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Ulam's Stability of Conformable Neutral Fractional Differential Equations

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ABSTRACT: This article is concerned with the existence and uniqueness of solutions of a nonlinear neutral fractional differential system with infinite delay, involving conformable fractional derivative. Additionally, we study the Ulam–Hyres stability, Ulam–Hyres–Mittag–Leffler stability, Ulam–Hyres–Mittag–Leffler–Rassias stability for the solutions of considered system using Picard operator. For application of the theory, we add an example at the end.

Key Words: Conformable fractional derivative, neutral fractional differential equations, Ulam–Hyres stability, Ulam–Hyers–Mittag–Leffler stability.

Contents

1. Introduction

Fractional differential equations (**FDEs**) have been considered as excellent tool in different fields of mechanics, electricity, biology, control and signal processing etc, to describe natural behavior and complex phenomena [6,18,19]. They also provide a better description over hereditary properties of various materials and processes, as a result many research papers and monographs have been published in this field see [7,17,16,10,25]. **FDEs** in which the highest fractional derivative of unknown term appears both with and without delays are known as neutral **FDEs**. In the last few years, the study of neutral **FDEs** have developed dramatically. This is due to the fact that the qualitative behavior of neutral **FDEs** is quite different from those of nonneutral **FDEs**. Neutral **FDEs** also play an important role and has many applications, for instance, it gives more better sketch of papulation fluctuations. Also, neutral **FDEs** with delay appear in models of electrical networks containing lossless transmission lines etc. [9].

In the literature there exists a number of definitions of fractional derivatives, but the most popular are Caupto and Riemann–Liouville fractional derivatives. It has been observed that these two types, and also some other fractional derivatives, dot not obey the classical chain rule. A new definition of fractional derivative known as conformable fractional derivative (CFD) was proposed by Khalil *et al.* [15], which obey all the properties satisfied by classical derivative. Some fundamental features of CFD have been discussed in [15,1] and for it's applications see [12,13].

The theory of investigating conformable fractional differential equations (**CFDEs**) is quite recent. Bayour and Torres [8] investigated the following problem for the existence of solutions using the criteria of tube solutions

$$\begin{cases} \mathcal{D}_0^{\alpha} z(t) = f(t, z(t)), \ t \in [a,b], a > 0 \ \alpha \in]0,1[, \\ x(a) = x_a, \end{cases}$$

where $\mathcal{D}_0^{\alpha}$ represent the CFD of order α.

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Ahmad *et al.* [2] obtained sufficient criteria for the existence of solutions to a new class of nonlinear neutral fractional differential equation involving two times Caputo-type fractional derivatives given by:

$$\begin{cases} {}^c\mathcal{D}_0^\alpha[{}^c\mathcal{D}_0^\beta x(t) - g(t,x(t))] = f(t,x(t)), t \in [0,T], T > 0, \\ x(0) = 0, {}^c\mathcal{D}_0^\gamma x(T) = \lambda(I^\delta x)(T) \end{cases}$$

where ${}^c\mathcal{D}_0^{(.)}$ is the Caputo fractional derivative each of order $\alpha, \beta, \gamma \in (0,1)$ and I^δ is the Riemann–Liouville fractional integral of order δ. The functions $f, g : [0,T] \times \mathbb{R} \to \mathbb{R}$ are continuous, $\lambda \neq \frac{\Gamma(1+\beta+\delta)}{T^{\gamma+\delta}\Gamma(1+\beta-\gamma)}$.

There exists a vast literature over existence and uniqueness of solutions related to **FDEs**, involving Caputo or Riemann–Liouville fractional derivatives [3,4,5,20,21,22,23,24], while in setting of **CFDEs** as for as we know existence, uniqueness and stabilities for neutral **CFDEs** have not been discussed.

Motivated by the work discussed above, in this article we study the existence and uniqueness, Ulam–Hyers (UH) stability, Ulam–Hyers–Mittag–Leffler (UHML) stability and Ulam–Hyers–Mittag–Leffler–Rassias (UHMLR) stability of the following nonlinear neutral **CFDEs**. The proposed system is given by:

$$\begin{cases} (\mathcal{D}_{0^+}^\beta - \lambda)[x(t) - g(t,x_t)] = f(t,x_t), t \in \mathrm{J} =]0,T], 0 < \beta \leq 1, \lambda, T > 0, \\ x_0 = \phi(t), t \in (-\infty, 0] \end{cases} \tag{1.1}$$

where $\mathcal{D}_{0^+}^\beta(.)$ represent the CFD of order β. The function $\phi \in \mathbf{B}$ with $\phi_t(t) = \phi(t)$, for $t \in (-\infty, 0]$ and $\mathbf{B}$ is a phase space of mappings from $(-\infty, 0]$ into the set of real numbers $\mathbb{R}$. The functions $f, g : \mathrm{J} \times \mathbf{B} \to \mathbb{R}$ are continuous satisfying certain conditions and $g(0, \phi(t)) = 0$.

The rest of the article is organized as follows: In section **2**, we recall some helpful definitions and results relating to both fractional derivatives and fractional integrals. Existence and uniqueness of solutions to the considered system (1.1) are discussed in section **3**. UH stability, UHML stability and UHMLR stability results are established in section **4**. A particular example is given in section **5**.

2. Auxiliary definitions and lemmas

This section is concerned with some notions, definitions and preliminary results used throughout this article.

Let $\mathcal{C}(\mathrm{J}, \mathbb{R})$ be the class of all real continuous functions and $\mathcal{L}^1(\mathrm{J}, \mathbb{R})$ be the space of all locally lebesgue integrable real functions. We also consider $\mathcal{C}_{1-\alpha}^\beta(\mathrm{J}, \mathbb{R})$, the Banach space of all continuous functions $x : \mathrm{J} \to \mathbb{R}$ such that $\lim_{t\to 0} x(t)$ exists with norm $\|x\|_{\mathcal{C}_{1-\alpha}^\beta} = \max\{|x(t)| : t \in \mathrm{J}\}$. If $x : (-\infty, T] \to \mathbb{R}$ is continuous, then $\forall\ t \in \mathrm{J}$, the function x_t is defined by $x_t(\tau) = x(t+\tau), \tau \in (-\infty, 0]$.

Consider the space

$$\Omega = \big\{x : (-\infty, T] \to \mathbb{R}, x|_{(-\infty,T]} \in \mathbf{B}, x|_{\mathrm{J}} \in \mathcal{C}_{1-\alpha}^\beta(\mathrm{J}, \mathbb{R})\big\},$$

$x|_{\mathrm{J}}$ is restriction of x into the interval J.

Definition 2.1. *[1,15] The* CFD *of a function $f : [0,\infty) \to \mathbb{R}$ starting from point 0 of order β is defined as*

$$\mathcal{D}_{0^+}^\beta f(t) = \lim_{\epsilon\to 0} \frac{f(t + \epsilon t^{1-\beta}) - f(t)}{\epsilon}.$$

If $\mathcal{D}_{0^+}^\beta f(t)$ exists on the interval (a,b), then $\mathcal{D}_{0^+}^\beta f(0) = \lim_{t\to 0} f(t)$.

Definition 2.2. *[1,15] The conformable fractional integral of a function $f : [0,\infty) \to \mathbb{R}$ starting from point 0 of order β is defined as*

$$\mathcal{I}_{0^+}^\beta f(t) = \int_0^t s^{\beta-1} f(s) ds.$$

Lemma 2.3. *[1,15] If $f : [a,\infty) \to \mathbb{R}$ is continuous, then $\forall\ t > a$*

$$\mathcal{D}_{0^+}^\beta \mathcal{I}_{0^+}^\beta f(t) = f(t).$$

Lemma 2.4. *[11] If $\mathcal{D}^{\beta}_{0+}f(t)$ is continuous on the interval $[0,\infty)$, then*

$$\mathcal{I}^{\beta}_{0+}\mathcal{D}^{\beta}_{0+}f(t) = f(t) - f(0).$$

Lemma 2.5. *[1,15] If the function f is $\beta-$differentiable at $t \in [0,\infty)$, then*

$$\mathcal{D}^{\beta}_{0+}f(t) = t^{1-\beta}f'(t).$$

Lemma 2.6. *[15] If the functions f and g are $\beta-$differentiable at $t \in [a,b]$, then fg is also $\beta-$differentiable at $t \in [a,b]$, and*

$$\mathcal{D}^{\beta}_{0+}(fg)(t) = f(t)\mathcal{D}^{\beta}_{0+}g(t) + g(t)\mathcal{D}^{\beta}_{0+}f(t).$$

Lemma 2.7. *[15] If $f : [a,b] \to \mathbb{R}$ is continuous on the interval $[c,d] \subset [a,b]$ and if $\mathcal{D}^{\beta}_{0+}f(t)$ exists on the open interval (c,d), then there exists a point $\xi \in (c,d)$ such that*

$$f(d) - f(c) = \frac{1}{\beta}\mathcal{D}^{\beta}_{0+}f(\xi)[(d-a)^{\beta} - (c-a)^{\beta}].$$

Lemma 2.8. *[14] (Banach fixed point theorem) Let $\mathcal{C} \neq \emptyset$ be a closed subset of a Banach space Y, then any contraction mapping $\mathcal{P} : \mathcal{C} \to \mathcal{C}$ has a unique fixed point.*

Lemma 2.9. *[14] (Schaefer's fixed point theorem) Let $\mathcal{P}$ be a completely continuous operator from a Banach space Y into itself. If the set*

$$\bar{Y} = \{y \in Y : y = \eta\mathcal{P}y\}$$

is bounded for some $0 < \eta < 1$, then $\mathcal{P}$ has fixed points.

In this paper we consider the phase space $(\mathbf{B}, \|.\|)$ which is a semi–normed linear space of mappings from $(-\infty, 0]$ into $\mathbb{R}$ fulfilling the following axioms.

$\mathcal{A}_0$ If $x : (-\infty, 0] \to \mathbb{R}$ is continuous on J, then $\forall\ t \in$ J and each $x_0 \in \mathbf{B}$ the following conditions are satisfied.

$\mathcal{A}_1$ $x_t \in \mathbf{B}$;

$\mathcal{A}_2$ There exists a constant $\mathcal{H} > 0$ such that $|x(t)| \leq \mathcal{H}\|x_t\|_{\mathbf{B}}$, $x : \mathrm{J} \to \mathbf{B}$ is continuous;

$\mathcal{A}_3$ There exists a continuous function $\mathcal{K}(t)$ and a locally bounded function $\mathcal{M}(t)$ both independent from $x(.)$ such that $\|x_t\|_{\mathbf{B}} \leq \mathcal{K}(t)\sup_{0\leq s\leq t}|x(s)| + \mathcal{M}(t)\|x_0\|_{\mathbf{B}}$. Take $\mathcal{K}_a = \sup\{\mathcal{K}(t), t \in \mathrm{J}\}$ and $\mathcal{M}_a = \sup\{\mathcal{M}(t), t \in \mathrm{J}\}$.

$\mathcal{A}_4$ The space $\mathbf{B}$ is complete.

It can be observed that $\mathcal{A}_4$ is equivalent to $|\phi(0)| \leq \mathcal{H}\|\phi\|_{\mathbf{B}}$, $\forall\ \phi \in \mathbf{B}$.

3. Main Results

In this section we demonstrate and exhibit the existence and uniqueness for the solution of the considered system on the interval $(-\infty, T]$ under Banach contraction principle and Schaefer's fixed point theorem. We also discuss the UHML stability and UMLR stability for the solution of considered problem (1.1). Before coming to the main results we assume some hypothesis as follows:

$\mathcal{H}_1$ Let $f, g : \mathrm{J} \times \mathbf{B} \to \mathbb{R}$ are continuous and there exist constants $\mathcal{Q}_f, \mathcal{Q}_g > 0$ such that $\forall\ t \in$ J and each $x, x^* \in \mathbf{B}$

$$|f(t,x) - f(t,x^*)| \leq \mathcal{Q}_f\|x - x^*\|_{\mathbf{B}}$$

and

$$|g(t,x) - g(t,x^*)| \leq \mathcal{Q}_g\|x - x^*\|_{\mathbf{B}};$$

$\mathcal{H}_2$ There exists constants $\mathcal{M}_f > 0$ such that $\forall\ x, y \in \mathbf{B}$ and $t \in \mathrm{J}$

$$|f(t,x)| \leq \mathcal{M}_f \|x\|_{\mathbf{B}}.$$

$\mathcal{H}_3$ Let g is completely continuous and for any bounded subset of Ω, $\{t \to g(t,x(t)) : x \in \mathbf{B}\}$ is equicontinuous in $\mathcal{C}^{\beta}_{1-\alpha}(\mathrm{J},\mathbb{R})$ and there exist constants $b_1 \in [0,1), b_2 > 0$ such that $|g(t,x)| \leq b_2 + b_1\|x\|_{\mathbf{B}}$, $t \in \mathrm{J}, x \in \mathbf{B}$.

Theorem 3.1. *Let $0 < \beta \leq 1$, then any solution $x \in \Omega$ of*

$$\begin{cases} (\mathcal{D}^{\beta}_{0+} - \lambda)[x(t) - g(t,x_t)] = f(t,x_t),\ t \in \mathrm{J}, \\ x_0 = \phi(t), t \in (-\infty, 0] \end{cases}$$

has the form

$$x(t) = g(t,x_t) + e^{\lambda\frac{t^\beta}{\beta}}\phi(0) + e^{\lambda\frac{t^\beta}{\beta}}\int_0^t s^{\beta-1}e^{-\lambda\frac{s^\beta}{\beta}}f(s,x_s)ds, t \in \mathrm{J}. \tag{3.1}$$

Proof. Consider

$$(\mathcal{D}^{\beta}_{0+} - \lambda)[x(t) - g(t,x_t)] = f(t,x_t),\ 0 < \beta \leq 1,\ t \in \mathrm{J}. \tag{3.2}$$

Using the approach of variation of constants method to represent the solutions of (3.2). Let any solution of (3.2) takes the form

$$x(t) = g(t,x_t) + e^{\lambda\frac{t^\beta}{\beta}}u(t), \tag{3.3}$$

where the unknown function $u(.)$ is continuously differentiable and should be determined. From equation (3.3) and Lemma 2.5, one has

$$\mathcal{D}^{\beta}_{0+}x(t) = \mathcal{D}^{\beta}_{0+}g(t,x_t) + \mathcal{D}^{\beta}_{0+}e^{\lambda\frac{t^\beta}{\beta}}u(t),$$

or

$$\begin{aligned} \mathcal{D}^{\beta}_{0+}[x(t) - g(t,x_t)] &= \mathcal{D}^{\beta}_{0+}e^{\lambda\frac{t^\beta}{\beta}}u(t) \\ &= t^{1-\beta}e^{\lambda\frac{t^\beta}{\beta}}\lambda\beta\frac{t^{\beta-1}}{\beta}u(t) + t^{1-\beta}e^{\lambda\frac{t^\beta}{\beta}}u'(t) \\ &= \lambda e^{\lambda\frac{t^{\beta-1}}{\beta}}u(t) + e^{\lambda\frac{t^\beta}{\beta}}t^{1-\beta}u'(t) \\ &= \lambda[x(t) - g(t,x_t)] + e^{\lambda\frac{t^\beta}{\beta}}\mathcal{D}^{\beta}_{0+}u(t). \end{aligned}$$

This yields

$$\begin{aligned} e^{\lambda\frac{t^\beta}{\beta}}\mathcal{D}^{\beta}_{0+}u(t) &= (\mathcal{D}^{\beta}_{0+} - \lambda)[x(t) - g(t,x_t)] \\ &= f(t,x_t), \end{aligned}$$

which gives

$$\mathcal{D}^{\beta}_{0+}u(t) = e^{-\lambda\frac{t^\beta}{\beta}}f(t,x_t).$$

Integrating from 0 to t, we obtained

$$u(t) = u(0) + \int_0^t e^{-\lambda\frac{s^\beta}{\beta}}s^{\beta-1}f(s,x_s), \tag{3.4}$$

where

$$u(0) = \phi(0).$$

Therefore equation (3.3) becomes

$$x(t) = g(t,x_t) + e^{\lambda\frac{t^\beta}{\beta}}\phi(0) + e^{\lambda\frac{t^\beta}{\beta}}\int_0^t e^{-\lambda\frac{s^\beta}{\beta}}s^{\beta-1}f(s,x_s)ds.$$

For $t \in (-\infty, 0]$, $x_0 = \phi(t)$. Thus $\phi(t) = x_0(t) = x(0+t) = x(t)$, which implies that $x(t) = \phi(t)$ for $t \in (-\infty, 0]$. □

Theorem 3.2. *Let $\mathcal{H}_1$ is true, then the system (1.1) has a unique solution on $(-\infty, T]$, provided that*

$$\left[\mathcal{Q}_g + \frac{\mathcal{Q}_f}{\lambda}\left(e^{\lambda\frac{T^\beta}{\beta}} - 1\right)\right]\mathcal{K}_a < 1. \tag{3.5}$$

Proof. Transform the considered system (1.1) into equivalent fixed point problem. Define an operator $\mathcal{T} : \Omega \to \Omega$ by:

$$(\mathcal{T}x)(t) = \begin{cases} g(t, x_t) + e^{\lambda\frac{t^\beta}{\beta}}\phi(0) + e^{\lambda\frac{t^\beta}{\beta}}\displaystyle\int_0^t e^{-\lambda\frac{s^\beta}{\beta}} s^{\beta-1} f(s, x_s)ds, \ t \in \mathrm{J}, \\ \phi(t), t \in (-\infty, 0]. \end{cases} \tag{3.6}$$

For every function $\hbar \in \mathcal{C}^\beta_{1-\alpha}(\mathrm{J}, \mathbb{R})$ along with the condition that $\hbar(0) = 0$, we consider $\widetilde{\hbar} : (-\infty, T] \to \mathbb{R}$, the extension of $\hbar$ into $(-\infty, T]$ defined by

$$\widetilde{\hbar}(t) = \begin{cases} \hbar(t), \quad t \in \mathrm{J}, \\ 0, \quad t \in (-\infty, 0]. \end{cases} \tag{3.7}$$

Also suppose $\widetilde{\phi} : (-\infty, T] \to \mathbb{R}$ be the extension of $\phi \in \mathbf{B}$ into $(-\infty, T]$ satisfying

$$\widetilde{\phi}(t) = \begin{cases} 0, \quad t \in \mathrm{J}, \\ \phi(t), \quad t \in (-\infty, 0]. \end{cases} \tag{3.8}$$

Then $\widetilde{\phi}_0 = \phi$ and $\widetilde{\hbar}_0 = 0$.

Now if $x(.)$ satisfies (3.1), then we can analyze $x(.)$ as $x(t) = \widetilde{\hbar}(t) + \widetilde{\phi}(t)$ for $t \in \mathrm{J}$, which implies that $x_t = \widetilde{\hbar}_t + \widetilde{\phi}_t$, where $\hbar$ satisfies

$$\hbar(t) = g(t, \widetilde{\hbar}_t + \widetilde{\phi}(t)) + \phi(0)e^{\lambda\frac{t^\beta}{\beta}} + e^{\lambda\frac{t^\beta}{\beta}}\int_0^t e^{-\lambda\frac{t^\beta}{\beta}} s^{\beta-1} f(s, \widetilde{\hbar}_s + \widetilde{\phi}_s)ds, \ t \in \mathrm{J}. \tag{3.9}$$

Setting

$$\underline{\Omega} = \{\hbar \in \Omega \ \text{ with } \ \hbar_0 = 0\}.$$

For any $\hbar \in \underline{\Omega}$, let $\|.\|_{\underline{\Omega}}$ denotes the semi-norm on $\underline{\Omega}$ described as

$$\|\hbar\|_{\underline{\Omega}} = \|\hbar_0\|_{\mathbf{B}} + \|\hbar\|_{\mathcal{C}^\alpha_{1-\beta}} = \sup_{t\in \mathrm{J}}\{|\hbar|, \ \hbar \in \Omega \ \texttt{with} \ \hbar_0 = 0\},$$

then $(\|\hbar\|_{\underline{\Omega}}, \|.\|_{\underline{\Omega}})$ form a Banach space.

Define an operator $\mathcal{P} : \underline{\Omega} \to \underline{\Omega}$ by

$$(\mathcal{P}\hbar)(t) = g(t, \widetilde{\hbar}_t + \widetilde{\phi}(t)) + e^{\lambda\frac{t^\beta}{\beta}}\phi(0) + e^{\lambda\frac{t^\beta}{\beta}}\int_0^t e^{-\lambda\frac{t^\beta}{\beta}} s^{\beta-1} f(s, \widetilde{\hbar}_s + \widetilde{\phi}_s)ds, \ t \in \mathrm{J}$$

with $(\mathcal{P}\hbar)(0) = 0$. Obviously the fixed point of the operator $\mathcal{T}$ is the solution of system (1.1) and it's equivalent fixed point of $\mathcal{P}$ is the solution of (3.9).

Now by means of Banach fixed point theorem we will show that $\mathcal{P}$ has a fixed point in $\underline{\Omega}$, equivalently $\mathcal{P} : \underline{\Omega} \to \underline{\Omega}$ is contraction mapping. For any $\hbar, \hbar^* \in \underline{\Omega}$ and $\forall \ t \in \mathrm{J}$, it follows that

$$\begin{aligned}
& |(\mathcal{P}\hbar)(t) - (\mathcal{P}\hbar^*)(t)| \\
\leq \ & \left|g(t, \widetilde{\hbar}_t + \widetilde{\phi}_t) - g(t, \widetilde{\hbar}^*_t + \widetilde{\phi}_t)\right| + e^{\lambda\frac{t^\beta}{\beta}}\int_0^t e^{-\lambda\frac{t^\beta}{\beta}} s^{\beta-1}\left|f(s, \widetilde{\hbar}_s + \widetilde{\phi}_s) - f(s, \widetilde{\hbar}^*_s + \widetilde{\phi}_s)\right|ds \\
\leq \ & \mathcal{Q}_g\|\widetilde{\hbar}_t - \widetilde{\hbar}^*_t\|_{\mathbf{B}} + \mathcal{Q}_f\|\widetilde{\hbar}_t - \widetilde{\hbar}^*_t\|_{\mathbf{B}} e^{\lambda\frac{t^\beta}{\beta}}\int_0^t e^{-\lambda\frac{t^\beta}{\beta}} s^{\beta-1}ds \\
= \ & \mathcal{Q}_g\|\widetilde{\hbar}_t - \widetilde{\hbar}^*_t\|_{\mathbf{B}} + \frac{\mathcal{Q}_f}{\lambda}\|\widetilde{\hbar}_t - \widetilde{\hbar}^*_t\|_{\mathbf{B}}\left(e^{\lambda\frac{t^\beta}{\beta}} - 1\right) \\
\leq \ & \mathcal{Q}_g\|\widetilde{\hbar}_t - \widetilde{\hbar}^*_t\|_{\mathbf{B}} + \frac{\mathcal{Q}_f}{\lambda}\|\widetilde{\hbar}_t - \widetilde{\hbar}^*_t\|_{\mathbf{B}}\left(e^{\lambda\frac{T^\beta}{\beta}} - 1\right), \forall \ t \in \mathrm{J}.
\end{aligned}$$

Now by $\mathcal{A}_1$ (iii) $\forall\, t\in \mathrm{J}$, we have

$$\begin{aligned}\|\widetilde{\hbar}_t-\widetilde{\hbar}^*_t\|_{\mathbf{B}} &= \|\hbar_t-\hbar^*_t\|_{\mathbf{B}}\\ &\le \mathcal{K}(t)\sup_{0\le\tau\le t}\big|\hbar(\tau)-\hbar^*(\tau)\big|+\mathcal{M}(t)\big\|\hbar_0-\hbar^*_0\big\|_{\mathbf{B}}\\ &\le \mathcal{K}_a\|\hbar-\hbar^*\|_{\underline{\Omega}}+\mathcal{M}(t)(0)\\ &= \mathcal{K}_a\|\hbar-\hbar^*\|_{\underline{\Omega}},\ \forall\, t\in\mathrm{J}.\end{aligned}$$

Therefore,

$$\sup_{t\in \mathrm{J}}|(\mathcal{P}\hbar)(t)-(\mathcal{P}\hbar^*)(t)|\le \mathcal{Q}_g\mathcal{K}_a\|\hbar-\hbar^*\|_{\underline{\Omega}}+\frac{\mathcal{Q}_f}{\lambda}\left(e^{\lambda\frac{T^\beta}{\beta}}-1\right)\mathcal{K}_a\|\hbar-\hbar^*\|_{\underline{\Omega}}.$$

Hence,

$$\|\mathcal{P}\hbar-\mathcal{P}\hbar^*\|_{\underline{\Omega}} \le \left[\mathcal{Q}_g+\frac{\mathcal{Q}_f}{\lambda}\left(e^{\lambda\frac{T^\beta}{\beta}}-1\right)\right]\mathcal{K}_a\|\hbar-\hbar^*\|_{\underline{\Omega}}.$$

It follows from the inequality (3.5) that $\mathcal{P}$ is a contraction mapping on $\underline{\Omega}$. The application of Banach fixed point theorem gives $\mathcal{P}$ has a unique fixed point which is the unique solution of (3.9) on J.

Set $x(t)=\widetilde{\hbar}(t)+\widetilde{\phi}(t)$, then $x(t)$ is the unique solution of system (1.1) on $(-\infty,T]$. □

Theorem 3.3. *If the hypothesis $\mathcal{H}_1$ to $\mathcal{H}_3$ are true with $1-\mathcal{Q}_{1a}\mathcal{K}_a\neq 0$, then the system (1.1) has at least one solution on the interval $(-\infty,T]$, where*

$$\mathcal{Q}_{1a}=\left[b_1+\frac{\mathcal{M}_f}{\lambda}\big(e^{\lambda\frac{T^\beta}{\beta}}-1\big)\right],$$
$$\mathcal{Q}_{2a}=b_2+\big(\mathcal{H}+\mathcal{Q}_{1a}\mathcal{M}_a\big)\|\phi\|_{\mathbf{B}}.$$

Proof. Consider the operator $\mathcal{P}:\underline{\Omega}\to\underline{\Omega}$ as defined in (3.6). We shall show that $\mathcal{P}$ has a fixed point. The proof will be completed in the following steps.

Step 1 : Let $\{\hbar_n\}_{n\in\mathbb{N}}$ be a sequence in $\underline{\Omega}$ such that $\hbar_n\to\hbar$ in $\underline{\Omega}$ as $n\to\infty$.

Now for any $t\in\mathrm{J}$, we have

$$\begin{aligned}|(\mathcal{P}\hbar_n)(t)-(\mathcal{P}\hbar^*)(t)| &\le \big|g(t,(\widetilde{\hbar}_n)_t+\widetilde{\phi}_t)-g(t,\widetilde{\hbar}_t+\widetilde{\phi}_t)\big|\\ &\quad +e^{\lambda\frac{t^\beta}{\beta}}\int_0^t e^{-\lambda\frac{t^\beta}{\beta}}s^{\beta-1}\big|f(s,((\widetilde{\hbar}_n)_s+\widetilde{\phi}_s))-f(s,\widetilde{\hbar}_s+\widetilde{\phi}_s)\big|ds\\ &\le \mathcal{Q}_g\|(\widetilde{\hbar}_n)_t-\widetilde{\hbar}_t\|_{\mathbf{B}}+\mathcal{Q}_f\|(\widetilde{\hbar}_n)_t-\hbar_t\|_{\mathbf{B}}e^{\lambda\frac{t^\beta}{\beta}}\int_0^t e^{-\lambda\frac{t^\beta}{\beta}}s^{\beta-1}ds\\ &\le \mathcal{Q}_g\mathcal{K}_a\|\hbar_n-\hbar\|_{\underline{\Omega}}+\frac{\mathcal{K}_a\mathcal{Q}_f}{\lambda}\left(e^{\lambda\frac{t^\beta}{\beta}}-1\right)\|\hbar_n-\hbar\|_{\underline{\Omega}}\\ &= \left[\mathcal{Q}_g+\frac{\mathcal{Q}_f}{\lambda}\left(e^{\lambda\frac{t^\beta}{\beta}}-1\right)\right]\mathcal{K}_a\|\hbar_n-\hbar\|_{\underline{\Omega}}\to 0,\ \text{as}\ \ n\to\infty,\forall\, t\in\mathrm{J}.\end{aligned}$$

This implies that

$$\|\mathcal{P}\hbar_n-\mathcal{P}\hbar\|_{\underline{\Omega}}\to 0,\ \text{as}\ n\to\infty.$$

So that $\mathcal{P}$ is continuous.

Step 2 : Consider a closed ball $\mathcal{B}_{r_0}=\{\hbar:\|\hbar\|_{\underline{\Omega}}\le r_0, r_0\ge\frac{\mathcal{Q}_{2a}}{1-\mathcal{Q}_{1a}}\}$. For any $\hbar\in\mathcal{B}_{r_0}$ and $t\in\mathrm{J}$, we have

$$\begin{aligned}|(\mathcal{P}\hbar)(t)| &\le \big|g(t,\widetilde{\hbar}_t+\widetilde{\phi}_t)\big|+e^{\lambda\frac{t^\beta}{\beta}}\big|\phi(0)\big|+e^{\lambda\frac{t^\beta}{\beta}}\int_0^t e^{-\lambda\frac{t^\beta}{\beta}}s^{\beta-1}\big|f(s,\widetilde{\hbar}_s+\widetilde{\phi}_s)\big|ds\\ &\le b_2+b_1\|\widetilde{\hbar}_t+\widetilde{\phi}_t\|_{\mathbf{B}}+\mathcal{H}\|\phi\|_{\mathbf{B}}+e^{\lambda\frac{t^\beta}{\beta}}\mathcal{M}_f\int_0^t e^{-\lambda\frac{t^\beta}{\beta}}s^{\beta-1}\|\widetilde{\hbar}_t+\widetilde{\phi}_t\|_{\mathbf{B}}ds\\ &\le b_2+b_1\|\widetilde{\hbar}_t+\widetilde{\phi}_t\|_{\mathbf{B}}+\mathcal{H}\|\phi\|_{\mathbf{B}}+\frac{\mathcal{M}_f}{\lambda}\big(e^{\lambda\frac{T^\beta}{\beta}}-1\big)\|\widetilde{\hbar}_t+\widetilde{\phi}_t\|_{\mathbf{B}}.\end{aligned}$$

Therefore,

$$\|\mathcal{P}\hbar\|_{\underline{\Omega}} \leq b_2 + \mathcal{H}\|\phi\|_{\mathbf{B}} + \Big[b_1 + \frac{\mathcal{M}_f}{\lambda}\big(e^{\lambda\frac{T^\beta}{\beta}} - 1\big)\Big]\|\widetilde{\hbar}_t + \widetilde{\phi}_t\|_{\mathbf{B}}, \quad t \in \mathrm{J}. \tag{3.10}$$

Since

$$\begin{aligned}
\|\widetilde{\hbar}_t + \widetilde{\phi}_t\|_{\mathbf{B}} &\leq \mathcal{K}(t)\sup_{0\leq\tau\leq t}|\widetilde{\hbar}(\tau) + \widetilde{\phi}(\tau)| + \mathcal{M}(t)\|\widetilde{\hbar}_0 + \widetilde{\phi}_0\|_{\mathbf{B}} \\
&\leq \mathcal{K}(t)\sup_{0\leq\tau\leq t}|\widetilde{\hbar}(\tau)| + \mathcal{K}(t)\sup_{0\leq\tau\leq t}|\widetilde{\phi}(\tau)| + \mathcal{M}_a\|\phi\|_{\mathbf{B}} \\
&\leq \mathcal{K}_a\sup_{0\leq\tau\leq t}|\hbar(\tau)| + \mathcal{M}_a\|\phi\|_{\mathbf{B}} \\
&= \mathcal{K}_a\|\hbar\|_{\underline{\Omega}} + \mathcal{M}_a\|\phi\|_{\mathbf{B}} \\
&\leq \mathcal{K}_a r_0 + \mathcal{M}_a\|\phi\|_{\mathbf{B}}.
\end{aligned}$$

Therefore the inequality (3.10) becomes

$$\begin{aligned}
\|\mathcal{P}h\|_{\underline{\Omega}} &\leq b_2 + \mathcal{H}\|\phi\|_{\mathbf{B}} + \Big[b_1 + \frac{\mathcal{M}_f}{\lambda}\big(e^{\lambda\frac{T^\beta}{\beta}} - 1\big)\Big](\mathcal{K}_a r_0 + \mathcal{M}_a\|\phi\|_{\mathbf{B}}) \\
&= b_2 + \big(\mathcal{H} + \mathcal{Q}_{1a}\mathcal{M}_a\big)\|\phi\|_{\mathbf{B}} + \mathcal{Q}_{1a}\mathcal{K}_a r_0 \\
&= \mathcal{Q}_{2a} + \mathcal{Q}_{1a}\mathcal{K}_a r_0 \\
&\leq r_0.
\end{aligned}$$

This implies that the operator $\mathcal{P}$ maps a bounded subset of Ω into a bounded subset of Ω.

Step 3 : For equicontinuity let $h \in \mathcal{B}_{r_0}$ and for any $t_1, t_2 \in \mathcal{J}$ with $t_1 < t_2$ and $t_2 - t_1 < \eta$, we have

$$\begin{aligned}
&\big|(\mathcal{P}\hbar)(t_2) - (\mathcal{P}\hbar)(t_1)\big| \\
\leq\ & \big|g(t_2, \widetilde{\hbar}_{t_2} + \widetilde{\phi}_{t_2}) - g(t_1, \widetilde{\hbar}_{t_1} + \widetilde{\phi}_{t_1})\big| \\
&+ \Big(e^{\lambda\frac{t_2^\beta}{\beta}} - e^{\lambda\frac{t_1^\beta}{\beta}}\Big)\Big(|\phi(0)| + \int_0^{t_1} e^{-\lambda\frac{t^\beta}{\beta}} s^{\beta-1}\big|f(s, \widetilde{\hbar}_s + \widetilde{\phi}_s)\big|ds\Big) \\
&+ \int_{t_1}^{t_2} e^{\lambda\frac{t_2^\beta}{\beta}} e^{-\lambda\frac{t^\beta}{\beta}} s^{\beta-1}\big|f(s, \widetilde{\hbar}_s + \widetilde{\phi}_s)\big|ds \\
\leq\ & \mathcal{Q}_g\mathcal{K}_a\|\hbar_{t_2} - \hbar_{t_1}\|_\Omega + \lambda\Big(\mathcal{H}\|\phi\|_{\mathbf{B}} + \frac{\mathcal{M}_f(\mathcal{K}_a r_0 + \mathcal{M}_a\|\phi\|_{\mathbf{B}})}{\beta} t_1^\beta\Big)\zeta^{\beta-1} e^{\frac{\zeta^\beta}{\beta}}(t_2 - t_1) \\
&+ \frac{\mathcal{M}_f}{\beta}(\mathcal{K}_a r_0 + \mathcal{M}_a\|\phi\|_{\mathbf{B}}) e^{\lambda\frac{t_2^\beta}{\beta}}(t_2 - t_1)^\beta, \quad \zeta \in (t_1, t_2).
\end{aligned}$$

The right hand side of last inequality approaches to zero as ζ tends to zero, it follows that $\big|(\mathcal{P}x)(t_2) - (\mathcal{P}x)(t_1)\big| \to 0$ as $\zeta \to 0$. By Arzelä−Ascoli theorem, we conclude that the operator $\mathcal{P}$ is continuous and completely continuous.

Step 4 : It remains to show that the set

$$Y = \{\hbar \in \underline{\Omega} : \hbar = \eta\mathcal{P}\hbar\}$$

is bounded for some $0 < \eta < 1$. On contrary suppose that Y is unbounded, that is for $\hbar \in Y$, $\|\hbar\|_{\underline{\Omega}} = K_1 \to \infty$. But for any $t \in \mathrm{J}$, we have

$$\begin{aligned}
\|\hbar\|_{\underline{\Omega}} &= \|\eta\mathcal{P}\hbar\|_{\underline{\Omega}} \\
&\leq \|\mathcal{P}\hbar\|_{\underline{\Omega}} \\
&\leq \mathcal{Q}_{1a} r_0 + \mathcal{Q}_{2a}.
\end{aligned}$$

Dividing both sides by $\|\hbar\|_{\underline{\Omega}}$, we get

$$1 \leq \frac{\mathcal{Q}_{1a} r_0 + \mathcal{Q}_{2a}}{\|\hbar\|_{\underline{\Omega}}} = \frac{\mathcal{Q}_{1a} r_0 + \mathcal{Q}_{2a}}{K_1} \to 0.$$

Which is a contradiction, thus the set Y is bounded. Schaefer's fixed point theorem guarantees that there exists at least one fixed point $\hbar$ of $\mathcal{P}$ in $\underline{\Omega}$. Then $\hbar = \widetilde{\hbar} + \widetilde{\phi}$ is at least one solution of (1.1) on $(-\infty, T]$. This completes the proof. □

4. Hyres–Ulam stability analysis

In this section we are establishing Hyres–Ulam stability and Hyres–Ulam–Mittag–Leffler stability results.

Definition 4.1. *The considered system (1.1) has Hyres–Ulam stability if we can find constant $\lambda^1 > 0$ such that for each $\epsilon > 0$ and every $x \in \Omega$ satisfying*

$$\left|(\mathcal{D}^{\beta}_{0+} - \lambda)[x(t) - g(t, x_t)] - f(t, x_t)\right| \leq \epsilon \tag{4.1}$$

there exists $y \in \Omega$ with

$$(\mathcal{D}^{\beta}_{0+} - \lambda)[y(t) - g(t, y_t)] = f(t, y_t) \tag{4.2}$$

satisfying

$$|x(t) - y(t)| \leq \lambda^1 \epsilon, t \in \mathrm{J}.$$

Definition 4.2. *The considered system (1.1) has Hyres–Ulam–Mittag–Leffler stability with respect to $\mathrm{E}_{\beta}(t^{\beta})$ if we can find constant $\lambda^* > 0$ such that for each $\epsilon^* > 0$ and every $x \in \Omega$ satisfying*

$$\left|(\mathcal{D}^{\beta}_{0+} - \lambda)[x(t) - g(t, x_t)] - f(t, x_t)\right| \leq \epsilon^* \mathrm{E}_{\beta}(t^{\beta}) \tag{4.3}$$

there exists $y \in \Omega$ with

$$(\mathcal{D}^{\beta}_{0+} - \lambda)[y(t) - g(t, y_t)] = f(t, y_t) \tag{4.4}$$

satisfying

$$|x(t) - y(t)| \leq \lambda^* \epsilon^* \mathrm{E}_{\beta}(t^{\beta}), t \in \mathrm{J},$$

where $\mathrm{E}_{\beta}(t^{\beta})$ is Mittag–Leffler function given by:

$$\mathrm{E}_{\beta}(t^{\beta}) = \sum_{m=0}^{\infty} \frac{t^{m\beta}}{\Gamma(m\beta + 1)}.$$

Definition 4.3. *The considered system (1.1) has UHMLR stability with respect to $\mathrm{E}_{\beta}(t^{\beta})$ if we can find constant $\lambda^{**} > 0$ and function $\varphi : \mathrm{J} \to \mathbb{R}_+$ such that for each $\epsilon^{**} > 0$ and every $x \in \Omega$ satisfying*

$$\left|(\mathcal{D}^{\beta}_{0+} - \lambda)[x(t) - g(t, x_t)] - f(t, x_t)\right| \leq \epsilon^* \varphi(t) \mathrm{E}_{\beta}(t^{\beta}) \tag{4.5}$$

there exists $y \in \Omega$ with

$$(\mathcal{D}^{\beta}_{0+} - \lambda)[y(t) - g(t, y_t)] = f(t, y_t) \tag{4.6}$$

satisfying

$$|x(t) - y(t)| \leq \lambda^{**} \epsilon^{**} \varphi(t) \mathrm{E}_{\beta}(t^{\beta}), t \in \mathrm{J}.$$

Remark 4.4. *A function $x \in \Omega$ satisfies the inequality (4.1) if and only there exists $\theta \in \mathcal{C}(\mathrm{J}, \mathbb{R})$ such that*

$$|\theta(t)| \leq \epsilon \text{ and } (\mathcal{D}^{\beta}_{0+} - \lambda)[x(t) - g(t, x_t)] = f(t, x_t) + \theta(t), t \in \mathrm{J}.$$

Remark 4.5. *A function $x \in \Omega$ satisfies the inequality (4.2), if and only there exists $\theta^* \in \mathcal{C}(\mathrm{J}, \mathbb{R})$ such that*

$$|\theta^*(t)| \leq \epsilon^* \mathrm{E}_{\beta}(t^{\beta}) \text{ and } (\mathcal{D}^{\beta}_{0+} - \lambda)[x(t) - g(t, x_t)] = f(t, x_t) + \theta^*(t), t \in \mathrm{J}.$$

A similar remark can be obtained for the inequality (4.5).

Lemma 4.6. *Let $0 < \beta \leq 1$ and $x \in \Omega$ be the solution of inequality (4.1), then x satisfies the following integral inequality*

$$\begin{aligned} & \left| x(t) - g(t, x_t) - e^{\lambda \frac{t^\beta}{\beta}} \phi(0) - e^{\lambda \frac{t^\beta}{\beta}} \int_0^t e^{-\lambda \frac{s^\beta}{\beta}} s^{\beta-1} f(s, x_s) ds \right| \\ \leq & \frac{\epsilon (e^{\lambda \frac{t^\beta}{\beta}} - 1)}{\lambda}, \ t \in \mathrm{J}. \end{aligned}$$

Proof. From Remark 4.4, we have

$$(\mathcal{D}_{0+}^\beta - \lambda)[x(t) - g(t, x_t)] = f(t, x_t) + \theta(t), t \in \mathrm{J}.$$

According to Theorem 3.1, the solution will be equivalent to:

$$x(t) = g(t, x_t) + e^{\lambda \frac{t^\beta}{\beta}} \phi(0) + e^{\lambda \frac{t^\beta}{\beta}} \int_0^t e^{-\lambda \frac{t^\beta}{\beta}} s^{\beta-1} f(s, x_s) ds + e^{\lambda \frac{t^\beta}{\beta}} \int_0^t e^{-\lambda \frac{s^\beta}{\beta}} \theta(s) ds.$$

Therefore,

$$\begin{aligned} & \left| x(t) - g(t, x_t) - e^{\lambda \frac{t^\beta}{\beta}} \phi(0) - e^{\lambda \frac{t^\beta}{\beta}} \int_0^t e^{-\lambda \frac{s^\beta}{\beta}} s^{\beta-1} f(s, x_s) ds \right| \\ \leq & e^{\lambda \frac{t^\beta}{\beta}} \int_0^t e^{-\lambda \frac{s^\beta}{\beta}} s^{\beta-1} |\theta(s)| ds \\ \leq & \epsilon e^{\lambda \frac{t^\beta}{\beta}} \int_0^t e^{-\lambda \frac{s^\beta}{\beta}} s^{\beta-1} ds \\ \leq & \frac{\epsilon (e^{\lambda \frac{t^\beta}{\beta}} - 1)}{\lambda}. \end{aligned}$$

□

Lemma 4.7. *Let $0 < \beta \leq 1$ and $x \in \mathcal{C}(\mathrm{J}, \mathbb{R})$ be the solution of inequality (4.2), then x satisfies the following integral inequality*

$$\begin{aligned} & \left| x(t) - g(t, x_t) - e^{\lambda \frac{t^\beta}{\beta}} \phi(0) - e^{\lambda \frac{t^\beta}{\beta}} \int_0^t e^{-\lambda \frac{s^\beta}{\beta}} s^{\beta-1} f(s, x_s) ds \right| \\ \leq & \frac{\epsilon^*}{\lambda} (e^{\lambda \frac{t^\beta}{\beta}} - 1) \mathrm{E}_\beta(t^\beta), \ t \in \mathrm{J}. \end{aligned}$$

Proof. From Remark 4.5, we have

$$(\mathcal{D}_{0+}^\beta - \lambda)[x(t) - g(t, x_t)] = f(t, x_t) + \theta^*(t), t \in \mathrm{J}.$$

According to Theorem 3.1, the solution will be equivalent to:

$$x(t) = g(t, x_t) + e^{\lambda \frac{t^\beta}{\beta}} \phi(0) + e^{\lambda \frac{t^\beta}{\beta}} \int_0^t e^{-\lambda \frac{s^\beta}{\beta}} s^{\beta-1} f(s, x_s) ds + e^{\lambda \frac{t^\beta}{\beta}} \int_0^t e^{-\lambda \frac{t^\beta}{\beta}} \theta(s) ds.$$

Therefore,

$$\begin{aligned} & \left| x(t) - g(t, x_t) - e^{\lambda \frac{t^\beta}{\beta}} \phi(0) - e^{\lambda \frac{t^\beta}{\beta}} \int_0^t e^{-\lambda \frac{s^\beta}{\beta}} s^{\beta-1} f(s, x_s) ds \right| \\ \leq & e^{\lambda \frac{t^\beta}{\beta}} \int_0^t e^{-\lambda \frac{s^\beta}{\beta}} s^{\beta-1} |\theta^*(s)| ds \\ \leq & e^{\lambda \frac{t^\beta}{\beta}} \int_0^t e^{-\lambda \frac{t^\beta}{\beta}} \epsilon^* s^{\beta-1} \mathrm{E}_\beta(s^\beta) ds \\ \leq & \frac{\epsilon^*}{\lambda} (e^{\lambda \frac{t^\beta}{\beta}} - 1) \mathrm{E}_\beta(t^\beta). \end{aligned}$$

□

Theorem 4.8. *Let the assumptions $\mathcal{H}_1$ to $\mathcal{H}_3$ hold along with condition that $1-\xi \neq 0$, where*

$$\xi = \Big[\mathfrak{Q}_g + \frac{\mathfrak{Q}_f}{\lambda}\big(e^{\lambda \frac{T^\beta}{\beta}} - 1\big)\Big]\mathcal{K}_a,$$

then (1.1) is UH stable.

Proof. Let x be the approximate solution of (1.1) and y be the unique solution, then by Theorem 3.1 for every $\phi \in \mathbf{B}$, y will have the form

$$y(t) = g(t, y_t) + e^{\lambda \frac{t^\beta}{\beta}}\phi(0) + e^{\lambda \frac{t^\beta}{\beta}}\int_0^t e^{-\lambda \frac{s^\beta}{\beta}} s^{\beta-1} f(s, y_s)ds, \;\; t \in \mathrm{J}.$$

Now consider

$$\begin{aligned}
&\big|x(t) - y(t)\big| \\
= &\Big|x(t) - g(t, y_t) - e^{\lambda \frac{t^\beta}{\beta}}\phi(0) - e^{\lambda \frac{t^\beta}{\beta}}\int_0^t e^{-\lambda \frac{s^\beta}{\beta}} s^{\beta-1} f(s, y_s)ds\Big| \\
= &\Big|x(t) - g(t, x_t) + e^{\lambda \frac{t^\beta}{\beta}}\phi(0) - e^{\lambda \frac{t^\beta}{\beta}}\int_0^t e^{-\lambda \frac{s^\beta}{\beta}} s^{\beta-1} f(s, x_s)ds \\
&+ g(t, x_t) + e^{\lambda \frac{t^\beta}{\beta}}\phi(0) + e^{\lambda \frac{t^\beta}{\beta}}\int_0^t e^{-\lambda \frac{s^\beta}{\beta}} s^{\beta-1} f(s, x_s)ds \\
&- g(t, y_t) - e^{\lambda \frac{t^\beta}{\beta}}\phi(0) - e^{\lambda \frac{t^\beta}{\beta}}\int_0^t e^{-\lambda \frac{s^\beta}{\beta}} s^{\beta-1} f(s, y_s)ds\Big| \\
\leq &\Big|x(t) - g(t, x_t) + e^{\lambda \frac{t^\beta}{\beta}}\phi(0) - e^{\lambda \frac{t^\beta}{\beta}}\int_0^t e^{-\lambda \frac{s^\beta}{\beta}} s^{\beta-1} f(s, x_s)ds\Big| \\
&+ |g(t, x_t) - g(t, y_t)| + e^{\lambda \frac{t^\beta}{\beta}}\int_0^t e^{-\lambda \frac{t^\beta}{\beta}} s^{\beta-1}\big|f(s, x_s) - f(s, y_s)\big|ds \\
\leq &\frac{\epsilon(e^{\lambda \frac{t^\beta}{\beta}} - 1)}{\lambda} + \mathfrak{Q}_g\|x_t - y_t\|_{\mathbf{B}} + \frac{\mathfrak{Q}_f}{\lambda}\big(e^{\lambda \frac{t^\beta}{\beta}} - 1\big)\|x_t - y_t\|_{\mathbf{B}} \\
\leq &\frac{\epsilon(e^{\lambda \frac{t^\beta}{\beta}} - 1)}{\lambda} + \mathfrak{Q}_g\mathcal{K}_a\|x - y\|_{\underline{\Omega}} + \frac{\mathfrak{Q}_f\mathcal{K}_a}{\lambda}\big(e^{\lambda \frac{t^\beta}{\beta}} - 1\big)\|x - y\|_{\underline{\Omega}}.
\end{aligned}$$

Or

$$\|x - y\|_{\underline{\Omega}} \leq \frac{\epsilon(e^{\lambda \frac{T^\beta}{\beta}} - 1)}{\lambda} + \Big[\mathfrak{Q}_g + \frac{\mathfrak{Q}_f}{\lambda}\big(e^{\lambda \frac{T^\beta}{\beta}} - 1\big)\Big]\mathcal{K}_a\|x - y\|_{\underline{\Omega}}.$$

From which we obtained

$$\|x - y\|_{\underline{\Omega}} \leq \lambda^1\epsilon, \;\; \lambda^1 = \frac{e^{\lambda \frac{T^\beta}{\beta}} - 1}{\lambda(1 - \xi)}, \;\; t \in \mathrm{J}.$$

The last inequality shows that the system (1.1) is UH stable. □

Theorem 4.9. *Let the assumptions $\mathcal{H}_1$ to $\mathcal{H}_3$ holds along with condition $1-\xi \neq 0$, where*

$$\xi = \Big[\mathfrak{Q}_g + \frac{\mathfrak{Q}_f}{\lambda}\big(e^{\lambda \frac{T^\beta}{\beta}} - 1\big)\Big]\mathcal{K}_a,$$

then (1.1) is UHML stable.

Proof. Let x be the approximate solution of (1.1) and y be the unique solution, then by Theorem 3.1 for every $\phi \in \mathbf{B}$ the solution of y will have the form

$$y(t) = g(t, y_t) + e^{\lambda \frac{t^\beta}{\beta}}\phi(0) + e^{\lambda \frac{t^\beta}{\beta}}\int_0^t e^{-\lambda \frac{s^\beta}{\beta}} s^{\beta-1} f(s, y_s)ds, \;\; t \in \mathrm{J}.$$

Now consider

$$\begin{aligned}
&\big|x(t)-y(t)\big| \\
=\;& \Big|x(t)-g(t,y_t)-e^{\lambda\frac{t^\beta}{\beta}}\phi(0)-e^{\lambda\frac{t^\beta}{\beta}}\int_0^t e^{-\lambda\frac{s^\beta}{\beta}}s^{\beta-1}f(s,y_s)ds\Big| \\
=\;& \Big|x(t)-g(t,x_t)+e^{\lambda\frac{t^\beta}{\beta}}\phi(0)-e^{\lambda\frac{t^\beta}{\beta}}\int_0^t e^{-\lambda\frac{s^\beta}{\beta}}s^{\beta-1}f(s,x_s)ds \\
& +g(t,x_t)+e^{\lambda\frac{t^\beta}{\beta}}\phi(0)+e^{\lambda\frac{t^\beta}{\beta}}\int_0^t e^{-\lambda\frac{s^\beta}{\beta}}s^{\beta-1}f(s,x_s)ds \\
& -g(t,y_t)-e^{\lambda\frac{t^\beta}{\beta}}\phi(0)-e^{\lambda\frac{t^\beta}{\beta}}\int_0^t e^{-\lambda\frac{s^\beta}{\beta}}s^{\beta-1}f(s,y_s)ds\Big| \\
\le\;& \Big|x(t)-g(t,x_t)+e^{\lambda\frac{t^\beta}{\beta}}\phi(0)-e^{\lambda\frac{t^\beta}{\beta}}\int_0^t e^{-\lambda\frac{s^\beta}{\beta}}s^{\beta-1}f(s,x_s)ds\Big| \\
& +|g(t,x_t)-g(t,y_t)|+e^{\lambda\frac{t^\beta}{\beta}}\int_0^t e^{-\lambda\frac{t^\beta}{\beta}}s^{\beta-1}\big|f(s,x_s)-f(s,y_s)\big|ds \\
\le\;& \frac{\epsilon^*}{\lambda}\big(e^{\lambda\frac{t^\beta}{\beta}}-1\big)\mathrm{E}_\beta(t^\beta)+\mathcal{Q}_g\|x_t-y_t\|_{\mathbf{B}}+\frac{\mathcal{Q}_f}{\lambda}\big(e^{\lambda\frac{t^\beta}{\beta}}-1\big)\|x_t-y_t\|_{\mathbf{B}} \\
\le\;& \frac{\epsilon^*}{\lambda}\big(e^{\lambda\frac{t^\beta}{\beta}}-1\big)\mathrm{E}_\beta(t^\beta)+\mathcal{Q}_g\mathcal{K}_a\|x-y\|_{\underline{\Omega}}+\frac{\mathcal{Q}_f\mathcal{K}_a}{\lambda}\big(e^{\lambda\frac{t^\beta}{\beta}}-1\big)\|x-y\|_{\underline{\Omega}}.
\end{aligned}$$

From which we obtain

$$\|x-y\|_{\underline{\Omega}}\le\lambda^*\epsilon\mathrm{E}_\beta(t^\beta),\quad \lambda^*=\frac{e^{\lambda\frac{T^\beta}{\beta}}-1}{\lambda(1-\xi)},\ t\in\mathrm{J}.$$

By Definition 4.2, the system (1.1) has UHML stability. This completes the proof.

Note: The UHMLR stability result can be obtained in similar way. □

5. Example

In this section, we illustrate the above results by an example.

Example 5.1. *Let $\alpha\in\mathbb{R}_+$ and define a space $\mathbf{B}_\alpha$ by:*

$$\mathbf{B}_\alpha=\{x\in\mathcal{C}((-\infty,0],\mathbb{R}):\lim_{t\to-\infty}e^{\alpha t}x(t)\ \textit{exists in}\ \mathbb{R}\},$$

with norm

$$\|x\|_\alpha=\sup_{-\infty<t\le0}\{e^{\alpha t}|x(t)|\}.$$

Then the functional space satisfies all the axioms from $\mathcal{A}_1$ to $\mathcal{A}_2$. Next consider the following nonlinear weighted neutral fractional differential system

$$\begin{cases}(\mathcal{D}_{0+}^{\frac{1}{2}}-\lambda)\big[x(t)-\dfrac{e^{-t-t\alpha}\|x_t\|}{10(e^t+e^{-t})}\big]=\dfrac{e^{-t\alpha}|x_t|}{(20+e^{-t})(1+\|x_t\|)},\ t\in\mathrm{J}=(0,1],\\ x(t)=\phi(t),t\in(-\infty,0].\end{cases}\tag{5.1}$$

Setting the functions as:

$$f(t,v)=\frac{e^{-t\alpha}v}{(20+e^{-t})(1+v)},\ t\in\mathrm{J}$$

and

$$g(t,v)=\frac{e^{-t-t\alpha}v}{10(e^t+e^{-t})},\ t\in\mathrm{J}.$$

For any v, v^* from $\mathbf{B}_\alpha$ and $t \in \mathrm{J}$, we have

$$\begin{aligned} |f(t,v) - f(t,v^*)| &\leq \frac{e^{-t\alpha}}{20+e^{-t}}\left|\frac{v}{1+v} - \frac{v^*}{1+v^*}\right| \\ &= \frac{e^{-t-t\alpha}}{20e^t+1}\left|\frac{v-v^*}{(1+v)(1+v^*)}\right| \\ &\leq \frac{e^{-t}}{20}e^{-t\alpha}|v-v^*| \\ &\leq \frac{1}{20}\|v-v^*\|_\alpha. \end{aligned}$$

$$\begin{aligned} |g(t,v) - g(t,v^*)| &\leq \frac{e^{-t-t\alpha}}{10(e^t+e^{-t})}|v-v^*| \\ &= \frac{e^{-t}}{10(e^t+e^{-t})}e^{-t\alpha}|v-v^*| \\ &\leq \frac{1}{10(e^{2t}+1)}\|v-v^*\|_\alpha \\ &\leq \frac{1}{10}\|v-v^*\|_\alpha. \end{aligned}$$

Therefore the conditions from $\mathcal{H}_1$ to $\mathcal{H}_2$ are satisfied with $\mathcal{Q}_f = \frac{1}{20}$, $\mathcal{Q}_g = \frac{1}{10}$. It can be observed that (3.5) is satisfied by considering $\mathcal{K}_a = \mathcal{M}_a = \lambda = t = 1$ and $\beta = \frac{1}{2}$. Actually

$$\left[\mathcal{Q}_g + \frac{\mathcal{Q}_f}{\lambda}(e^{\lambda\frac{T^\beta}{\beta}} - 1)\right]\mathcal{K}_a = 0.1324 < 1.$$

Thus by Theorem 3.2, the system (5.1) has a unique solution on $(-\infty, 1]$. Also $1 - \xi = 0.8676 \neq 0$, the solution of (5.1) has UH stability and UHML stability on J.

Conclusion

In this manuscript, we exercised the Banach contraction principle and Schaefer's fixed point theorem, to achieve the necessary conditions for the existence and uniqueness of solution to a nonlinear weighted neutral fractional differential equation system. Likewise under specific assumptions and conditions, we have found the UH stability, UHML stability and UHMLR stability results for the solution of (1.1).

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