret vectors as [17]

$$\vec{D} = \tau \vec{t} - \kappa \vec{b} \tag{2.5}$$

There are two cases corresponding to the causal character of Darboux vector $\vec{D}$.

Case I. If $|\kappa| > |\tau|$, then $\vec{D}$ is a spacelike vector. In this situation, we can write

$$\begin{array}{l} \kappa = \left\|\vec{D}\right\| \cosh\theta \\ \tau = \left\|\vec{D}\right\| \sinh\theta \end{array}, \quad \left\|\vec{D}\right\|^2 = \left\langle \vec{D}, \vec{D} \right\rangle = \kappa^2 - \tau^2 \tag{2.6}$$

and the unit vector $\vec{D}_0$ of direction $\vec{D}$ is

$$\vec{D}_0 = \frac{1}{\left\|\vec{D}\right\|}\vec{D} = \cosh\theta\, \vec{t} - \sinh\theta\, \vec{b} \tag{2.7}$$

Case II. If $|\kappa| < |\tau|$, then $\vec{D}$ is a timelike vector. In this situation, we have

$$\begin{array}{l} \kappa = \left\|\vec{D}\right\| \cosh\theta \\ \tau = \left\|\vec{D}\right\| \sinh\theta \end{array}, \quad \left\|\vec{D}\right\|^2 = -\left\langle \vec{D}, \vec{D} \right\rangle = \tau^2 - \kappa^2 \tag{2.8}$$

and the unit vector $\vec{D}_0$ of direction $\vec{D}$ is

$$\vec{D}_0 = \frac{1}{\left\|\vec{D}\right\|}\vec{D} = \sinh\theta\, \vec{t} - \cosh\theta\, \vec{b} \tag{2.9}$$

where θ is the Lorentzian timelike angle between $\vec{t}$ and $\vec{D}_0$.

Remark 1. We can easily see from the equations of Case I and Case II that $\frac{\tau}{\kappa} = \tanh\theta$ $\left(or\ \frac{\tau}{\kappa} = \coth\theta\right)$, and if θ is an arbitrary constant, then $\vec{r}(s)$ is a general helix.

The angle between two vectors in Minkowski 3-space is defined by [18]:

Definition 1. Let $\vec{u}$ and $\vec{v}$ be spacelike vectors in $\mathbb{R}^3_1$ that span a spacelike vector subspace. We then have $\left|\left\langle \vec{u}, \vec{v} \right\rangle\right| \leq \left\|\vec{u}\right\| \left\|\vec{v}\right\|$, and hence, there is a unique real number φ such that

$$\left\langle \vec{u}, \vec{v} \right\rangle = \left\|\vec{u}\right\| \left\|\vec{v}\right\| \cos\varphi. \tag{2.10}$$

The real number φ is called the Lorentzian spacelike angle between $\vec{u}$ and $\vec{v}$.

Definition 2. Let $\vec{u}$ and $\vec{v}$ be spacelike vectors in $\mathbb{R}^3_1$ that span a timelike vector subspace. We then have $\left\langle \vec{u}, \vec{v} \right\rangle > \left\|\vec{u}\right\| \left\|\vec{v}\right\|$, and hence, there is a unique positive real number φ such that

$$\left|\left\langle \vec{u}, \vec{v} \right\rangle\right| = \left\|\vec{u}\right\| \left\|\vec{v}\right\| \cosh\varphi. \tag{2.11}$$

The real number φ is called the Lorentzian timelike angle between $\vec{u}$ and $\vec{v}$.

Definition 3. Let $\vec{u}$ be a spacelike vector and $\vec{v}$ a positive timelike vector in $\mathbb{R}^3_1$. Then, there is a unique non-negative real number φ such that

$$\left|\left\langle \vec{u}, \vec{v} \right\rangle\right| = \left\|\vec{u}\right\| \left\|\vec{v}\right\| \sinh\varphi. \tag{2.12}$$

The real number φ is called the Lorentzian timelike angle between $\vec{u}$ and $\vec{v}$.

Definition 4. Let $\vec{u}$ and $\vec{v}$ be positive (negative) timelike vectors in $\mathbb{R}^3_1$. Then there is a unique non-negative real number φ such that

$$\left\langle \vec{u}, \vec{v} \right\rangle = \left\|\vec{u}\right\| \left\|\vec{v}\right\| \cosh\varphi. \tag{2.13}$$

The real number φ is called the Lorentzian timelike angle between $\vec{u}$ and $\vec{v}$.

Let $\vec{\gamma}(s)$ be spacelike involute of a given timelike space curve $\vec{r}(s)$ in $\mathbb{R}^3_1$. Then we have

$$\vec{\gamma}(s) = \vec{r}(s) + (c-s)\vec{t}(s), \tag{2.14}$$

where c is an arbitrary constant and $\dot{\vec{r}}(s) = \vec{t}(s)$. In this study, we assume that $c - s \neq 0$ for convenience. Let the Frenet frames of $\vec{r}(s)$ and $\vec{\gamma}(s)$ be $\left\{\vec{t}(s), \vec{n}(s), \vec{b}(s)\right\}$, and $\left\{\vec{t^*}(s),\ \vec{n^*}(s),\ \vec{b^*}(s)\right\}$, respectively. Consider the Frenet frame $\left\{\vec{t^*}(s),\ \vec{n^*}(s),\ \vec{b^*}(s)\right\}$ attached to the spacelike involute $\vec{\gamma}(s)$ such that the tangent vector $\vec{t^*}(s)$ is a unit spacelike vector, the principal normal vector $\vec{n}^*(s)$ is a unit timelike vector, and the binormal vector $\vec{b}^*(s)$ is a unit spacelike vector. In this case, the following can be written:

$$\left\langle \vec{t^*}, \vec{t^*} \right\rangle = -\left\langle \vec{n^*}, \vec{n^*} \right\rangle = \left\langle \vec{b^*}, \vec{b^*} \right\rangle = 1,\ \left\langle \vec{t^*}, \vec{n^*} \right\rangle = \left\langle \vec{n^*}, \vec{b^*} \right\rangle = \left\langle \vec{b^*}, \vec{t^*} \right\rangle = 0 \tag{2.15}$$

For the Frenet trihedron, the vectoral product is given by

$$\vec{t^*} \times \vec{n^*} = \vec{b^*},\ \vec{n^*} \times \vec{b^*} = \vec{t^*},\ \vec{b^*} \times \vec{t^*} = -\vec{n^*}. \tag{2.16}$$

On the other hand, the relationships between the Frenet frames of curve couple $\left(\vec{r},\ \vec{\gamma}\right)$ can be given by [19]

1) If $\vec{D}$ is a spacelike vector, then

$$\begin{pmatrix} \vec{t^*}(s) \\ \vec{n^*}(s) \\ \vec{b^*}(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -\cosh\theta & 0 & \sinh\theta \\ -\sinh\theta & 0 & \cosh\theta \end{pmatrix} \begin{pmatrix} \vec{t}(s) \\ \vec{n}(s) \\ \vec{b}(s) \end{pmatrix}. \tag{2.17}$$

2) If $\vec{D}$ is a timelike vector, then

$$\begin{pmatrix} \vec{t^*}(s) \\ \vec{n^*}(s) \\ \vec{b^*}(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \sinh\theta & 0 & -\cosh\theta \\ -\cosh\theta & 0 & \sinh\theta \end{pmatrix} \begin{pmatrix} \vec{t}(s) \\ \vec{n}(s) \\ \vec{b}(s) \end{pmatrix}. \tag{2.18}$$

The trace of an $\vec{X}$ oriented line along space curve $\vec{\gamma}(s)$ is generally a trajectory ruled surface. A parametric equation of this trajectory ruled surface generated by an $\vec{X}$ oriented line is given by

$$\Psi(s,k) = \vec{\gamma}(s) + k\vec{d}(s),\ s,k \in I \subset \mathbb{R},$$

where $\vec{d}$ is the unit direction vector of $\vec{X}$ the oriented line. The distribution parameter (or drall) of the $\Psi(s,k)$ trajectory ruled surface is given as

$$\delta_d = \frac{\det\left(\dot{\vec{\gamma}}, \vec{d}, \dot{\vec{d}}\right)}{\left\|\dot{\vec{d}}\right\|^2}. \tag{2.19}$$

A developable trajectory ruled surface is characterized by $\delta_d = 0$. The parameterization of the striction curve on a trajectory ruled surface is given by

$$\vec{C}(s) = \vec{\gamma}(s) - \frac{\left\langle \dot{\vec{\gamma}}, \dot{\vec{d}} \right\rangle}{\left\|\dot{\vec{d}}\right\|^2}\vec{d}. \tag{2.20}$$

3. Involute Trajectory Timelike Ruled Surfaces

Let $\vec{\gamma}(s)$ be the spacelike involute of a timelike space curve $\vec{r}(s)$ with spacelike $\vec{D}$, and let $\left\{\vec{t^*}(s), \vec{n^*}(s), \vec{b^*}(s)\right\}$ be its Frenet frame, defined as in Eq. (17). Also, we consider that a timelike oriented line $\vec{X}(s)$ in $\mathbb{R}^3_1$, such that it is firmly connected to the Frenet frame of the involute $\vec{\gamma}(s)$, is represented uniquely with respect to this frame, in the form

$$\vec{X}(s) = x_1(s)\vec{t^*}(s) + x_2(s)\vec{n^*}(s) + x_3(s)\vec{b^*}(s), \quad \left\langle \vec{X}, \vec{X} \right\rangle < 0, \quad \left\|\vec{X}\right\| = 1 \tag{3.1}$$

where $x_i(s)$ (i=1, 2, 3) are scalar functions of the arc length parameter of the involute $\vec{\gamma}(s)$. The trajectory ruled surfaces generated by line $\vec{X}(s)$, $\vec{t}^*(s)$, $\vec{n}^*(s)$, and $\vec{b}^*(s)$ are

$$\Psi_{\vec{X}} : \Psi(s, v) = \vec{\gamma}(s) + v\vec{X}(s), \tag{3.2}$$

$$\Psi_{\vec{t^*}} : \Psi(s, u) = \vec{\gamma}(s) + u\vec{t^*}(s), \tag{3.3}$$

$$\Psi_{\vec{n^*}} : \Psi(s, z) = \vec{\gamma}(s) + z\vec{n^*}(s), \tag{3.4}$$

and

$$\Psi_{\vec{b^*}} : \Psi(s, w) = \vec{\gamma}(s) + w\vec{b^*}(s), \tag{3.5}$$

respectively. We can obtain the distribution parameter of the involute trajectory timelike ruled (I.T.R.S) surface generated by $\vec{X}(s)$ in $\mathbb{R}^3_1$. Analytically, from equations (17), (21) and Frenet formulas

$$\begin{aligned} \dot{\vec{X}} &= \left(x_1\kappa - \dot{x}_2\cosh\theta - x_2\dot{\theta}\sinh\theta - \dot{x}_3\sinh\theta - x_3\dot{\theta}\cosh\theta\right)\vec{t} - \left(\dot{x}_1 - x_2\left\|\vec{D}\right\|\right)\vec{n} + \\ &\left(x_1\tau + \dot{x}_2\sinh\theta + x_2\dot{\theta}\cosh\theta + \dot{x}_3\cosh\theta + x_3\dot{\theta}\sinh\theta\right)\vec{b}. \end{aligned} \tag{3.6}$$

By differentiating Eq. (14) with respect to the arc length parameter s, we have

$$\dot{\vec{\gamma}}(s) = (c - s)\kappa(s)\vec{n}(s). \tag{3.7}$$

By substituting Eqs. (26) and (27) into Eq. (19), the distribution parameter of this surface is

$$\delta_X = \frac{\det\left(\dot{\vec{\gamma}}, \vec{X}, \dot{\vec{X}}\right)}{\left\|\dot{\vec{X}}\right\|^2}, \tag{3.8}$$

$$\delta_X = \frac{(c-s)\kappa\left\{\left(x_2^2 - x_3^2\right)\dot{\theta} + x_1x_3\left\|\vec{D}\right\| + x_2\dot{x}_3 - \dot{x}_2x_3\right\}}{\left|\begin{array}{c} x_1^2 - \dot{x}_2^2 + \dot{x}_3^2 + \left(x_2^2 - x_1^2\right)\left\|\vec{D}\right\|^2 + \left(x_2^2 - x_3^2\right)\dot{\theta}^2 \\ +2\left\|\vec{D}\right\|\left(x_1\dot{x}_2 - \dot{x}_1x_2 + x_1x_3\dot{\theta}\right) + 2\dot{\theta}\left(x_2\dot{x}_3 - x_1x_2 - \dot{x}_2x_3\right)\end{array}\right|}. \tag{3.9}$$

The ruled surface is developable if and only if δ_X is zero. From Eq. (29) we have

$$\left(x_2^2 - x_3^2\right)\dot{\theta} + x_1x_3\left\|\vec{D}\right\| + x_2\dot{x}_3 - \dot{x}_2x_3 = 0. \tag{3.10}$$

Thus, we state the following theorem.

Theorem 1. The involute trajectory timelike ruled surface $\Psi_{\vec{X}}$ is developable if and only if the Lorentzian timelike angle θ between $\vec{t}$ and $\vec{D}_0$ of space curve $\vec{r}(s)$ satisfies the following equality:

$$\theta = \int \frac{\left(x_2\dot{x}_3 + x_1x_3\left\|\vec{D}\right\| - \dot{x}_2x_3\right)}{x_3^2 - x_2^2} ds + \lambda, \tag{3.11}$$

where λ is an arbitrary constant.

Special Cases

The Case Where $\vec{X}(s) = \vec{t^*}(s)$

In this case, $x_1 = 1$, $x_2 = x_3 = 0$. Thus, from Eq. (29),

$$\delta_{\vec{t}^*} = 0. \tag{3.12}$$

We can therefore give the following result.

Result 1. The ruled surface $\Psi_{\vec{t^*}}$ given by (23) is developable.

The Case Where $\vec{X}(s) = \vec{n^*}(s)$

In this case, $x_2 = 1$, $x_1 = x_3 = 0$. Thus, from Eq. (29),

$$\delta_{\vec{n^*}} = \frac{(c-s)\kappa\dot{\theta}}{\left\|\vec{D}\right\|^2 + \dot{\theta}^2}. \tag{3.13}$$

So, we can give the following result.

Result 2. If space curve $\vec{r}(s)$ is a general helix, then the ruled surface $\Psi_{\vec{n^*}}$ given by (24) is developable.

The Case Where $\vec{X}(s) = \vec{b^*}$

In this case, $x_3 = 1$, $x_1 = x_2 = 0$, and from Eq. (29),

$$\delta_{\vec{b^*}} = \frac{(c-s)\kappa}{\dot{\theta}}. \tag{3.14}$$

So, we can give the following result.

Result 3. The ruled surface $\Psi_{\overrightarrow{b^*}}$ given by (25) is never developable.

From Eqs. (33) and (34) we have

$$\frac{\delta_{\overrightarrow{n^*}}}{\delta_{\overrightarrow{b^*}}} = -\frac{\dot{\theta}^2}{\left\|\overrightarrow{D}\right\|^2 + \dot{\theta}^2}. \tag{3.15}$$

Thus, the following theorem can be given.

Theorem 2. If $\overrightarrow{\gamma}(s)$ represents the spacelike involute of a timelike space curve $\overrightarrow{r}(s)$ with spacelike $\overrightarrow{D}$ and θ is the Lorentzian timelike angle between $\overrightarrow{t}$ and $\overrightarrow{D}_0$, then there is a relationship (35) between the Darboux vector of $\overrightarrow{r}(s)$ and the distribution parameters of the ruled surfaces generated by $\overrightarrow{n}^*$ and $\overrightarrow{b}^*$ in $\mathbb{R}^3_1$.

From Eq. (35), if θ is constant, then $\frac{\delta_{\overrightarrow{n^*}}}{\delta_{\overrightarrow{b^*}}} = 0$. On the contrary, if $\frac{\delta_{\overrightarrow{n^*}}}{\delta_{\overrightarrow{b^*}}} = 0$, then θ is constant. Therefore, with respect to this condition, we can give the following theorem.

Theorem 3. The space curve $\overrightarrow{r}(s)$ is a general helix if and only if $\frac{\delta_{\overrightarrow{n^*}}}{\delta_{\overrightarrow{b^*}}} = 0$.

Next we consider three special cases: a) X lies in normal plane $T^{*\perp}$, b) X lies in osculating plane $B^{*\perp}$, c) X lies in rectifying plane $N^{*\perp}$.

a) If X lies in normal plane of γ, then $x_1(s) = 0$. From equation (29) the distribution parameter of the involute trajectory timelike ruled surface $\Psi_{\overrightarrow{X}}$ is

$$\delta_X = \frac{(c-s)\kappa \left\{ x_2\dot{x_3} - \dot{\theta} - \dot{x_2}x_3 \right\}}{\left| \dot{x_3}^2 - \dot{x_2}^2 + x_2^2 \left\|\overrightarrow{D}\right\|^2 - \dot{\theta}^2 + 2\dot{\theta}\left(x_2\dot{x_3} - \dot{x_2}x_3\right) \right|}. \tag{3.16}$$

Therefore, we state the following theorem.

Theorem 4. The involute trajectory timelike ruled surface $\Psi_{\overrightarrow{X}}$ is developable on the normal plane if and only if the Lorentzian timelike angle θ between $\overrightarrow{t}$ and $\overrightarrow{D}_0$ of space curve $\overrightarrow{r}(s)$ satisfies the following equality:

$$\theta = x_2x_3 - 2\int \dot{x_2}x_3 ds + \lambda, \tag{3.17}$$

where λ is an arbitrary constant.

b) If X lies in osculating plane of γ, then $x_3(s) = 0$. From equation (29) the distribution parameter of the involute trajectory timelike ruled surface $\Psi_{\overrightarrow{X}}$ is

$$\delta_X = \frac{(c-s)\kappa\dot{\theta}x_2^2}{\left| x_1^2 - \dot{x_2}^2 - \left\|\overrightarrow{D}\right\|^2 - \dot{\theta}^2 x_2^2 + 2\left\|\overrightarrow{D}\right\|\left(x_1\dot{x_2} - \dot{x_1}x_2\right) - 2\dot{\theta}x_1x_2 \right|}. \tag{3.18}$$

Thus we may give following result.

Result4. If the space curve $\overrightarrow{r}(s)$ is a general helix, then the involute trajectory timelike ruled surface $\Psi_{\overrightarrow{X}}$ is generated by a timelike oriented line $\overrightarrow{X}(s)$ is developable on the osculating plane in $\mathbb{R}^3_1$.

c) If X lies in rectifying plane of γ, then $x_2(s) = 0$. From equation (29) the distribution parameter of the involute trajectory timelike ruled surface $\Psi_{\overrightarrow{X}}$ is

$$\delta_X = \frac{(c-s)\kappa\left\{x_1x_3\left\|\vec{D}\right\| - \dot{\theta}x_3^2\right\}}{\left|x_1^2 - \dot{x_3}^2 - x_1^2\left\|\vec{D}\right\|^2 - x_3^2\dot{\theta}^2 + 2\left\|\vec{D}\right\|x_1x_3\dot{\theta}\right|}. \tag{3.19}$$

Hence, we can give the following theorem to characterize the developability on the rectifying plane in $\mathbb{R}_1^3$.

Theorem 5. The involute trajectory timelike ruled surface $\Psi_{\vec{X}}$ is developable on the rectifying plane if and only if the Lorentzian timelike angle θ between $\vec{t}$ and $\vec{D}_0$ of space curve $\vec{r}(s)$ satisfies the following equality:

$$\theta = \int \frac{x_1}{x_3}\left\|\vec{D}\right\| ds + \lambda, \tag{3.20}$$

where λ is an arbitrary constant.

The Case Where the Base Curve $\vec{\gamma}(s)$ is the Striction Curve $\vec{C}(s)$

From Eq. (20) the parametrization of the striction curve on the involute trajectory timelike ruled surface generated by timelike oriented line $\vec{X}(s)$ is given by

$$\vec{C}(s) = \vec{\gamma}(s) + \frac{(c-s)\kappa\left(\dot{x_1} - x_2\left\|\vec{D}\right\|\right)}{\left\|\dot{\vec{X}}\right\|^2}\vec{X}. \tag{3.21}$$

From here, if the base curve $\vec{\gamma}(s)$ is the striction curve $\vec{C}(s)$, then we have $\dot{x_1} = x_2\left\|\vec{D}\right\|$ Hence, the following theorem can be given.

Theorem 6. The base curve $\vec{\gamma}(s)$ is the same as the striction curve $\vec{C}(s)$ if and only if then the scalar functions of the arc length parameter of the involute $\vec{\gamma}(s)$ satisfy the following equality:

$$x_1 = \int x_2\left\|\vec{D}\right\| ds + \lambda, \tag{3.22}$$

where λ is an arbitrary constant.

3.1. Example

Let $\vec{r}(s) = \left(\sqrt{3}\sinh(s), \sqrt{3}\cosh(s), \sqrt{2}s\right)$ be a unit speed timelike helix with spacelike Darboux vector $\vec{D}$such that $\kappa = \sqrt{3}$and $\tau = \sqrt{2}$. The short calculations give

$$\begin{aligned} \vec{t}(s) &= \left(\sqrt{3}\cosh(s), \sqrt{3}\sinh(s), \sqrt{2}\right) \\ \vec{n}(s) &= (\sinh(s), \cosh(s), 0) \\ \vec{b}(s) &= \left(\sqrt{2}\cosh(s), \sqrt{2}\sinh(s), \sqrt{3}\right). \end{aligned} \tag{3.23}$$

In this situation, from Eq. (19), the involute $\vec{\gamma}(s)$ of the curve $\vec{r}(s)$ can be given by the equation

$$\vec{\gamma}(s) = \left(\sqrt{3}\sinh(s) + (c-s)\sqrt{3}\cosh(s), \sqrt{3}\cosh(s) + (c-s)\sqrt{3}\sinh(s), \sqrt{2}c\right). \tag{3.24}$$

From Eqs. (9) and (10) we have

$$\overrightarrow{D} = \sqrt{2}\overrightarrow{t} - \sqrt{3}\overrightarrow{b} \tag{3.25}$$

and

$$\begin{array}{l} \cosh\theta = \sqrt{3} \\ \sinh\theta = \sqrt{2} \end{array}, \tag{3.26}$$

respectively. Using Eq. (17), we have the Frenet trihedron of the involute $\overrightarrow{r}(s)$

$$\begin{array}{rcl} \overrightarrow{t^*}(s) & = & (\sinh(s), \cosh(s), 0) \\ \overrightarrow{n^*}(s) & = & (-\cosh(s), -\sinh(s), 0) \\ \overrightarrow{b^*}(s) & = & (0, 0, 1). \end{array} \tag{3.27}$$

Thus, we obtain the involute trajectory ruled surfaces generated by $\overrightarrow{t}^*$, $\overrightarrow{n}^*$ and $\overrightarrow{b}^*$ as

$$\begin{aligned} \Psi_{\overrightarrow{t^*}}(s,v) &= (\sqrt{3}\sinh(s) + (c-s)\sqrt{3}\cosh(s) + v\sinh(s), \\ &\quad \sqrt{3}\cosh(s) + (c-s)\sqrt{3}\sinh(s) + v\cosh(s), \sqrt{2}c) \qquad (3.28) \\ \Psi_{\overrightarrow{n^*}}(s,v) &= (\sqrt{3}\sinh(s) + (c-s)\sqrt{3}\cosh(s) - v\cosh(s), \\ &\quad \sqrt{3}\cosh(s) + (c-s)\sqrt{3}\sinh(s) - v\sinh(s), \sqrt{2}c) \qquad (3.29) \\ \Psi_{\overrightarrow{b^*}}(s,v) &= (\sqrt{3}\sinh(s) + (c-s)\sqrt{3}\cosh(s), \\ &\quad \sqrt{3}\cosh(s) + (c-s)\sqrt{3}\sinh(s), \sqrt{2}c + v) \qquad (3.30) \end{aligned}$$

respectively, where $v\epsilon[-5,5]$ and $c = 8$ (Figs. 1-3).

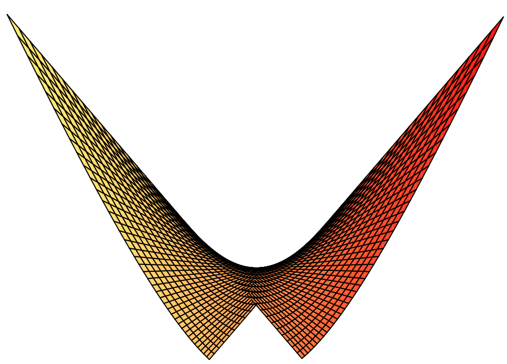

Figure 1: I.T.R.S $\Psi_{\overrightarrow{t^*}}(s,v)$ generated by $\overrightarrow{t^*}$ for $s\epsilon[-0.2, 0, 2]$

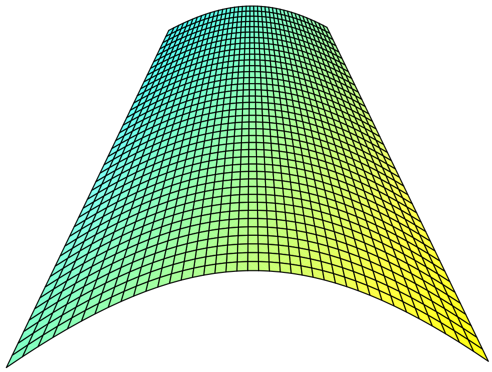

Figure 2: I.T.R.S $\Psi_{\overrightarrow{n^*}}(s,v)$ generated by $\overrightarrow{n^*}$ for $s\epsilon\,[-0.5,0,5]$

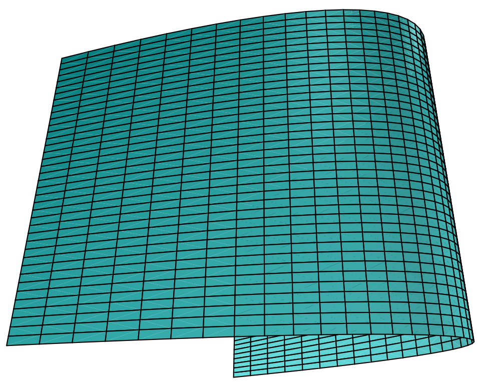

Figure 3: I.T.R.S $\Psi_{\overrightarrow{b^*}}(s,v)$ generated by $\overrightarrow{b^*}$ for $s\epsilon\,[-5,5]$

4. Conclusions

In this study, involute trajectory ruled surfaces $\Psi_{\overrightarrow{t}^*}(s,v)$, $\Psi_{\overrightarrow{n}^*}(s,v)$ and $\Psi_{\overrightarrow{b}^*}(s,v)$ generated by the Frenet trihedron $\left\{\overrightarrow{t}^*(s), \overrightarrow{n}^*(s), \overrightarrow{b}^*(s)\right\}$ moving along spacelike involute $\overrightarrow{\gamma}(s)$ of a given timelike space curve $\overrightarrow{r}(s)$ are stated according to Lorentzian timelike angle θ between the binormal vector $\overrightarrow{b}$ and the unit vector $\overrightarrow{D}_0$ of direction of Darboux vector $\overrightarrow{D}$ of the timelike evolute curve $\overrightarrow{r}(s)$. Also, some new results and theorems related to the developability of these surfaces are obtained. I hope that this paper will provide good motivation for new studies and contribute to the study of new special types of ruled surfaces.

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