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A Modified Reproducing Kernel Hilbert Space Method for Solving Fuzzy Fractional Integro-differential Equations

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abstract: The aim of this paper is to extend the application of the reproducing kernel Hilbert space method (RKHSM) to solve linear and non-linear fuzzy integro-differential equations of fractional order under Caputo's H-differentiability. The analytic and approximate solutions are given in series form in term of their parametric form in the space $W_2^2[a,b] \bigoplus W_2^2[a,b]$. Several examples are carried out to show the effectiveness and the absence of complexity of the proposed method.

Key Words: Fuzzy fractional integro-differential equation, Caputo's H-differentiability, Reproducing kernel Hilbert space method.

Contents

1. Introduction

The theory of fractional calculus has been a subject of interest, not only among mathematicians, but also among physicists and engineers for its considerable importance in many fields of science, see [1,2,3]. But when modeling real world phenomena, information about the behavior of a dynamical system may be uncertain. So, fuzzy set theory was established to describe uncertainty in mathematical models. It was originally introduced by Zadeh in 1965 [4]. In 1978, Dubois and Prade introduced the notion of fuzzy real numbers and established some of their basic properties [5]. The term "fuzzy differential equations" was coined in the same year by Kandel and Byatt [6]. The fuzzy set theory has been used in various other fields, i.e., fuzzy fixed-point theory, fuzzy topology, fuzzy control systems, fuzzy automata, etc. There are many suggestions to define a fuzzy derivative and then to study fuzzy differential equations [7,8,9,10,11].

The most popular approach is using Hukuhara differentiability [8,9,10,11,12] which was generalized by Bede and Gal in 2005 [13]. In 2010, the concept of fuzzy fractional differential equations (FFDEs) has been introduced [14]. In [15], the authors considered a generalization of H-differentiability for the fractional case. This concept of fuzzy derivative was used in [16] to introduce fuzzy fractional Volterra-Fredholm integro-differential equations and prove the existence and uniqueness of the solutions of this class of fractional equations.

In this paper, we present an effective numerical method to obtain approximate solutions of fuzzy fractional integro-differential equations (FFIDEs) based on the reproducing kernel theory. Among the last works that applied RKHSM to solve differential equations are: solving fuzzy Fredholm-Volterra integro-differential equations [38], solving integro-differential equations of fractional order [18], solving fourth

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order boundary value problem [19], solving systems of fractional integro-differential equations [17], solving fuzzy singular integral equation with Abel's type kernel using a novel hybrid method [20] and solving non-linear Fredholm integro-differential equations [21].
This paper is organized as follows: In section 2, we give some basic concepts related to fuzzy calculus and fractional calculus. In section 3, we convert a FFIDE into a system of fractional integro-differential equations. A description of the RKHSM for solving linear and non-linear FFIDEs is given in section 4. Some numerical examples are carried out in section 5. This paper ends with a conclusion in section 6.

2. Preliminaries

In this section, we introduce some necessary definitions and mathematical preliminaries of fuzzy calculus and fractional calculus.

Definition 2.1. *[23] A fuzzy number u is a mapping $u : R \longrightarrow [0,1]$ that satisfies the following properties:*

1. *u is normal. That is, there is $x \in R$ with $u(x) = 1$.*
2. *u is fuzzy convex. That is, $u(\gamma x + (1-\gamma)y) \geq \min\{u(x), u(y)\}$ for all $x, y \in R$ and $\gamma \in [0,1]$.*
3. *u is upper semicontinuous.*
4. *The set $\overline{\{x \in R : u(x) > 0\}}$ is bounded.*

The set of all fuzzy numbers is denoted by R_F. An effective way to present a fuzzy number u is by using its r-cut representation which is given by $[u]^r = \{x \in R : u(x) \geq r\}$ for $r \in (0,1]$ and $[u]^0 = \overline{\{x \in R : u(x) > 0\}}$. $[u]^0$ is called the support of A.The core of A is the crisp set of all points x in R such that $u(x) = 1$. Obviously, if u is a fuzzy number, then $[u]^r = [u_1(r),\ u_2(r)]$ where $u_1(r) = u_{1r} = \min\{x|x \in [u]^r\}$ and $u_2(r) = u_{2r} = \max\{x|x \in [u]^r\},\ \forall r \in [0,1]$.
The following theorem gives the conditions that must be satisfied by two real valued functions $u_1,\ u_2$ defined on $[0,1]$ so that $[u_1(r), u_2(r)]$ is the parametric form of a fuzzy number u for each $r \in [0,1]$.

Theorem 2.2. *[24] Suppose that $u_1,\ u_2 : [0,1] \longrightarrow R$ satisfy the conditions; first, u_1 is a bounded monotonic nondecreasing left continuous function $\forall r \in (0,1]$ and right continuous for $r = 0$; second, u_2 is a bounded monotonic non increasing left continuous function $\forall r \in (0,1]$ and right continuous for $r = 0$; third,$u_1(1) \leq u_2(1)$. Then $u : R \longrightarrow [0,1]$ which is defined by $u(x) = \sup\{r|u_1(r) \leq x \leq u_2(r)\}$ is a fuzzy number with parameterization $[u]^r = [u_1(r), u_2(r)]$. Moreover, if u is a fuzzy number with $[u]^r = [u_1(r), u_2(r)]$ (or simply, $[u_{1r}, u_{2r}]$), then the functions $u_1,\ u_2 : [0,1] \longrightarrow R$ satisfy the above conditions. In this case, we can represent a fuzzy number by an ordered pair of functions (u_1, u_2).*

Arithmetic operations in R_F can be defined as those on intervals of R. So for any $\gamma \in R - \{0\}$,and $u, v \in R_F$ with $[u]^r = [u_{1r}, u_{2r}]$ and $[v]^r = [v_{1r}, v_{2r}]$, we have $[u+v]^r = [u]^r + [v]^r = [u_{1r} + v_{1r}, u_{2r} + v_{2r}]$, and $[\gamma u]^r = \gamma[u]^r = [\min\{\gamma u_{1r}, \gamma u_{2r}\},\ \max\{\gamma u_{1r}, \gamma u_{2r}\}]$.
Obviously, R_F does not form a vector space with the zero element $\{0\}$. Hence, additive simplification is not valid, that is $u + v = u + w$ does not imply that $v = w$ for fuzzy numbers u, v and w. To overcome this situation, we will use the Hukuhara difference (H-difference).The H-difference of $u, v \in R_F$, denoted by $u \ominus v = w$, is the fuzzy number that satisfies $u = v + w$. Its r-cut representation is $[u \ominus v]^r = [u_{1r} - v_{1r}, u_{2r} - v_{2r}]$.

Definition 2.3. *[25] The Housdorff metric D on R_F is defined by $D : R_F \times R_F \longrightarrow R^+ \cup \{0\}$ such that $D(u,v) = \sup\limits_{r \in [0,1]} \max\{|u_{1r} - v_{1r}|,\ |u_{2r} - v_{2r}|\}$ for any fuzzy numbers $u = (u_1, u_2)$ and $v = (v_1, v_2)$. A fuzzy function on an interval T is a mapping $F : T \longrightarrow R_F$. If for fixed $t_0 \in T$ and $\varepsilon > 0, \exists \delta > 0$ such that $|t - t_0| < \delta \Longrightarrow D(F(t), F(t_0)) < \varepsilon$, then we say that F is continuous at t_0. If F is continuous $\forall t \in T$, then F is continuous on T [26]. A natural way for extending a crisp mapping $f : R \longrightarrow R$ to a mapping $F : R_F \longrightarrow R_F$ is Zadeh's extension principle [27]. An important result related to fuzzy functions is Nguyen Theorem which says that Zadeh's extension of a continuous real valued function $f : R \times R \longrightarrow R$, say $F : R_F \times R_F \longrightarrow R_F$, is a well-defined fuzzy function with r-cuts $[F(u,v)]^r = f([u]^r, [v]^r) = \{f(x,y) : x \in [u]^r,\ y \in [v]^r\},\ \forall r \in [0,1]$ and $u, v \in R_F$ [28].*

For the differentiation of a fuzzy function, we are interested with the following definition:

Definition 2.4. *[13] Let $F:(a,b)\longrightarrow R_F$ and $t_0\in(a,b)$. We say that F is strongly generalized differentiable at t_0 if there exists a fuzzy number $F'(t_0)$ such that:*

1. *There exist $F(t_0+h)\ominus F(t_0)$ and $F(t_0)\ominus F(t_0-h)$ and $\lim\limits_{h\longrightarrow 0^+}\dfrac{F(t_0+h)\ominus F(t_0)}{h}$ and $\lim\limits_{h\longrightarrow 0^+}\dfrac{F(t_0)\ominus F(t_0-h)}{h}=F'(t_0)$, or*
2. *There exist $F(t_0)\ominus F(t_0+h)$ and $F(t_0-h)\ominus F(t_0)$ and $\lim\limits_{h\longrightarrow 0^+}\dfrac{F(t_0)\ominus F(t_0+h)}{-h}$ and $\lim\limits_{h\longrightarrow 0^+}\dfrac{F(t_0-h)\ominus F(t_0)}{-h}=F'(t_0)$*

The limits here are taken in the metric space (R_F,D).

We say that F is (n)-differentiable for $n=1,2$ if F is strongly generalized differentiable in the nth form and denote the (n)-derivative of F at t_0 by $F'(t_0)=D_n^1F(t_0)$. However, if $D_1^1F(t_0)$ exists, then $D_2^1F(t_0)$ does not exist [29].

Remark 2.1. In [13], the authors suggested four cases for the generalized H-derivative and proved that two of them are reduced to a crisp element. So, they are missing here.

Theorem 2.5. *[30] Let $F:[a,b]\longrightarrow R_F$ be a strongly generalized differentiable function at $t_0\in[a,b]$. Then F_{1r} and F_{2r} are differentiable at $t_0,\forall r\in[0,1]$. Moreover, $[F'(t_0)]^r=[F'_{1r}(t_0),F'_{2r}(t_0)]$ if F is (1)-differentiable and $[F'(t_0)]^r=[F'_{2r}(t_0),F'_{1r}(t_0)]$ under (2)-differentiability.*

For integration of a fuzzy valued function, we will consider the following definition.

Definition 2.6. *[31] Let $F:[a,b]\longrightarrow R_F$. Then F is said to be fuzzy-Riemann integrable to $I\in R_F$ if $\forall\varepsilon>0$, there is $\delta>0$ such that for any division $P:a=x_0<x_1<...<x_n=b$ of $[a,b]$ with norm $\triangle(P)<\delta$, and for any points $\xi_i\in[x_i,x_{i+1}]$, $i=0,1,...,n-1$, we have $D(\sum_{i=0}^{n-1}F(\xi_i)(x_{i+1}-x_i),I)<\varepsilon$. The fuzzy-Riemann integral is denoted by $I=\int_a^b F(x)\,dx$.*

Remark 2.2. If $F:[a,b]\longrightarrow R_F$ is fuzzy-Riemann integrable, then the r-cut representation of its integral is $\Big[\int_a^b F_{1r}(x)\,dx,\ \int_a^b F_{2r}(x)\,dx\Big]$, see [32].

In this paper, we deal with Caputo's H-derivative. Before presenting its definition, we give the following definitions for crisp functions: The Riemann-Liouville fractional integral of order $\alpha\in(0,1]$ over the interval $[a,b]$ for a function $f\in L[a,b]$ is defined by $(J_a^\alpha f)(x)=\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_a^x\dfrac{f(t)}{(x-t)^{1-\alpha}\,dt}$, $x>a$. The Caputo fractional derivative of order $\alpha\in(0,1]$ is given by $(D_{a+}^\alpha f)(x)=\dfrac{1}{\Gamma(1-\alpha)}\displaystyle\int_a^x\dfrac{f'(t)}{(x-t)^\alpha}\,dt$. The Caputo's derivative for a continuous function f on $[a,b]$ satisfies the property $(J_{a+}^\alpha D_{a+}^\alpha f)(x)=f(x)-f(a)$. For details, see [33]. Now, we define some notations which are used for fuzzy fractional calculus:

$C^F[a,b]=$ The space of continuous fuzzy valued functions on $[a,b]$. $L_p^F[a,b]=\{F:[a,b]\longrightarrow R_F;F$ is measurable and $\int_a^b D(F(x),0)^p\,dx<\infty\}$, $1\le p<\infty$. The generalized H-differentiability was used to expand the definitions of fractional derivatives in the crisp sense for the fuzzy space as follows.

Definition 2.7. *[34] Let $0<\alpha\le 1$, $F:[a,b]\longrightarrow R_F$ and $F\in C^F[a,b]\cap L^F[a,b]$. The fuzzy Riemann-Liouville fractional integral of order α is defined by $(J_{a+}^\alpha F)(x)=\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_a^x\dfrac{F(t)}{(x-t)^{1-\alpha}}\,dt$, $x>a$. It can be written in parametric form as $[(J_{a+}^\alpha F)(x)]^r=\Big[\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_a^x\dfrac{F_{1r}(t)}{(x-t)^{1-\alpha}}\,dt,\ \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_a^x\dfrac{F_{2r}(t)}{(x-t)^{1-\alpha}}\,dt\Big]$.*

Definition 2.8. *[35] Let $0<\alpha<1$, $F:[a,b]\longrightarrow R_F$ and $F\in C^F[a,b]\cap L^F[a,b]$. Then F is said to be Caputo's H-differentiable at x if $(D_{a+}^\alpha F)(x)=\dfrac{1}{\Gamma(1-\alpha)}\displaystyle\int_a^x\dfrac{F'(t)}{(x-t)^\alpha}\,dt$ exists. We say that F is $[(1)-\alpha]$-differentiable if F is (1)-differentiable, and F is $[(2)-\alpha]$-differentiable if F is (2)-differentiable.*

3. Fuzzy Fractional Integro-differential Equations

Consider the following Fredholm-Volterra FFIDE:

$$(D^{\beta}_{a+}x)(t) = p(t)x(t) + (Tx)(t) + f(t),\ x(a) = \alpha,\ 0 < \beta \leq 1,\ a \geq 0 \tag{3.1}$$

where $(Tx)(t) = \int_a^b H(t, \tau, x(\tau))\, d\tau + \int_a^t K(t, \tau, x(\tau))\, d\tau$ with continuous kernel functions $H(t, \tau, x(\tau))$ and $K(t, \tau, x(\tau))$. Also, $p(t)$ is a continuous real valued function with nonnegative or nonpositive values on $[a, b]$, $f : [a, b] \longrightarrow R_F$ is continuous, and $\alpha \in R_F$. Without loss of generality, we assume that $H(t, \tau, x(\tau)) = h(t, \tau)x(\tau)$ for $\tau \in [a, b]$, $h(t, \tau) \geq 0$ for $\tau \in [a, c_1]$ and $h(t, \tau) \leq 0$ for $\tau \in [c_1, b]$. Similarly, $K(t, \tau, x(\tau)) = k(t, \tau)x(\tau)$ for $\tau \in [a, t]$, $k(t, \tau) \geq 0$ for $\tau \in [a, c_2]$ and $k(t, \tau) \leq 0$ for $\tau \in [c_2, t]$. Also, we suppose that $p(t) \geq 0$ on $[a, b]$. We write $f(t)$ and $x(t)$ in term of their r-cut representations: $[f(t)]^r = [f_{1r}(t), f_{2r}(t)]$, $[x(t)]^r = [x_{1r}(t), x_{2r}(t)]$ and $[x(a)]^r = [x_{1r}(a), x_{2r}(a)] = [x_{1r}, x_{2r}] = [\alpha_{1r}, \alpha_{2r}]$. Hence, (3.1) has the form:

$$[(D^{\beta}_{a^+}x)(t)]^r = p(t)[x(t)]^r + \int_a^b [h(t, \tau)x(\tau)]^r\ d\tau + \int_a^t [k(t, \tau)x(\tau)]^r\ d\tau + [f(t)]^r, [x(a)]^r = [\alpha]^r.$$

Therefore, we can translate (3.1) into one of the following systems:
If $x(t)$ is $[(1) - \beta]$- differentiable, the system is:

$$(D^{\beta}_{a^+}x_{1r})(t) = p(t)x_{1r}(t) + \int_a^{c_1} h(t, \tau)x_{1r}(t)\ d\tau + \int_{c_1}^b h(t, \tau)x_{2r}(t)\ d\tau + \int_a^{c_2} k(t, \tau)x_{1r}(t)\ d\tau$$
$$+ \int_{c_2}^t k(t, \tau)x_{2r}(t)d\tau + f_{1r}(t)$$

$$(D^{\beta}_{a^+}x_{2r})(t) = p(t)x_{2r}(t) + \int_a^{c_1} h(t, \tau)x_{2r}(t)\ d\tau + \int_{c_1}^b h(t, \tau)x_{1r}(t)\ d\tau + \int_a^{c_2} k(t, \tau)x_{2r}(t)\ d\tau$$
$$+ \int_{c_2}^t k(t, \tau)x_{1r}(t)d\tau + f_{2r}(t) \tag{3.2}$$

subject to

$$x_{1r}(a) = \alpha_{1r},\ x_{2r}(a) = \alpha_{2r} \tag{3.3}$$

While if $x(t)$ is $[(2) - \beta]$ - differentiable, the system is:

$$(D^{\beta}_{a^+}x_{1r})(t) = p(t)x_{2r}(t) + \int_a^{c_1} h(t, \tau)x_{2r}(t)\ d\tau + \int_{c_1}^b h(t, \tau)x_{1r}(t)\ d\tau + \int_a^{c_2} k(t, \tau)x_{2r}(t)\ d\tau$$
$$+ \int_{c_2}^t k(t, \tau)x_{1r}(t)d\tau + f_{2r}(t)$$

$$(D^{\beta}_{a^+}x_{2r})(t) = p(t)x_{1r}(t) + \int_a^{c_1} h(t, \tau)x_{1r}(t)\ d\tau + \int_{c_1}^b h(t, \tau)x_{2r}(t)\ d\tau + \int_a^{c_2} k(t, \tau)x_{1r}(t)\ d\tau$$
$$+ \int_{c_2}^t k(t, \tau)x_{2r}(t)d\tau + f_{1r}(t) \tag{3.4}$$

subject to

$$x_{1r}(a) = \alpha_{1r},\ x_{2r}(a) = \alpha_{2r} \tag{3.5}$$

4. The RKHSM for Solving FFIDEs

To apply the RKHSM, we need to construct reproducing kernel functions of certain spaces, see [36, 37,38,39]:

- $W_2^1[a,b] = \{u : [a,b] \longrightarrow R : u \in AC[a,b],\ u' \in L_2[a,b]\}$ with inner product for $u, v \in W_2^1[a,b]$ given by $\langle u, v\rangle_{W_2^1} = u(a)v(a) + \int_a^b u'(t)v'(t)\ dt$ and norm $||u||_{W_2^1} = \sqrt{\langle u(t), u(t)\rangle_{W_2^1}}$. Its reproducing function has the form
$$G_t^2(s) = \begin{cases} 1 - a + t & , \ s \le t \\ 1 - a + s & , \ s > t \end{cases}$$
It was shown by [37] that the space $W_2^1[a,b]$ is a complete RKHS with the reproducing kernel function $G_t^2(s)$.

- $W_2^2[a,b] = \{u : u, u' \in AC[a,b], u'' \in L_2[a,b], u(a) = 0\}$ with inner product for $u, v \in W_2^2[a,b]$ given by $\langle u, v\rangle_{W_2^2} = u'(a)v'(a) + \int_a^b u''(t)v''(t)\ dt$, and norm: $||u||_{W_2^2} = \sqrt{\langle u(t), u(t)\rangle_{W_2^2}}$. The reproducing function of $W_2^2[a,b]$ is
$$K_t(s) = \frac{1}{6}\begin{cases} (s-a)(2a^2 - s^2 + 3t(2+s) - a(6+3t+s)) & , \ s \le t \\ (t-a)(2a^2 - t^2 + 3s(2+t) - a(6+3s+t)) & , \ s > t \end{cases}$$

- $H[a,b] = W_2^1[a,b] \oplus W_2^1[a,b] = \{(u_1(t), u_2(t))^T : u_1, u_2 \in W_2^1[a,b]\}$. The inner product and the norm of $u(t) = (u_1(t), u_2(t))^T,\ v(t) = (v_1(t), v_2(t))^T$ in $H[a,b]$ are given by : $\langle u(t), v(t)\rangle_H = \sum_{i=1}^{2}\langle u_i(t), v_i(t)\rangle_{W_2^1}$ and $||u||_H = \sqrt{\sum_{i=1}^{2} ||u_i||^1_{W_2^2}}$.

- $W[a,b] = W_2^2[a,b] \oplus W_2^2[a,b] = \{(u_1(t), u_2(t))^T : u_1, u_2 \in W_2^2[a,b]\}$.The inner product and the norm of $u(t) = (u_1(t), u_2(t))^T, v = (v_1(t), v_2(t))^T$ in $W[a,b]$ are given by: $\langle u(t), v(t)\rangle_W = \sum_{i=1}^{2}\langle u_i(t), v_i(t)\rangle_{W_2^2}$ and $||u||_W = \sqrt{\sum_{i=1}^{2} ||u_i||^2_{W_2^2}}$, respectively.

Now, we apply the RKHS method to obtain approximate and analytic solutions of (3.1) under $[(1) - \beta]$ -differentiability. The same manner can be employed to obtain solutions under $[(2) - \beta]$ -differentiability. In order to get the solutions of (3.2) and (3.3), consequently, the solutions of (3.1) under (1)-differentiability, we define the integro-differential operators $v_{ij} : W_2^2[a,b] \longrightarrow W_2^1[a,b],\ i, j = 1, 2$ such that:

$$v_{ij}z(t) = \begin{cases} (D^{\beta}_{a^+}z)(t) - p(t)z(t) - \int_a^{c_1} h(t,\tau)z(t)\ d\tau - \int_a^{c_2} k(t,\tau)z(t)d\ \tau & , \ i = j \\ -\int_{c_1}^{b} h(t,\tau)z(t)\ d\tau - \int_{c_2}^{t} k(t,\tau)z(t)\ d\tau & , \ i \ne j \end{cases}$$

Put $F_r = \begin{pmatrix} f_{1r} \\ f_{2r} \end{pmatrix}$, $x_r = \begin{pmatrix} x_{1r} \\ x_{2r} \end{pmatrix}$, $\alpha_r = \begin{pmatrix} \alpha_{1r} \\ \alpha_{2r} \end{pmatrix}$, $V = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}$. So, $V : W[a,b] \longrightarrow H[a,b]$. Also, we homogenize the I.C.s (3.3) using the transform $y_{1r} = x_{1r} - \alpha_{1r}, y_{2r} = x_{2r} - \alpha_{2r}$, to obtain the system:

$$Vy_r = F_r(t) \tag{4.1}$$

subject to $y_r(a) = 0$.

Theorem 4.1. *The operator $V : W[a,b] \longrightarrow H[a,b]$ is bounded and linear.*

Proof: It is enough to prove that the operators $v_{ij} : W_2^2[a,b] \longrightarrow W_2^1[a,b],\ j = 1, 2$, are bounded and linear. The linearity of $v_{ij},\ j = 1, 2$ is clear. For boundedness, we have $\forall y_{jr} \in W_2^2[a,b],\ j = 1, 2$:

$$||v_{ij}y_{jr}||^2_{W_2^1} = \langle v_{ij}y_{jr}, v_{ij}y_{jr}\rangle_{W_2^1} = \Big[(v_{ij}y_{jr})(a)\Big]^2 + \int_a^b \Big[(v_{ij}y_{jr})'(t)\Big]^2 \, dt$$

By the reproducing property of $K_t(s)$, and using Schwarz Inequality and the fact that $D^{\beta}_{a^+}K_t(s)$ is continuous and uniformly bounded in s and t, we get

$$|(v_{ij}y_{jr})(t)| = \Big|\langle y_{jr}(.), (v_{ij}K_t)(.)\rangle_{W_2^2}\Big| \leq ||y_{jr}||_{W_2^2}||(v_{ij}K_t)(.)||_{W_2^1} \leq M_{ij}||y_{jr}||_{W_2^2},$$

and

$$|(v_{ij}y_{jr})'(t)| = \Big|\langle y_{jr}(.), (v_{ij}K_t)'(.)\rangle_{W_2^2}\Big| \leq ||y_{jr}||_{W_2^2}||(v_{ij}K_t)'(.)||_{W_2^1} \leq N_{ij}||y_{jr}||_{W_2^2},$$

where M_{ij} and $N_{ij} \in R^+$. So, $||v_{ij}y_{jr}||^2_{W_2^1} \leq (M^2_{ij} + N^2_{ij}(b-a))||y_{jr}||^2_{W_2^2} = L_{ij}||y_{jr}||^2_{W_2^2}$, where $L_{ij} = \sqrt{(M^2_{ij} + N^2_{ij}(b-a))}$. Hence, $L_{jr},\ j = 1,2$ are bounded.

Now, let $\{t_k\}_{k=1}^{\infty}$ be a countable dense set in $[a,b]$ and v^*_{ij} be the adjoint operator of v_{ij}. Then we have $[v^*_{ij}G^2_{t_k}(.)](t) = \langle [v^*_{ij}G^2_{t_k}(.)](s), K_t(s)\rangle_{W_2^2} = \langle G^2_{t_k}(.), [v_{ij}K_t(s)]\rangle_{W_2^1} = v_{ij}K_t(s)|_{s=t_k}$.
Define $\psi_{ij}(t)$ as $\psi_{ij}(t) = \Big[v_{j1}K_t(s)|_{s=t_i}, v_{j2}K_t(s)|_{s=t_i}\Big]^T,\ i = 1,2,3,...,\ j = 1,2.$

Theorem 4.2. *Assume that the system (3.2) and (3.3) has a unique solution and $\{t_i\}_{i=1}^{\infty}$ is dense in $[a,b]$, then $\{\psi_{ij}(t)\}_{(i,j)=(1,1)}^{(\infty,2)}$ is the complete function system of $W[a,b]$.*

Proof: See [38].
For simplicity, we rearrange the sequence , $\{\psi_{ij}(t)\}_{(i,j)=(1,1)}^{(\infty,2)}$ as $\{\mu_k(t)\}_{k=1}^{\infty}$ such that $\mu_{2(i-1)+j}(t) = \psi_{ij}(t),\ i = 1,2,...,\ j = 1,2$, and apply the Gram-Schmidt process on $\{\mu_k(t)\}_{k=1}^{\infty}$ to obtain the orthonormal function system $\{\overline{\mu}_k(t)\}_{k=1}^{\infty}$ of $W[a,b]$ that satisfies $\overline{\mu}_i(t) = \sum_{k=1}^{i}\beta_{ik}\mu_k(t),\ i = 1,2,3,...$, where β_{ik} are the orthogonalization coefficients.

Theorem 4.3. *If $\{t_i\}_{i=1}^{\infty}$ is dense on $[a,b]$ and the solution of (4.1) is unique, then it has the form:*

$$y_r(t) = \sum_{i=1}^{\infty}(\sum_{k=1}^{i}\beta_{ik}\gamma_k)\overline{\mu}_i(t),$$

where

$$\gamma_k = \langle y_r(t), \mu_k(t)\rangle_W = \begin{cases} f_{1r}\Big(t_{\frac{k+1}{2}}\Big) & ,\ k \text{ is odd} \\ f_{2r}\Big(t_{\frac{k}{2}}\Big) & ,\ k \text{ is even} \end{cases}$$

Proof: See [38].

The n-term approximate solution $y^n_r(t)$ of (4.1) is given by the finite sum $y^n_r(t) = \sum_{i=1}^{n}(\sum_{k=1}^{i}\beta_{ik}\gamma_k)\overline{\mu}_i(t)$.
We now define another operator in order to solve non-linear FFIDEs of the form

$$(D^{\beta}_{a^+}x_r)(t) = F_r(t, x_{1r}(t), x_{2r}(t)),\ x(a) = \alpha$$

which is equivalent to the system

$$\begin{cases} (D^{\beta}_{a^+}x_{1r})(t) = F_{1r}(t, x_{1r}(t), x_{2r}(t)) \\ (D^{\beta}_{a^+}x_{2r})(t) = F_{2r}(t, x_{1r}(t), x_{2r}(t)) \\ x_{1r}(a) = \alpha_{1r},\ x_{2r}(a) = \alpha_{2r}. \end{cases} \tag{4.2}$$

under (1)-differentiability, and to the system:

$$\begin{cases} (D^{\beta}_{a+}x_{1r})(t) = F_{2r}(t, x_{1r}(t), x_{2r}(t)) \\ (D^{\beta}_{a+}x_{2r})(t) = F_{1r}(t, x_{1r}(t), x_{2r}(t)) \\ x_{1r}(a) = \alpha_{1r},\ x_{2r}(a) = \alpha_{2r}. \end{cases} \tag{4.3}$$

under (2)-differentiability. Applying the Riemann-Liouville fractional integral J^{β}_{a+} to the two equations in (4.2) after homogenizing the initial conditions to get:

$$J^{\beta}_{a+}[(D^{\beta}_{a+}x_{1r}](t) = J^{\beta}_{a+}[F_{1r}(t, x_{1r}(t), x_{2r}(t))] \text{ and } J^{\beta}_{a+}[(D^{\beta}_{a+}x_{2r})](t) = J^{\beta}_{a+}[F_{2r}(t, x_{1r}(t), x_{2r}(t))],$$

which are equivalent to

$$\begin{cases} x_{1r}(t) = \frac{1}{\Gamma(\beta)}\int_a^t \frac{F_{1r}(s,x_{1r}(s),x_{2r}(s))}{(t-s)^{1-\beta}}\ dt = Q_{1r}(t, x_{1r}(t), x_{2r}(t)),\ t > a \\ x_{2r}(t) = \frac{1}{\Gamma(\beta)}\int_a^t \frac{F_{2r}(s,x_{1r}(s),x_{2r}(s))}{(t-s)^{1-\beta}}\ dt = Q_{2r}(t, x_{1r}(t), x_{2r}(t)),\ t > a. \end{cases} \tag{4.4}$$

Define the operator $I_{jr} : W_2^2[a, b] \longrightarrow W_2^1[a, b]$ by $I_{jr}x_{jr}(t) = x_{jr}(t), j = 1, 2$, and let $I_r = diag(I_{1r}, I_{2r})$. Obviously, I_r is a bounded linear operator such that $I_r : W[a, b] \longrightarrow H[a, b]$ and (4.4) can be rewritten as $I_r x_r(t) = Q_r(t, x_r(t)) = Q_r(t, x_{1r}(t), x_{2r}(t))$, where $Q_r = (Q_{1r}, Q_{2r})^T$.
Applying the RKHSM with the operator I_r, we can obtain the approximate solution of (4.2) of the form

$$x_r^n(t) = \sum_{i=1}^{n}\sum_{j=1}^{2}\sum_{l=1}^{i}\sum_{k=1}^{j} \beta_{jil} Q_{kr}(t_l, x_r(t_l))\overline{\psi}_{ij}(t),$$

which converges to the analytic solution:

$$x_r(t) = \sum_{i=1}^{\infty}\sum_{j=1}^{2}\sum_{l=1}^{i}\sum_{k=1}^{j} \beta_{jil} Q_{kr}(t_l, x_r(t_l))\overline{\psi}_{ij}(t).$$

Here, β_{jil} are the orthogonalization coefficients.

5. Numerical Examples

Example 5.1. *Consider the following FFIDE of Volterra type:*

$$(D^{\beta}_{0+}x)(t) = -x(t) + \sin(t) - \int_0^t x(\tau)\ d\tau,\ 0 < \beta \le 1,\ t \in [0, 1],\ x(0) = \alpha \tag{5.1}$$

where $[\alpha]^r = \Big[\frac{24}{25} + \frac{r}{25}, \frac{101}{100} - \frac{r}{100}\Big]$.

Case1: Under $[(1) - \beta]$-differentiability, the equivalent system is

$$(D^{\beta}_{0+}x_{1r})(t) = -x_{2r}(t) + \sin(t) - \int_0^t x_{2r}(\tau)\ d\tau,\ (D^{\beta}_{0+}x_{2r})(t) = -x_{1r}(t) + \sin(t) - \int_0^t x_{1r}(\tau)\ d\tau,$$

$$x_{1r}(0) = \frac{24}{25} + \frac{r}{25},\ x_{2r}(0) = \frac{101}{100} - \frac{r}{100}$$

If $\beta = 1$, then the exact solution of this system is

$$x_{1r}(t) = a_1(r)e^{(\frac{1-\sqrt{5}}{2})t} + a_2(r)e^{(\frac{1+\sqrt{5}}{2})t} + e^{-0.5t}\Big(a_3(r)\cos(\frac{\sqrt{3}}{2}t) + a_4(r)\sin(\frac{\sqrt{3}}{2}t)\Big) + \sin t$$

$$x_{2r}(t) = b_1(r)e^{(\frac{1-\sqrt{5}}{2})t} + b_2(r)e^{(\frac{1+\sqrt{5}}{2})t} + e^{-0.5t}\Big(b_3(r)\cos(\frac{\sqrt{3}}{2}t) + b_4(r)\sin(\frac{\sqrt{3}}{2}t)\Big) + \sin t$$

where $a_1(r) = \frac{5-\sqrt{5}}{20}\Big(x_{1r}(0) - x_{2r}(0)\Big)$, $a_2(r) = \frac{5+\sqrt{5}}{20}\Big(x_{1r}(0) - x_{2r}(0)\Big)$, $a_3(r) = \frac{1}{2}\Big(x_{1r}(0) + x_{2r}(0)\Big)$, $a_4(r) = \frac{\sqrt{3}}{6}\Big(-x_{2r}(0) - x_{1r}(0) - 4\Big)$, $b_1(r) = -a_1(r)$, $b_2(r) = a_2(r)$, $b_3(r) = a_3(r)$, $b_4(r) = -a_4(r)$.

Case2: Under $[(2)-\beta]$-differentiability,(5.1) is equivalent to the system:

$$D^{\beta}_{0+}x_{1r}(t) = -x_{1r}(t) - \int_0^t x_{1r}(\tau)\ d\tau + \sin t, D^{\beta}_{0+}x_{2r}(t) = -x_{2r}(t) - \int_0^t x_{2r}(\tau)\ d\tau + \sin t$$

$$x_{1r}(0) = \frac{24}{25} + \frac{r}{25},\ x_{2r}(0) = \frac{101}{100} - \frac{r}{100}$$

If $\beta = 1$, then the exact solution of this system is

$$x_{1r}(t) = \sin t + \Big(\frac{24}{25} + \frac{r}{25}\Big)e^{-0.5t}\cos(\frac{\sqrt{3}}{2}t) + e^{-0.5t}\sin(\frac{\sqrt{3}}{2}t)\Big(\frac{-2}{\sqrt{3}} - \frac{1}{\sqrt{3}}(\frac{24}{25} + \frac{r}{25})\Big)$$

$$x_{2r}(t) = \sin t + \Big(\frac{101}{100} - \frac{r}{100}\Big)e^{-0.5t}\cos(\frac{\sqrt{3}}{2}t) + e^{-0.5t}\sin(\frac{\sqrt{3}}{2}t)\Big(\frac{-2}{\sqrt{3}} - \frac{1}{\sqrt{3}}(\frac{101}{100} - \frac{r}{100})\Big)$$

Applying the RKHS method, with $n = 50$ and $m = 5$, some numerical results are given in TABLE 1, 2 and FIGURE 1 .

Table 1: The fuzzy approximate $(1)-$solution $[x_{1r}(t), x_{2r}(t)]$ at different values of t and r of example 5.1

t	r	$\beta = 1$	Error $\beta = 1$	$\beta = 0.9$	$\beta = 0.8$
0.6	0	[0.51156229, 0.61655959]	5.40893182×10^{-5}	[0.49595447, 0.61223616]	[0.48358722, 0.61461544]
	0.5	[0.5410404, 0.59353910]	4.004646025×10^{-5}	[0.52807641, 0.58621725]	[0.51925885, 0.58477296]
	0.75	[0.55577952, 0.58202885]	3.302503127×10^{-5}	[0.54413738, 0.57320780]	[0.53709466, 0.56985172]
	1	[0.57051860, 0.57051860]	2.60036023×10^{-5}	[0.56019834, 0.56019834]	[0.55493047, 0.55493047]
0.9	0	[0.37622029, 0.53923971]	7.419506231×10^{-5}	[0.37810912, 0.56349221]	[0.37933748, 0.5946745]
	0.5	[0.41843591, 0.49994563]	4.660477954×10^{-5}	[0.4259355, 0.51862714]	[0.43471020, 0.54237872]
	0.75	[0.43954373, 0.48029858]	3.280963816×10^{-5}	[0.44984882, 0.49619460]	[0.46239656, 0.51623082]
	1	[0.46065154, 0.46065154]	1.901449677×10^{-5}	[0.47376206, 0.47376206]	[0.49008292, 0.49008292]

Table 2: The fuzzy approximate $(2)-$solution $[x_{1r}(t), x_{2r}(t)]$ at different values of t and r of example 5.1

t	r	$\beta = 1$	Error $\beta = 1$	$\beta = 0.9$	$\beta = 0.8$
0.4	0	[0.65359329, 0.6840618]	7.623925024×10^{-5}	[0.62976210, 0.65844928]	[0.60932730, 0.63637377]
	0.25	[0.65968701, 0.68253845]	7.565488371×10^{-5}	[0.61951492, 0.67299953]	[0.59752145, 0.65223660]
	0.5	[0.66578073, 0.68101502]	7.507051718×10^{-5}	[0.64123697, 0.65558057]	[0.62014589, 0.63366912]
	0.75	[0.67187444, 0.67949159]	7.448615065×10^{-5}	[0.64697441, 0.65414621]	[0.62555518, 0.63231680]
	1	[0.67796816, 0.67796816]	7.390178412×10^{-5}	[0.65271185, 0.65271185]	[0.63096448, 0.63096448]
0.8	0	[0.48083881, 0.49426469]	1.029130177×10^{-6}	[0.48691471, 0.50001255]	[0.49700106, 0.51005328]
	0.25	[0.48352398, 0.49359339]	9.572314812×10^{-6}	[0.46185765, 0.52703428]	[0.47190067, 0.53711150]
	0.5	[0.48620916, 0.49292210]	8.853327853×10^{-6}	[0.49215384, 0.49870276]	[0.50222195, 0.50874805]
	0.75	[0.48889433, 0.49225080]	8.134340894×10^{-6}	[0.49477341, 0.49804787]	[0.50483239, 0.50809544]
	1	[0.49157951, 0.49157951]	7.415353936×10^{-6}	[0.49739298, 0.49739298]	[0.5074428, 0.50744283]

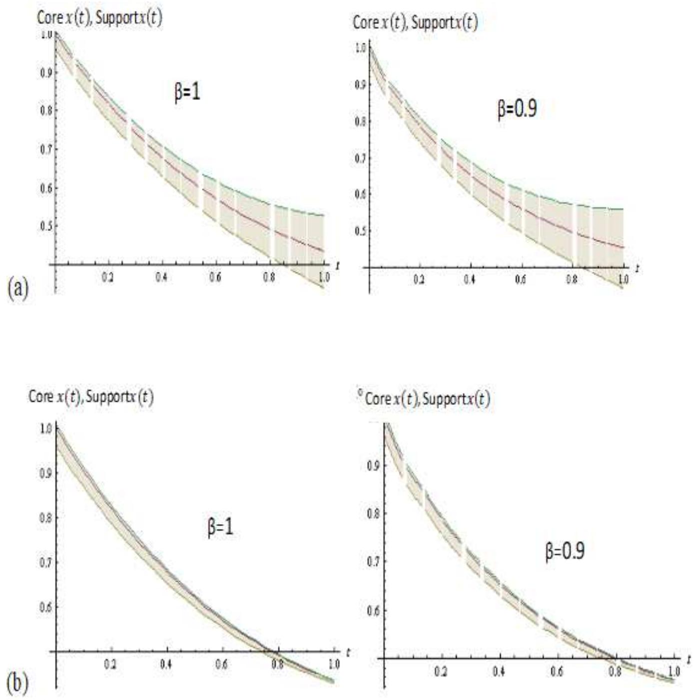

Figure 1: a) The core and the support of the fuzzy approximate $(1)-$solution, b) The core and the support of the fuzzy approximate $(2)-$solution

Example 5.2. *Consider the following FFIDE of Frdholm type:*

$$(D^{\beta}_{0+}x)(t) + 2e^t x(t) = \Big(\sinh(t)(1-t) + e^{2t} + e^{-1}\Big)\alpha + \int_0^1 \tau x(\tau)\ d\tau + \int_0^t tx(\tau)\ d\tau, \tag{5.2}$$

where $0 < \beta \leq 1,\ t \in [0,1],\ x(0) = \alpha$ *and* $[\alpha]^r = \Big[-\sqrt{1-r}, \sqrt{1-r}\Big]$.

Case1: Under $[(1)-\beta]$-differentiability, the equivalent system is

$$(D^{\beta}_{0+}x_{1r})(t) + 2e^t x_{1r}(t) = -\Big(\sinh(t)(1-t) + e^{2t} + e^{-1}\Big)\sqrt{1-r} + \int_0^1 \tau x_{1r}(\tau)\ d\tau + \int_0^t tx_{1r}(\tau)\ d\tau,$$

$$(D^{\beta}_{0+}x_{2r})(t) + 2e^t x_{2r}(t) = \Big(\sinh(t)(1-t) + e^{2t} + e^{-1}\Big)\sqrt{1-r} + \int_0^1 \tau x_{2r}(\tau)\ d\tau + \int_0^t tx_{2r}(\tau)\ d\tau$$

$x_{1r}(0) = -\sqrt{1-r},\ x_{2r}(0) = \sqrt{1-r}$.
If $\beta = 1$, then the exact solution of this system is $x_{1r}(t) = -\sqrt{1-r}\cosh(t),\ x_{2r}(t) = \sqrt{1-r}\cosh(t)$.
Applying the RKHSM, with $n = 100$ and $m = 5$, some numerical results are given in Table 3 and Figure 2.

Table 3: The fuzzy exact and approximate $(1)-$solution $[x_{1r}(t), x_{2r}(t)]$ at different values of t and r of example 5.2

t	r	$\beta = 1$	Error $\beta = 1$	$\beta = 0.9$	$\beta = 0.8$
0.5	0	[-1.12764366, 1.12764366]	1.769585551×10^{-5}	[-1.14200986, 1.14200986]	[-1.15547570, 1.15547570]
	0.25	[-0.97656805, 0.97656805]	1.532506041×10^{-5}	[-0.98900955, 0.98900955]	[-1.00067131, 1.00067131]
	0.5	[-0.79736447, 0.79736447]	1.251285943×10^{-5}	[-0.80752292, 0.80752292]	[-0.81704470, 0.81704470]
	0.75	[-0.56382183, 0.56382183]	8.847927754×10^{-6}	[-0.57100493, 0.57100493]	[-0.57773785, 0.57773785]
1	0	[-1.54309532, 1.54309532]	1.469289793×10^{-5}	[-1.55929451, 1.55929451]	[-1.57399929, 1.57399929]
	0.25	[-1.33635975, 1.33635975]	1.272442287×10^{-5}	[-1.35038866, 1.35038866]	[-1.36312337, 1.36312337]
	0.5	[-1.09113317, 1.09113317]	1.038944776×10^{-5}	[-1.10258772, 1.10258772]	[-1.11298557, 1.11298557]
	0.75	[-0.77154766, 0.77154766]	7.346448967×10^{-6}	[-0.77964725, 0.77964725]	[-0.78699964, 0.78699964]

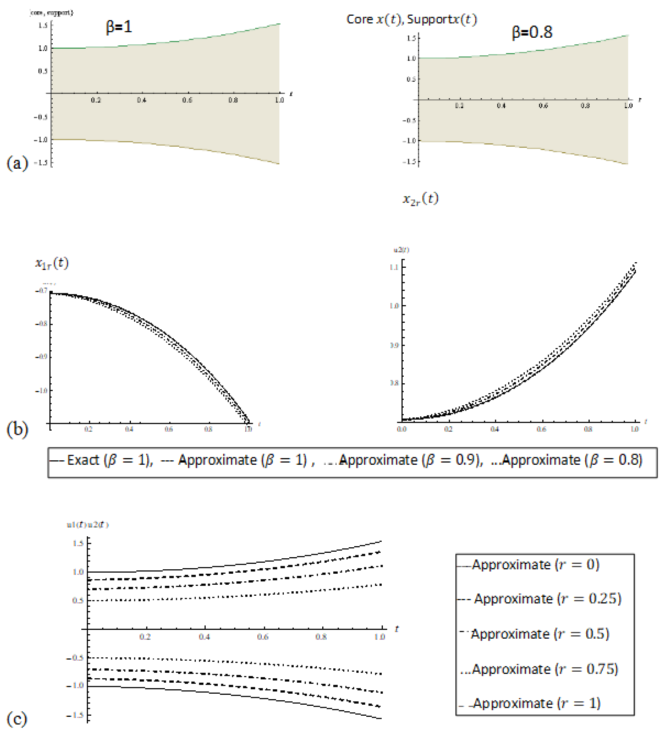

Figure 2: a) The core and the support of the fuzzy approximate solution, b) Approximate solutions $x_{1r}(t)$ and $x_{2r}(t)$ for different values of β at $r = 0.5$, c) Approximate solutions $x(t)$ at different values of r when $\beta = 0.9$ for example 5.2, case1

Case2: Under $[(2) - \beta]$-differentiability, the system is:

$$(D_{0+}^{\beta}x_{1r})(t) + 2e^t x_{2r}(t) = \left(\sinh(t)(1-t) + e^{2t} + e^{-1}\right)\sqrt{1-r} + \int_0^1 \tau x_{2r}(\tau)\,d\tau + \int_0^t t x_{2r}(\tau)\,d\tau,$$

$$(D_{0+}^{\beta}x_{2r})(t) + 2e^t x_{1r}(t) = -\left(\sinh(t)(1-t) + e^{2t} + e^{-1}\right)\sqrt{1-r} + \int_0^1 \tau x_{1r}(\tau)\,d\tau + \int_0^t t x_{1r}(\tau)\,d\tau$$

$x_{1r}(0) = -\sqrt{1-r},\ x_{2r}(0) = \sqrt{1-r}$

Applying the RKHS method, with $n = 25$ and $\beta = 1$, the resulting approximate solution is not acceptable since $x_{1r} > x_{2r}$ for some $t \in [0, 1]$.

Example 5.3. *Consider the following FFIDE of Volterra type:*

$$(D^{\beta}_{0+}x)(t) = \int_0^t x(\tau)\ d\tau + (1+t)\alpha,\ \ 0<\beta\le 1,\ \ t\in[0,1],\ \ x(0)=0 \tag{5.3}$$

where $[\alpha]^r = \Big[r-1, 1-r\Big]$.

Case1: Under $[(1)-\beta]$-differentiability, the equivalent system is

$$(D^{\beta}_{0+}x_{1r})(t) = (1+t)(r-1) + \int_0^t x_{1r}(\tau)\ d\tau,\ \ (D^{\beta}_{0+}x_{2r})(t) = (1+t)(1-r) + \int_0^t x_{2r}(\tau)\ d\tau$$

$$x_{1r}(0)=0,\ \ x_{2r}(0)=0$$

If $\beta = 1$, then the exact solution of this system is

$$x_{1r}(t) = (r-1)(e^t-1),\ \ x_{2r}(t) = (1-r)(e^t-1).$$

Applying the RKHS method with $n = 100$ and $m = 6$, some numerical results are given in Table 4 and Figure3.

Table 4: The fuzzy exact and approximate solutions at different values of t and r of example 5.3, case1.

r	t	$\beta = 1$	Error $\beta = 1$	$\beta = 0.9$	$\beta = 0.8$
0	0.1	[-0.08413588, 0.08413588]	8.456648354×10^{-7}	[-0.17527338, 0.17527338]	[-0.13570074, 0.13570074]
	0.2	[-0.17712092, 0.17712092]	1.27691154×10^{-6}	[-0.32904351, 0.32904351]	[-0.27015534, 0.27015534]
	0.3	[-0.27988553, 0.27988553]	1.509760871×10^{-6}	[-0.48662083, 0.48662083]	[-0.41305344, 0.41305344]
	0.4	[-0.39345821, 0.39345821]	1.54005965×10^{-6}	[-0.65417798, 0.65417798]	[-0.56775614, 0.56775614]
	0.5	[-0.51897565, 0.51897565]	1.35854611×10^{-6}	[-0.83523550, 0.83523550]	[-0.73662785, 0.73662785]
	0.6	[-0.65769408, 0.65769408]	9.505664934×10^{-7}	[-1.03255728, 1.03255728]	[-0.9217967, 0.9217967]
	0.7	[-0.81100187, 0.81100187]	2.956949585×10^{-7}	[-1.24867876, 1.24867876]	[-1.12538282, 1.12538282]
	0.8	[-0.98043337 0.98043337]	6.32750391×10^{-7}	[-1.48612193, 1.48612193]	[-1.34959976, 1.34959976]
	0.9	[-1.16768435, 1.16768435]	1.868304202×10^{-6}	[-1.74750771, 1.74750771]	[-1.59681544, 1.59681544]
	1	[-1.37462891, 1.37462891]	3.452073773×10^{-6}	[-2.03568221, 2.03568221]	[-1.86963027, 1.86963027]
0.6	0.1	[-0.04206794, 0.04206794]	4.228324177×10^{-7}	[-0.07010935, 0.07010935]	[-0.05428029, 0.05428029]
	0.2	[-0.08856046, 0.08856046]	6.384557709×10^{-7}	[-0.13161740, 0.13161740]	[-0.10806213, 0.10806213]
	0.3	[-0.13994276, 0.13994276]	7.548804354×10^{-7}	[-0.19464833, 0.19464833]	[-0.16522137, 0.16522137]
	0.4	[-0.19672910, 0.19672910]	7.700298252×10^{-7}	[-0.26167119, 0.26167119]	[-0.22710245, 0.22710245]
	0.5	[-0.25948782, 0.25948782]	6.792730553×10^{-7}	[-0.33409420, 0.33409420]	[-0.29465114, 0.29465114]
	0.6	[-0.32884704, 0.32884704]	4.752832468×10^{-7}	[-0.41302291, 0.41302291]	[-0.36871870, 0.36871870]
	0.7	[-0.40550093, 0.40550093]	1.478474794×10^{-7}	[-0.49947150, 0.49947150]	[-0.45015313, 0.45015313]
	0.8	[-0.49021668, 0.49021668]	3.163751954×10^{-7}	[-0.59444877, 0.59444877]	[-0.53983990, 0.53983990]
	0.9	[-0.58384217, 0.58384217]	9.341521008×10^{-7}	[-0.69900308, 0.69900308]	[-0.63872617, 0.63872617]
	1	[-0.68731445, 0.68731445]	1.726036886×10^{-6}	[-0.81427288, 0.81427288]	[-0.74785210, 0.74785210]

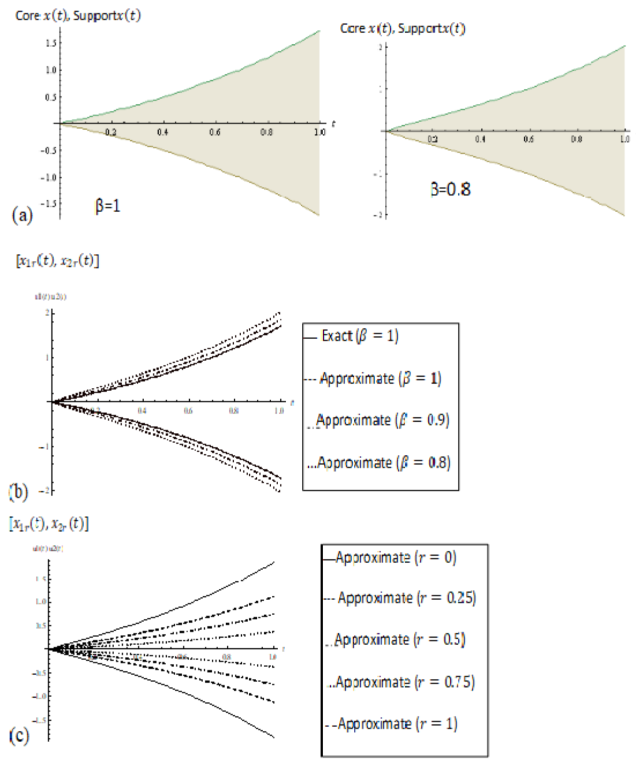

Figure 3: a) The core and the support of the fuzzy approximate solution, b) Approximate solution $x(t)$ for different values of β at $r = 0.2$, and c) Approximate solutions for different values of r at $\beta = 0.9$ for example 5.3, case1.

Case2: Under $[(2)-\beta]$-differentiability,(5.3) is equivalent to the system:

$$D^{\beta}_{0+}x_{2r}(t) = (1+t)(r-1) + \int_0^t x_{1r}(\tau)\,d\tau,\ \ D^{\beta}_{0+}x_{1r}(t) = (1+t)(1-r) + \int_0^t x_{2r}(\tau)\,d\tau$$

$x_{1r}(0) = 0,\ x_{2r}(0) = 0$ whose solution is not $(2)-$ differentiable.

Example 5.4. *Consider the following non linear FFIDE:*

$$(D^{\beta}_{0+}x)(t) = f(t) + 0.1t^2 \int_0^1 \tau x^2(\tau)\,d\tau,\ 0 < \beta \le 1,\ t \in [0,1],\ x(0) = 0 \tag{5.4}$$

where $[f(t)]^r = \left[r - \dfrac{(rt)^2}{40}, 2 - r - \dfrac{((2-r)t)^2}{40}\right]$.

Assuming that $x(t)$ is $[(1)-\beta]-$ differentiable, then the FFIDE is equivalent to the system

$$(D^{\beta}_{0+}x_{1r})(t) = r - \frac{(rt)^2}{40} + 0.1t^2 \int_0^1 \tau x^2_{1r}(\tau)\,d\tau,\ (D^{\beta}_{0+}x_{2r})(t) = 2 - r - \frac{((2-r)t)^2}{40} + 0.1t^2 \int_0^1 \tau x^2_{2r}(\tau)\,d\tau$$

$x_{1r}(0) = 0,\ x_{2r}(0) = 0$. For $\beta = 1$, the exact solution is $x_{1r}(t) = rt$ and $x_{2r}(t) = (2-r)t$.
Applying the RKHS method with $n = 40$ and $m = 6$, some numerical results are in Table 5 and Figure 4.

Table 5: The fuzzy exact and approximate solutions at different values of t and r of example 5.4

r	t	$\beta = 1$	Error $\beta = 1$	$\beta = 0.9$	$\beta = 0.8$
0.25	0.1	[0.02492374, 0.17446466]	7.625625322×10^{-5}	[0.03272358,0.22903388]	[0.04254030, 0.29773781]
	0.2	[0.04992374, 0.34946451]	$7.6259231922\times10^{-5}$	[0.06106021,0.42718901]	[0.07406147, 0.51812231]
	0.3	[0.07492373, 0.524464118]	$7.6267296882\times10^{-5}$	[0.11391203, 0.79567582]	[0.10242648, 0.71603335]
	0.4	[0.09992371, 0.699463342]	$7.6282996492\times10^{-5}$	[0.08794104, 0.61483721]	[0.12891137, 0.90028348]
	0.5	[0.12492369, 0.874462059]	$7.6308883082\times10^{-5}$	[0.13922167, 0.97136179]	[0.15407526, 1.07473360]
	0.6	[0.14992365, 1.04946048]	$7.6340701372\times10^{-5}$	[0.16401345, 1.14289813]	[0.178232070, 1.24164542]
	0.7	[0.17492362, 1.22445890]	$7.6372519652\times10^{-5}$	[0.18838362, 1.31109837]	[0.201585794, 1.40270101]
	0.8	[0.19992359, 1.39945732]	$7.6404337942\times10^{-5}$	[0.21240592, 1.47684445]	[0.224285385, 1.55948472]
	0.9	[0.22492356, 1.57445574]	$7.6436156222\times10^{-5}$	[0.23614524, 1.64129525]	[0.246452591, 1.71380307]
	1	[0.24992353, 1.74945416]	$7.6467974512\times10^{-5}$	[0.25966706, 1.80609914]	[0.26819888, 1.86796217]
0.75	0.1	[0.07477101, 0.12461798]	2.289892228×10^{-4}	[0.09816628,0.16360305]	[0.12761457, 0.21268041]
	0.2	[0.14977098, 0.24961790]	2.290160679×10^{-4}	[0.18314742, 0.30519035]	[0.22214041, 0.37016070]
	0.3	[0.22477091, 0.37461770]	2.290888003×10^{-4}	[0.26371597, 0.43934803]	[0.30714344, 0.51167906]
	0.4	[0.29977076, 0.49961731]	2.292305055×10^{-4}	[0.34149205, 0.56874665]	[0.38643468, 0.64355873]
	0.5	[0.37477053, 0.62461665]	2.294643766×10^{-4}	[0.41720935, 0.69458942]	[0.46168397, 0.76857014]
	0.6	[0.44977024, 0.74961585]	2.297519607×10^{-4}	[0.49129827, 0.81759330]	[0.53384241, 0.88831377]
	0.7	[0.52476996, 0.87461505]	2.300395448×10^{-4}	[0.56406778, 0.93830653]	[0.60355870, 1.00393128]
	0.8	[0.59976967, 0.99961425]	2.303271289×10^{-4}	[0.63579194, 1.05727299]	[0.67135794, 1.11642682]
	0.9	[0.67476938, 1.12461345]	2.30614713×10^{-4}	[0.706767291, 1.17515465]	[0.73774318, 1.22686600]
	1	[0.74976909, 1.24961264]	2.309022972×10^{-4}	[0.77736204, 1.292848569]	[0.80326866, 1.33653588]

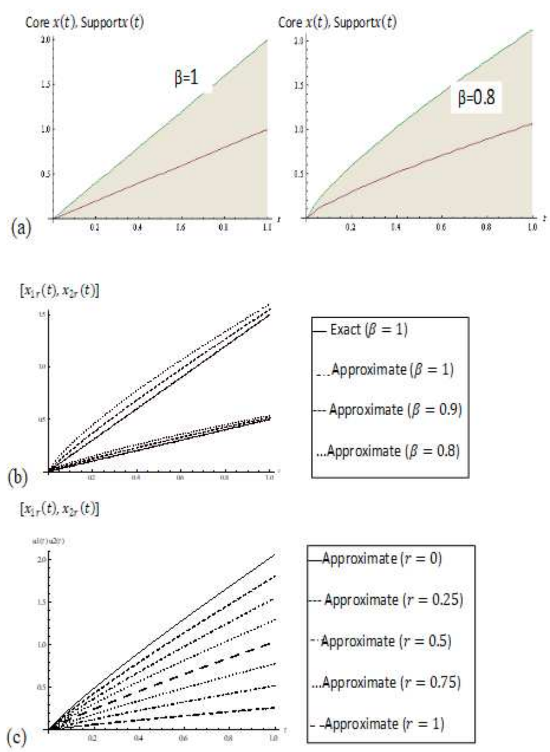

Figure 4: a) The core and the support of the fuzzy approximate solution, b) Approximate solutions $x(t)$ for different values of β at $r = 0.5$, and c) Approximate solutions for different values of r at $\beta = 0.9$ for example 5.4.

6. Conclusion

In this work, we introduced modified algorithms based on the RKHS method to obtain approximate solutions of fuzzy fractional integro-differential equations under Caputo's H-differentiability. The analytic solution and its approximate solution are represented in series form in term of their parametric form in the space $W_2^2[a,b] \oplus W_2^2[a,b]$. Moreover, the approximate solution and its derivatives are uniformly convergent to the analytic solution and its derivatives, respectively.
Several examples of linear and non-linear FFIDEs were given to show the effectiveness of the proposed method. By comparing our results with the exact solutions, we observe that the RKHS method yields accurate approximations. To see the effects of the fractional derivative on the solution, we solved the same fuzzy integro-differential equation (FIDEs) for different values of the fractional order. The results showed that the solutions of FFIDEs approach the solution of FIDEs as the fractional order approaches the integer order. The RKHS method has several advantages; it is accurate and applicable for linear and non-linear differential equations. Also, it is possible to pick any point in the interval of integration.

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