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General Decay for a Timoshenko-type System for Thermoelasticity of Type III with Delay, Past History and Distributed Delay

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ABSTRACT: In this studies, we are interested to the one dimensional Timoshenko system of thermoelasticity of type III with past history, delay and distributive delay. We show the well-posedness and we give a general stability result of the system, under suitable conditions on the kernel function of infinite history in the both cases equal and nonequal speeds of wave probagation. Our result improves the few existing works in this direction.

Key Words: Timoshenko system, past history, relaxation function, distributed delay, energy decay.

Contents

1. Introduction

In the present work, we are concerned with the following problem,

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi\right)_x + \beta\theta_{tx} + \mu_1\varphi_t\left(x,t\right) \\ +\mu_2\varphi_t\left(x,t-\tau\right) = 0, \text{ in } (0,1)\times(0,\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi\right) - \beta\theta_t + f\left(\psi\right) \\ +\int_0^\infty g\left(s\right)\psi_{xx}\left(x,t-s\right)ds = 0, \text{ in } (0,1)\times(0,\infty), \\ \rho_3\theta_{tt} - \delta\theta_{xx} + \gamma\varphi_{tx} + \gamma\psi_t - \ell\theta_{txx} \\ -\int_{\tau_1}^{\tau_2}\mu\left(\zeta\right)\theta_{txx}\left(x,t-\zeta\right)d\zeta = 0, \text{ in } (0,1)\times\times(\tau_1,\tau_2)\times(0,\infty), \\ \varphi\left(x,0\right) = \varphi_0\left(x\right), \ \varphi_t\left(x,0\right) = \varphi_1\left(x\right), \ \theta\left(x,0\right) = \theta_0\left(x\right), \\ \theta_t\left(x,0\right) = \theta_1\left(x\right), \ x\in(0,1), \\ \psi\left(x,-t\right) = \psi_0\left(x,t\right), \ x\in(0,1), \ t>0, \\ \varphi\left(0,t\right) = \varphi\left(1,t\right) = \psi\left(0,t\right) = \psi\left(1,t\right) = 0, \\ \theta\left(0,t\right) = \theta\left(1,t\right) = 0, \ t>0, \\ \varphi_t\left(x,t-\tau\right) = h_0\left(x,t-\tau\right) \ , \ x\in(0,1), \ \tau\in(0,\tau_2), \ t>0, \\ \theta_{tx}\left(x,-t\right) = f_0\left(x,t\right), \ x\in(0,1), \ t>0. \end{cases} \tag{1.1}$$

Where φ is the longitudinal displacement, ψ is the volume fraction, θ is the difference in temperature, the coefficients ρ_1, ρ_2, ρ_3, k, b, ℓ, β, δ, γ, μ_1, μ_2 are positive constants with $\mu_1 \geq \mu_2$ and $\tau_1 < \tau_2$ are non-negative constants such that $\mu : [\tau_1, \tau_2] \to R$ represents distributed time delay, f is a forcing term.

In 1921, Timoshenko [19] considered the system (1.2) as a simple model describing the transverse vibration of a beam. Where t denotes the time variable and x is the space variable along the beam of length L, in its equilibrium configuration, u is the transverse displacement of the beam and φ is the rotation angle of the filament of the beam. The coefficients ρ, I_ρ, E, I and K are respectively the density (the mass per unit length), the polar moment of inertia of a cross section,Young's modulus of elasticity, the moment of inertia of a cross section, and the shear modulus.

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$$\begin{cases} \rho u_{tt} = k\left(u_x + \varphi\right)_x, \\ I_\rho \varphi_{tt} = \left(EI\varphi_x\right)_x + K\left(u_x - \varphi\right). \end{cases} \tag{1.2}$$

In recent years, the problem of existence and stability of the Timoshenko system and hyperbolic system with past history has attracted considerable attention of a lot mathematicians and many results have been established concerning existence and asymptotic behavior. (e.g. [4,5,6,7,9,11,15,18,20,22]). Guesmia and Messaoudi [7] considered the following Timoshenko-type system

$$\begin{cases} \rho_1 \varphi_{tt} - b\varphi_{xx} + k\left(\varphi_x + \psi\right) + \gamma h\left(\varphi\left(t\right)\right) + \int_0^\infty g\left(s\right)\left(a\left(x\right)\varphi_x\left(x, t-s\right)\right)_x ds = 0, \\ \rho_2 \psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi\right) = 0. \end{cases} \tag{1.3}$$

showed that the complementary controls guarantees the stability of the system in the both cases equal and nonequal speeds propagation.

E. Jaime et all [18] considered the similar system

$$\begin{cases} \rho_1 \varphi_{tt}) - k\left(\varphi_x + \psi\right)_x = 0, \\ \rho_2 \psi_{tt} - b\psi_{xx} + \int_0^\infty g\left(s\right)\left(\varphi_x\left(x, t-s\right)\right)_x ds + k\left(\varphi_x + \psi\right) = 0. \end{cases} \tag{1.4}$$

They gave some general decay results for the case of equal speeds propagation, under some conditions that satisfies the relaxation function g which are given by

$$\begin{aligned} k_0 g(t) &\leq g'(t) \leq -k_1 g(t),\ \left|g''(t)\right| \leq -k_1\left|g(t)\right|, \\ \overline{b} &= b - \int_0^\infty g\left(s\right) ds > 0. \end{aligned}$$

In our case we suppose that the relaxation function g satisfies the following condition

$$g'(t) \leq -\xi(t)g(t), \tag{1.5}$$

for which we will assure a general decay for the case of equal speeds of wave propagation as well as in the opposite one.

Time delays arise in many application, because the model where the time delay appears takes into consideration the past of the phenomena, in order to be more precise when they treat it. (e.g. [3,4,8,9, 10,12,14,21,22,23]). Houasni et all [10] considered a thermo-viscoelastic system of Timoshenko-type with nonlinear damping and a distributed delay acting on transverse displacement,

$$\begin{cases} \rho_1 \varphi_{tt} - K(\varphi_x + \psi)_x + \mu_1 \varphi_t + \int_{\tau_1}^{\tau_2} \mu_2(s)\varphi_t\left(x, t-s\right) ds = 0, \\ \rho_2 \psi_{tt} - \beta\psi_{xx} + \int_0^t g\left(t-s\right)\left(a(x)\psi_x(s)\right)_x ds + K(\varphi_x + \psi) \\ +\mu_3\left(t\right) b(x) f(\psi_t) + \gamma\theta_x = 0, \\ \rho_3 \theta_t + Kq_x + \gamma\psi_{tx} = 0, \\ \rho_4 q_t + \delta q + K\theta_x = 0. \end{cases} \tag{1.6}$$

Under suitable assumptions on the weight of the delay and that of frictional damping, the authors established the well-posedness result and proved that the system is exponentially stable regardless of the speeds of wave propagation.

In the same direction Feng and Pelicer [4] considered a Timoshenko system with time delay

$$\begin{cases} \rho_1 \varphi_{tt} - k\left(\varphi_x + \psi\right)_x = 0 \\ \rho_2 \psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi\right) + \mu_1 \psi_t\left(x, t\right) + \mu_2 \psi_t\left(x, t-\tau\right) + f\left(\psi\right) = 0 \end{cases} \tag{1.7}$$

where $\mu_2 \psi_t(x, t-\tau)$ is time delay. They established the well-posedness of the problem with respect to weak solutions under suitable assumptions and the exponential stability of the system under the usual equal wave speed assumption.

Kafini et al. [9] were interested by the following Timoshenko-type system of thermoelasticity of type III with distributive delay, precisely they considered the system

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \beta\theta_x = 0, \\ \rho_3\theta_{tt} - \delta\theta_{xx} - k\theta_{txx} - \int_{\tau_1}^{\tau_2} g(s)\,\theta_{txx}(x, t-s)\,ds + \gamma\psi_{tx} = 0, \end{cases} \tag{1.8}$$

where $\tau_1 < \tau_2$ are non-negative constants such that $g : [\tau_1, \tau_2] \to R^+$ represents the distributive time delay. They proved an exponential decay in the case of equal wave speeds and they gave a polynomial decay result in the case of nonequal wave speeds for some smooth initial data.

In the present work we consider (1.1), we prove the well-posedness and we establish the energy decay rate in case of the equal-speed propagation as well as in the opposite case.

In the next section we present some basic results, in section 2 by the aid of the semigroup theory we show that the problem (1.1) is well-posed, section 3, is devoted to establish some technical lemmas that we need. in section 4 we give our stability result.

2. Some useful preliminary results

In this section we present some basic results which will be used later, for easy reading we introduce the new variables

$$\bar{z}(x, \rho, t) = \varphi_t(x, t - \tau\rho), \quad x \in (0,1),\ \rho \in (0,1),\ t > 0,$$

$$z(x, \rho, \varsigma, t) = \theta_{tx}(x, t - \varsigma\rho), \quad x \in (0,1),\ \rho \in (0,1),\ \varsigma \in (\tau_1, \tau_2),\ t > 0.$$

leads to

$$\tau\bar{z}_t(x, \rho, t) + \bar{z}_\rho(x, \rho, t) = 0, \quad x \in (0,1),\ \rho \in (0,1),\ t > 0.$$

So we have

$$\varsigma z_t(x, \rho, \varsigma, t) + z_\rho(x, \rho, \varsigma, t) = 0, \quad x \in (0,1),\ \rho \in (0,1),\ \varsigma \in (\tau_1, \tau_2),\ t > 0.$$

Substituting the last expressions we get the equivalent system

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi)_x + \beta\theta_{tx} + \mu_1\varphi_t(x,t) \\ +\mu_2\bar{z}(x,1,t) = 0 \text{ in } (0,1)\times(0,\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) - \beta\theta_t + f(\psi) \\ + \int_0^\infty g(s)\,\psi_{xx}(x, t-s)\,ds = 0 \text{ in } (0,1)\times(0,\infty), \\ \rho_3\theta_{tt} - \delta\theta_{xx} - \ell\theta_{txx} + \gamma\varphi_{tx} + \gamma\psi_t \\ - \int_{\tau_1}^{\tau_2} \mu(\varsigma)\,z_x(x,1,\varsigma,t)\,d\varsigma = 0 \text{ in } (0,1)\times(\tau_1,\tau_2)\times(0,\infty), \\ \tau\bar{z}(x,\rho,t) + \bar{z}_\rho(x,\rho,t) = 0, \quad x,\rho \in (0,1),\ t > 0, \\ \varsigma z_t(x,\rho,\varsigma,t) + z_\rho(x,\rho,\varsigma,t) = 0, \quad x,\rho \in (0,1),\ \varsigma \in (\tau_1,\tau_2),\ t > 0, \\ \varphi(x,0) = \varphi_0(x),\ \varphi_t(x,0) = \varphi_1(x),\ \theta(x,0) = \theta_0(x), \\ \theta_t(x,0) = \theta_1(x) \text{ in } (0,1), \\ \psi(x,-t) = \psi_0(x,t) \text{ in } (0,1)\times(0,\infty), \\ \varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = 0, \\ \theta(0,t) = \theta(1,t) = 0 \text{ in } (0,\infty), \\ \bar{z}(x,\rho,0) = h_0(x,-\tau\rho), \quad x \in (0,1),\ \rho \in (0,1),\ \tau \in (0,\tau_2) \\ z(x,\rho,\varsigma,0) = f_0(x,\rho\varsigma),\ x,\rho \in (0,1),\ \varsigma \in (\tau_1,\tau_2)\ . \end{cases} \tag{2.1}$$

Suppose that (H_1,H_2,H_3,H_4,H_5,H_6) hold.

(H1) $\mu : [\tau_1, \tau_2] \to R$ is a bounded function and

$$\ell - \int_{\tau_1}^{\tau_2} |\mu(\varsigma)|d\varsigma > 0. \tag{2.2}$$

(H2) $g : R_+ \to R_+$ is a C^1 function satisfying

$$g(0) > 0, \quad b - \int_0^\infty g(s)ds = l > 0, \quad \int_0^\infty g(s)ds = g_0. \tag{2.3}$$

(H3) There exists a positive nonincreasing differentiable function $\xi : R_+ \to R_+$ satisfying

$$g'(t) \leq -\xi(t)g(t), \ t \geq 0. \tag{2.4}$$

(H4) $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$|f(\psi^2) - f(\psi^1)| \leq k_0(|\psi^1|^\varrho + |\psi^2|^\varrho)|\psi^1 - \psi^2|, \quad \psi^1, \psi^2 \in R, \tag{2.5}$$

where $k_0 > 0$, $\varrho > 0$. In addition we assume that

$$0 \leq \widehat{f}(\psi) \leq f(\psi)\psi, \quad \psi \in \mathbb{R}, \tag{2.6}$$

with $\widehat{f}(\psi) = \int_0^\psi f(s)ds$.

(H5) Assume that the constant ξ satisfies the following inequality

$$\tau\gamma\mu_2 < \xi < \tau\gamma\left(2\mu_1 - \mu_2\right). \tag{2.7}$$

The first-order energy associated to the system (1.2) is given by

$$\begin{aligned} E(t) = {} & \frac{\gamma}{2}\int_0^1 \left[\rho_1\varphi_t^2 + k(\varphi_x + \psi)^2 + \rho_2\psi_t^2 + \left(b - \int_0^\infty g(s)ds\right)\psi_x^2\right]dx \\ & + \frac{\gamma}{2}(g \circ \psi_x) + \gamma\int_0^1 \widehat{f}(\psi)dx + \frac{\beta}{2}\int_0^1 \left(\rho_3\theta_t^2 + \delta\theta_x^2\right)dx \\ & + \frac{\xi}{2}\int_0^1\int_0^1 \bar{z}^2(x,\rho,t)\, d\rho\, dx + \frac{\beta}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \varsigma|\mu(\varsigma)|z^2(x,\rho,\varsigma,t)\, d\varsigma\, d\rho\, dx, \end{aligned} \tag{2.8}$$

where

$$(g \circ \nu)(t) = \int_0^1\int_0^\infty g(s)(\nu(x,t) - \nu(x,t-s))^2\, ds\, dx.$$

3. Well-posedness of the problem

Concerning problem's (1.2) solution, we give an existence and uniqueness result by using semigroup theory. Before that we denote by

$$\eta^t(x,s) = \psi(x,t) - \psi(x,t-s), \quad t \in R_+, \ (x,t,s) \in (0,1) \times R_+ \times R_+, \tag{3.1}$$

and we have

$$\begin{aligned} \eta_t^t + \eta_s^t - \psi_t &= 0, \ (x,t,s) \in (0,1) \times R_+ \times R_+ \\ \eta^t(0,s) &= \eta^t(1,s) = 0, \ (t,s) \in R_+ \times R_+ \\ \eta^t(x,0) &= 0, \ (x,t) \in (0,1) \times R_+ \end{aligned} \tag{3.2}$$

Using the new notation, the second equation of the system (1.2) can be formulated as follow

$$\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + g_0\psi_{xx}(x,t) - \int_0^\infty g(s)\eta_{xx}^t(x,s)ds - \beta\theta_t + f(\psi) = 0.$$

Let

$$\eta_0(x,s) := \eta^0(x,s) = \psi_0(x,0) - \psi_0(x,s), \quad (x,s) \in (0,1) \times R_+.$$

and

$$u=\varphi_t,\quad v=\psi_t,\quad \omega=\theta_t.$$

Let $\mathcal{U}=(\varphi,u,\psi,v,\theta,\omega,\eta^t,\bar{z},z)$, we define the linear operator $\mathcal{A}:D(\mathcal{A})\subset\mathcal{H}\to\mathcal{H}$ and the vector F as follow

$$\mathcal{A}U=\begin{pmatrix} -u \\ -\dfrac{k}{\rho_1}(\varphi_x+\psi)_x+\dfrac{\beta}{\rho_1}\omega_x+\dfrac{\mu_1}{\rho_1}u+\dfrac{\mu_2}{\rho_1}\tilde{z}\,(x,1,t) \\ -v \\ -\dfrac{b}{\rho_2}\psi_{xx}+\dfrac{k}{\rho_2}(\varphi_x+\psi)+\dfrac{g_0}{\rho_2}\psi_{xx}-\dfrac{1}{\rho_2}\int_0^\infty g(s)\eta^t_{xx}(x,t,s)ds-\frac{\beta}{\rho_2}\omega \\ -\omega \\ -\dfrac{\delta}{\rho_3}\theta_{xx}-\dfrac{\ell}{\rho_3}\omega_{xx}+\dfrac{\gamma}{\rho_3}u_x+\dfrac{\gamma}{\rho_3}v-\dfrac{1}{\rho_3}\int_{\tau_1}^{\tau_2}\mu(\varsigma)z_x(x,1,\varsigma,t)d\varsigma \\ \eta^t_s-v \\ \dfrac{1}{\tau}\bar{z}_\rho \\ \dfrac{1}{\varsigma}z_\rho \end{pmatrix}, \tag{3.3}$$

$$F(U)=\begin{pmatrix} 0\\0\\0\\-\frac{1}{\rho_2}f(\psi)\\0\\0\\0\\0\\0 \end{pmatrix}. \tag{3.4}$$

then, the system(1.2) takes the abstract formulation (3.5)

$$\begin{cases} \dfrac{d}{dt}\mathcal{U}+\mathcal{A}\mathcal{U}=F\left(\mathcal{U}\right), \\ \mathcal{U}\left(0\right)=\mathcal{U}_0=\left(\varphi_0,\varphi_1,\psi_0,\psi_1,\theta_0,\theta_1,\eta_0,h_0,f_0\right). \end{cases} \tag{3.5}$$

We will use the following standard energy space

$$\begin{aligned}\mathcal{H}=&H_0^1(0,1)\times L^2(0,1)\times H_0^1(0,1)\times L^2(0,1)\times H_0^1(0,1)\times L^2(0,1)\times L_g\\ &\times L^2((0,1),L^2(0,1))\times L^2((0,1)\times(0,1)\times(\tau_1,\tau_2)),\end{aligned}$$

where

$$L_g=\left\{\phi:\mathbb{R}_+\to H_0^1(0,1),\ \int_0^1\int_0^\infty g(s)\phi_x^2\,ds\,dx<\infty\right\},$$

and the inner product is

$$\langle\phi_1,\phi_2\rangle_{L_g}=\int_0^1\int_0^\infty g(s)\phi_{1x}(s)\phi_{2x}(s)\,ds\,dx.$$

Suppose that $\ell-\int_{\tau_1}^{\tau_2}|\mu(\varsigma)|d\varsigma>0$.

For $U=(\varphi,u,\psi,v,\theta,\omega,\eta^t,\overline{z},z)^T\in H$, and $\widetilde{U}=(\widetilde{\varphi},\widetilde{u},\widetilde{\psi},\widetilde{v},\widetilde{\theta},\widetilde{\omega},\widetilde{\eta}^t,\widetilde{\overline{z}},\widetilde{z})^T\in H$ we define the inner product in $\mathcal{H}$ as follow

$$\begin{aligned}\langle U,\widetilde{U}\rangle_{\mathcal{H}} &= \gamma\int_0^1(\rho_1 u\tilde{u}+\rho_2 v\widetilde{v}+k(\varphi_x+\psi)(\widetilde{\varphi_x}+\widetilde{\psi})+b\psi_x\widetilde{\psi_x}-g_0\psi_x\widetilde{\psi_x})dx\\ &\quad+\gamma<\eta^t,\widetilde{\eta^t}>_{L_g}+\beta\int_0^1(\rho_3\omega\widetilde{\omega}+\delta\theta_x\widetilde{\theta_x})dx\\ &\quad+\xi\int_0^1\int_0^1\bar{z}\widetilde{\bar{z}}(x,\rho)\,d\rho\,dx\\ &\quad+\beta\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}\varsigma|\mu(\varsigma)|z\widetilde{z}(x,\rho,\varsigma,\cdot)\,d\varsigma\,d\rho\,dx.\end{aligned}$$

The domain of $\mathcal{A}$ is

$$\begin{aligned}D(\mathcal{A})=\Big\{&U\in\mathcal{H}:\varphi,\psi,\theta\in H^2(0,1)\cap H_0^1(0,1),u,v,\omega\in H_0^1(0,1),\\ &\eta^t\in L_g,\bar{z},\bar{z}_\rho\in L^2((0,1)\times(0,1)),z,z_\rho\in L^2((0,1)\times(0,1)\times(\tau_1,\tau_2))\Big\},\end{aligned}$$

which is dense in $\mathcal{H}$, and we have the following existence theorem.

Theorem 3.1. *Assume $U_0\in\mathcal{H}$ and (H1)–(H5) hold. Then, there exists a unique solution $U\in(R_+,\mathcal{H})$ of problem (2.1). Moreover, if $U_0\in D(\mathcal{A})$ then*

$$U\in C(R_+,D(\mathcal{A}))\cap C^1(R_+,\mathcal{H}).$$

To prove Theorem (3.1), we use the semigroup approach see for example [21].

4. Stability results

In this section we give a stability result for the solution of the system (2.1) , by using the multiplier technic. To achieve our goal, we need the following lemmas.

Lemma 4.1. *Let $(\varphi,\ \psi,\ \theta)$ be the solution of (2.1), then we have*

$$\begin{aligned}E'(t)\le\ &\frac{\gamma}{2}(g'\circ\psi_x)-\beta\left(\ell-\int_{\tau_1}^{\tau_2}|\mu(\varsigma)|d\varsigma\right)\int_0^1\theta_{tx}^2dx-\left(\mu_1\gamma-\frac{\xi}{2\tau}-\frac{\mu_2\gamma}{2}\right)\int_0^1\varphi_t^2dx\\ &-\left(\frac{\xi}{2\tau}-\frac{\mu_2\gamma}{2}\right)\int_0^1\bar{z}^2\,(x,1,t)\,dx.\end{aligned}\tag{4.1}$$

Proof: Multiplying $(2.1)_1$ by $\gamma\varphi_t$, $(2.1)_2$ by $\gamma\psi_t$, $(2.1)_3$ by $\beta\theta_t$, integrating over $(0,1)$ with respect to x, multiplying equation $(2.1)_4$ by $\frac{\xi}{\tau}z$ and integrating over $(0,1)\times(0,1)$, multiplying $(2.1)_5$ by $\beta|\mu(\varsigma)|z$ and integrating over $(0,1)\times(0,1)\times(\tau_1,\tau_2)$ with respect to ρ,x and ς, summing them up, we obtain

$$\begin{aligned}E'(t)=\ &\frac{\gamma}{2}(g'\circ\psi_x)+\left(\frac{\xi}{2\tau}-\mu_1\gamma+\frac{\mu_2\gamma}{2}\right)\int_0^1\varphi_t^2dx\\ &+\left(\frac{\mu_2\gamma}{2}-\frac{\xi}{2\tau}\right)\int_0^1\bar{z}^2\,(x,1,t)\,dx-\ell\beta\int_0^1\theta_{tx}^2dx\\ &+\beta\int_0^1\int_{\tau_1}^{\tau_2}\mu(\varsigma)z_x(x,1,\varsigma,t)d\varsigma\theta_t(x,t)dx\\ &-\frac{\beta}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu(\varsigma)|z^2(x,1,\varsigma,t)d\varsigma\,dx\\ &+\frac{\beta}{2}\int_{\tau_1}^{\tau_2}|\mu(\varsigma)|d\varsigma\int_0^1\theta_{tx}^2(x,t)dx.\end{aligned}\tag{4.2}$$

Integrating by parts and using Young's inequality, we get

$$
\begin{aligned}
&\beta \int_0^1 \int_{\tau_1}^{\tau_2} \mu(\varsigma) z_x(x,1,\varsigma,t) d\varsigma \theta_t(x,t) dx \\
&= -\beta \int_0^1 \int_{\tau_1}^{\tau_2} \mu(\varsigma) z(x,1,\varsigma,t) d\varsigma \theta_{tx}(x,t) dx \\
&\leq \frac{\beta}{2} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu(\varsigma)| z^2(x,1,\varsigma,t) d\varsigma \, dx + \frac{\beta}{2} \int_{\tau_1}^{\tau_2} |\mu(\varsigma)| d\varsigma \int_0^1 \theta_{tx}^2(x,t) dx. \qquad (4.3)
\end{aligned}
$$

A combination of (4.2) and (4.3) gives

$$
\begin{aligned}
E'(t) \leq \frac{\gamma}{2}(g' \circ \psi_x) - \beta \left(\ell - \int_{\tau_1}^{\tau_2} |\mu(\varsigma)| d\varsigma\right) \int_0^1 \theta_{tx} dx, \\
- \left(\mu_1 \gamma - \frac{\xi}{2\tau} - \frac{\mu_2 \gamma}{2}\right) \int_0^1 \varphi_t^2 dx, \\
- \left(\frac{\xi}{2\tau} - \frac{\mu_2 \gamma}{2}\right) \int_0^1 \bar{z}^2 (x,1,t) \, dx.
\end{aligned}
$$

Thus (4.1) follows. □

Let $E(t) = E(t,\varphi,\psi,\theta,z) = E_1(t)$, the energy defined in (2.8) then

$$
E_2(t) = E(t,\varphi_t,\psi_t,\theta_t,z_t),
$$

and we have

$$
E_2'(t) \leq \frac{\gamma}{2}(g' \circ \psi_{xt}) - c \int_0^1 \theta_{tx}^2 dx,
$$

where c is some positive constant. It is easy to obtain to show the following inequalities.

Lemma 4.2. *For the functions g, ψ, g_0, we have,*

$$
\int_0^1 \left(\int_0^\infty g(s)\psi_x(t-s) ds\right)^2 dx \leq 2g_0 (g \circ \psi_x)(t) + 2g_0 \int_0^1 \psi_x^2 dx, \qquad (4.4)
$$

$$
\int_0^1 \left(\int_0^\infty g(s)(\psi_x(t) - \psi_x(t-s)) ds\right)^2 dx \leq g_0 (g \circ \psi_x)(t), \qquad (4.5)
$$

$$
\int_0^1 \left(\int_0^\infty g(s)(\psi(t) - \psi(t-s)) ds\right)^2 dx \leq d_1 (g \circ \psi_x)(t), \qquad (4.6)
$$

$$
\int_0^1 \left(\int_0^\infty g'(s)(\psi(t) - \psi(t-s)) ds\right)^2 dx \leq -d_2 (g \circ \psi_x)(t), \qquad (4.7)
$$

$$
\int_0^1 \left(\int_0^\infty g'(s)(\psi_x(t) - \psi_x(t-s)) ds\right)^2 dx \leq -g(0)(g' \circ \psi_x)(t), \qquad (4.8)
$$

where d_1 and d_2 are positive constants.

Lemma 4.3. *Let (φ,ψ,θ) be the solution of (2.1). Then, the functional*

$$
J_1(t) := -\rho_1 \int_0^1 \varphi_t \varphi \, dx - \rho_2 \int_0^1 \psi_t \psi dx - \frac{\mu_1}{2} \int_0^1 \varphi^2 dx
$$

satisfies

$$\frac{dJ_1(t)}{dt} \leq -\rho_2 \int_0^1 \psi_t^2 dx + \frac{\beta^2}{4k} \int_0^1 \theta_{tx}^2 dx + g_0 (g \circ \psi_x)(t) + 3k \int_0^1 (\varphi_x + \psi)^2 dx + \frac{\mu_2^2}{2k} \int_0^1 \bar{z}^2 (x,1,t)\, dx + c \int_0^1 \psi_x^2 dx. \tag{4.9}$$

Proof: By a simple computations, and using (2.1), we obtain

$$\frac{dJ_1(t)}{dt} = -\rho_1 \int_0^1 \varphi_t^2 dx + k \int_0^1 (\varphi_x + \psi)^2 dx - \rho_2 \int_0^1 \psi_t^2 dx + b \int_0^1 \psi_x^2 dx - \beta \int_0^1 \theta_t (\varphi_x + \psi)\, dx - \int_0^1 \psi_x \int_0^\infty g(s) \psi_x (x, t-s)\, ds\, dx + \int_0^1 f(\psi)\psi\, dx + \mu_2 \int_0^1 \bar{z}(x,1,t)\, \varphi dx. \tag{4.10}$$

Using Young's and Poincaré inequalities, then

$$-\beta \int_0^1 \theta_t (\varphi_x + \psi)\, dx \leq k \int_0^1 (\varphi_x + \psi)^2 dx + \frac{\beta^2}{4k} \int_0^1 \theta_{tx}^2 dx. \tag{4.11}$$

By using Young's inequality and (4.4), we get

$$- \int_0^1 \psi_x \int_0^\infty g(s) \psi_x (x, t-s)\, ds\, dx \leq \left(\frac{1}{2} + g_0 \right) \int_0^1 \psi_x^2\, dx + g_0 (g \circ \psi_x)(t). \tag{4.12}$$

Cauchy-Schwarz and Poincaré inequalities lead to

$$\int_0^1 |f(\psi)\psi| dx \leq \int_0^1 |\psi|^\varrho |\psi| |\psi| \leq \|\psi\|_{2(\varrho+1)}^\varrho \|\psi\|_{2(\varrho+1)} \|\psi\| dx \leq c \int_0^1 \psi_x^2 dx. \tag{4.13}$$

By using Young's and Poincaré inequalities, we get

$$\mu_2 \int_0^1 \bar{z}(x,1,t)\, \varphi dx \leq \frac{k}{2} \int_0^1 \varphi_x^2 dx + \frac{\mu_2^2}{2k} \int_0^1 \bar{z}^2 (x,1,t)\, dx. \tag{4.14}$$

On the other hand, we have

$$\int_0^1 \varphi_x^2 dx \leq 2 \int_0^1 (\varphi_x + \psi)^2 dx + 2 \int_0^1 \psi^2 dx \leq 2 \int_0^1 (\varphi_x + \psi)^2 dx + 2 \int_0^1 \psi_x^2 dx. \tag{4.15}$$

So, (4.14) becomes

$$\mu_2 \int_0^1 \bar{z}(x,1,t)\, \varphi dx \leq k \int_0^1 (\varphi_x + \psi)^2 dx + k \int_0^1 \psi_x^2 dx + \frac{\mu_2^2}{2k} \int_0^1 \bar{z}^2 (x,1,t)\, dx. \tag{4.16}$$

The substitution of (4.11), (4.13) and (4.16) into (4.10) gives (4.9). □

Lemma 4.4. *Let (φ, ψ, θ) be the solution of (2.1). Then, the functional*

$$J_2(t) := \rho_3 \int_0^1 \theta_t \theta dx + \frac{\ell}{2} \int_0^1 \theta_x^2 dx + \gamma \int_0^1 \varphi_x \theta dx, \tag{4.17}$$

satisfies

$$\begin{aligned} \frac{dJ_2(t)}{dt} \leq & -\frac{\delta}{2} \int_0^1 \theta_x^2 dx + \frac{16\gamma^2}{k} \int_0^1 \theta_{tx}^2 dx + \frac{k}{32} \int_0^1 \psi_x^2 dx \\ & + \frac{\gamma^2}{\delta} \int_0^1 \psi_t^2 dx + \frac{k}{32} \int_0^1 (\varphi_x + \psi)^2 dx \\ & + \frac{1}{\delta} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu(\varsigma)| z^2(x, 1, \varsigma, t) d\varsigma dx. \end{aligned} \tag{4.18}$$

Proof: Taking the derivative $\frac{dJ_2(t)}{dt}$ and using (2.1), we get

$$\begin{aligned} \frac{dJ_2(t)}{dt} = & \rho_3 \int_0^1 \theta_t^2 dx + \gamma \int_0^1 \varphi_x \theta_t dx - \delta \int_0^1 \theta_x^2 dx - \gamma \int_0^1 \psi_t \theta dx \\ & - \int_0^1 \int_{\tau_1}^{\tau_2} \mu(\varsigma) z(x, 1, \varsigma, t) d\varsigma \theta_x \, dx. \end{aligned} \tag{4.19}$$

By using Young's, Poincaré inequalities and (4.15), we have

$$\gamma \int_0^1 \varphi_x \theta_t dx \leq \frac{k}{32} \int_0^1 (\varphi_x + \psi)^2 dx + \frac{k}{32} \int_0^1 \psi_x^2 dx + \frac{16\gamma^2}{k} \int_0^1 \theta_{tx}^2 dx, \tag{4.20}$$

$$\gamma \int_0^1 \psi_t \theta dx \leq \frac{\delta}{4} \int_0^1 \theta_x^2 dx + \frac{\gamma^2}{\delta} \int_0^1 \psi_t^2 dx. \tag{4.21}$$

Young's and Cauchy Schawrz inequalities, give us

$$\begin{aligned} & \int_0^1 \theta_x \left(\int_{\tau_1}^{\tau_2} \mu(\varsigma) z(x, 1, \varsigma, t) d\varsigma \right) dx \\ & \leq \frac{\delta}{4} \int_0^1 \theta_x^2 dx + \frac{1}{\delta} \int_0^1 \left(\int_{\tau_1}^{\tau_2} \mu(\varsigma) z(x, 1, \varsigma, t) d\varsigma \right)^2 dx \\ & \leq \frac{\delta}{4} \int_0^1 \theta_x^2 dx + \frac{1}{\delta} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu(\varsigma)| z^2(x, 1, \varsigma, t) d\varsigma dx. \end{aligned} \tag{4.22}$$

The substitution of (4.20), (4.21) and (4.22) into (4.19), then (4.18) is established. □

Lemma 4.5. *Let $(\varphi,\ \psi,\ \theta)$ be the solution of (2.1). Then, for any positive constant ε_1 the functional*

$$J_3(t) := -\rho_2 \int_0^1 \psi_t \int_0^\infty g(s)(\psi(t) - \psi(t-s)) \, ds \, dx, \tag{4.23}$$

satisfies

$$\begin{aligned} \frac{dJ_3(t)}{dt} \leq & \varepsilon_1 c \int_0^1 \psi_x^2 dx - (\rho_2 g_0 - \varepsilon_1 \rho_2) \int_0^1 \psi_t^2 dx \\ & + c(1 + \varepsilon_1 + \frac{1}{\varepsilon_1})(g \circ \psi_x)(t) + \varepsilon_1 k \int_0^1 (\varphi_x + \psi)^2 dx \\ & + \frac{\beta}{2} \int_0^1 \theta_{tx}^2 dx - \frac{\rho_2 d_2}{4\varepsilon_1} (g' \circ \psi_x)(t). \end{aligned} \tag{4.24}$$

Proof: First, we note that

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\int_0^\infty g(s)(\psi(t)-\psi(t-s))ds\right)\\
&=\frac{\partial}{\partial t}\left(\int_{-\infty}^t g(t-s)(\psi(t)-\psi(s))ds\right)\\
&=\int_{-\infty}^t g'(t-s)(\psi(t)-\psi(s))ds+\int_{-\infty}^t g(t-s)\psi_t(t)ds\\
&=g_0\psi_t(t)+\int_0^\infty g'(s)(\psi(t)-\psi(t-s))ds.
\end{aligned}$$

By simple computation and using (2.1), we find

$$\begin{aligned}
\frac{dJ_3(t)}{dt}=&\,b\int_0^1\psi_x\left(\int_0^\infty g(s)(\psi_x(t)-\psi_x(t-s))\,ds\right)dx-g_0\rho_2\int_0^1\psi_t^2dx\\
&-\rho_2\int_0^1\psi_t\left(\int_0^\infty g'(s)(\psi(t)-\psi(t-s))\right)ds\,dx\\
&+k\int_0^1(\varphi_x+\psi)\left(\int_0^\infty g(s)(\psi(t)-\psi(t-s))\,ds\right)dx\\
&-\beta\int_0^1\theta_t\left(\int_0^\infty g(s)(\psi(t)-\psi(t-s))\,ds\right)dx\\
&+\int_0^1 f(\psi)\left(\int_0^\infty g(s)(\psi(t)-\psi(t-s))\,ds\right)dx\\
&-\int_0^1\left(\int_0^\infty g(s)\psi_x(t-s)ds\right)\left(\int_0^\infty g(s)(\psi_x(t)-\psi_x(t-s))\,ds\right)dx. \qquad (4.25)
\end{aligned}$$

Young's inequality, (4.5), (4.7) and (4.6), lead to

$$\begin{aligned}
&b\int_0^1\psi_x\left(\int_0^\infty g(s)(\psi_x(t)-\psi_x(t-s))\,ds\right)dx\\
&\le\frac{b}{2}\varepsilon_1\int_0^1\psi_x^2dx+\frac{b}{2\varepsilon_1}\int_0^1\left(\int_0^\infty g(s)(\psi_x(t)-\psi_x(t-s))ds\right)^2dx\\
&\le\frac{b}{2}\varepsilon_1\int_0^1\psi_x^2dx+\frac{bg_0}{2\varepsilon_1}(g\circ\psi_x)(t), \qquad (4.26)
\end{aligned}$$

$$\begin{aligned}
&-\rho_2\int_0^1\psi_t\left(\int_0^\infty g'(s)(\psi(t)-\psi(t-s))\,ds\right)dx\\
&\le\rho_2\varepsilon_1\int_0^1\psi_t^2dx+\frac{\rho_2}{4\varepsilon_1}\int_0^1\left(\int_0^\infty g'(s)(\psi(t)-\psi(t-s))ds\right)^2dx\\
&\le\rho_2\varepsilon_1\int_0^1\psi_t^2dx-\frac{\rho_2d_2}{4\varepsilon_1}(g'\circ\psi_x)(t), \qquad (4.27)
\end{aligned}$$

$$\begin{aligned}
&k\int_0^1(\varphi_x+\psi)\left(\int_0^\infty g(s)(\psi(t)-\psi(t-s))\,ds\right)dx\\
&\le\varepsilon_1k\int_0^1(\varphi_x+\psi)^2dx+\frac{k}{4\varepsilon_1}\int_0^1\left(\int_0^\infty g(s)(\psi(t)-\psi(t-s))ds\right)^2dx\\
&\le\varepsilon_1k\int_0^1(\varphi_x+\psi)^2dx+\frac{kd_1}{4\varepsilon_1}(g\circ\psi_x)(t), \qquad (4.28)
\end{aligned}$$

Young's, Poincaré inequalities and (4.6), give us

$$\begin{aligned} &\beta \int_0^1 \theta_t \int_0^\infty g(s)(\psi(t)-\psi(t-s))\, ds\, dx \\ &\leq \frac{\beta}{2}\int_0^1 \theta_{tx}^2 dx + \frac{\beta}{2}\int_0^1 \left(\int_0^\infty g(s)(\psi(t)-\psi(t-s))ds\right)^2 dx \\ &\leq \frac{\beta}{2}\int_0^1 \theta_{tx}^2 dx + \frac{\beta d_1}{2}(g\circ\psi_x)(t), \end{aligned} \tag{4.29}$$

$$\begin{aligned} &\int_0^1 f(\psi)\left(\int_0^\infty g(s)(\psi(t)-\psi(t-s))\, ds\right) dx \\ &\leq k_0 \int_0^1 |\psi|^{\varrho}|\psi| \left(\int_0^\infty g(s)(\psi(t)-\psi(t-s))\, ds\right) dx \\ &\leq k_0 \|\psi\|_{2(\varrho+1)}^{\varrho} \|\psi\|_{2(\varrho+1)} \left(\int_0^1 (\int_0^\infty g(s)(\psi(t)-\psi(t-s))ds)^2 dx\right)^{1/2} \\ &\leq \varepsilon_1 c \int_0^1 \psi_x^2 dx + \frac{d_1}{4\varepsilon_1}(g\circ\psi_x)(t), \end{aligned} \tag{4.30}$$

$$\begin{aligned} &\int_0^1 \left(\int_0^\infty g(s)\psi_x(t-s)ds\right)\left(\int_0^\infty g(s)(\psi_x(t)-\psi_x(t-s))\, ds\right) dx \\ &\leq \varepsilon_1 \int_0^1 \left(\int_0^\infty g(s)\psi_x(t-s)ds\right)^2 dx \\ &+ \frac{1}{4\varepsilon_1}\int_0^1 \left(\int_0^\infty g(s)(\psi_x(t)-\psi_x(t-s))ds\right)^2 dx \\ &\leq \left(2\varepsilon_1 + \frac{1}{4\varepsilon_1}\right) g_0 (g\circ\psi_x) + 2\varepsilon_1 g_0 \int_0^1 \psi_x^2 dx. \end{aligned} \tag{4.31}$$

By substituting (4.26)-(4.31) into (4.25), we obtain (4.24). □

Let the multiplier ω defined as a solution of the following differential equation

$$-w_{xx} = \psi_x, \quad w(0) = w(1) = 0. \tag{4.32}$$

Lemma 4.6. *The solution w of (4.32) satisfies*

$$\int_0^1 w_x^2 dx \leq \int_0^1 \psi^2 dx \leq \int_0^1 \psi_x^2 dx, \tag{4.33}$$

$$\int_0^1 w_t^2 dx \leq \int_0^1 w_{xt}^2 dx \leq \int_0^1 \psi_t^2 dx. \tag{4.34}$$

Lemma 4.7. *Let (φ, ψ, θ) be the solution of (2.1). Then, the functional*

$$J_4(t) := \int_0^1 (\rho_1 \varphi_t w + \rho_2 \psi_t \psi) dx,$$

satisfies

$$\begin{aligned} \frac{dJ_4(t)}{dt} \leq &-\frac{l}{2}\int_0^1 \psi_x^2 dx + \left(\frac{5\beta^2}{2l}\right)\int_0^1 \theta_{tx}^2 dx + \left(\frac{l}{10} + \frac{5\mu_1^2}{2l}\right)\int_0^1 \varphi_t^2 dx \\ &+ \left(\frac{5\rho_1^2}{2l} + \rho_2\right)\int_0^1 \psi_t^2 dx + \frac{5g_0}{2l}(g\circ\psi_x)(t) \\ &+ \frac{5\mu_2^2}{2l}\int_0^1 \bar{z}^2(x,1,t)\, dx - \int_0^1 \widehat{f}(\psi) dx. \end{aligned} \tag{4.35}$$

Proof: A simple differentiation of $J_4(t)$, using (2.1) and 4.6, we obtain

$$\begin{aligned}\frac{dJ_4(t)}{dt} &\leq \beta\int_0^1 \theta_t w_x dx + \rho_1\int_0^1 \varphi_t w_t dx + \beta\int_0^1 \theta_t \psi dx \\ &- b\int_0^1 \psi_x^2 dx + \rho_2\int_0^1 \psi_t^2 dx \\ &+ \int_0^1 \psi_x \left(\int_0^\infty g(s)\psi_x(x,t-s)ds\right) dx - \int_0^1 f(\psi)\psi\, dx \\ &- \mu_1\int_0^1 \varphi_t w dx - \mu_2\int_0^1 \bar{z}\,(x,1,t)\, w dx, \end{aligned} \tag{4.36}$$

where we have used integration by parts, (4.32) and the boundary conditions in (2.1). By using Young's, Poincaré inequalities, Lemma 4.2 and Lemma 4.6, we have

$$\int_0^1 \beta\theta_t w_x dx \leq \sigma\int_0^1 w_x^2 dx + \frac{1}{4\sigma}\int_0^1 \beta^2\theta_t^2 dx \leq \sigma\int_0^1 \psi_x^2 dx + \frac{\beta^2}{4\sigma}\int_0^1 \theta_{tx}^2 dx \tag{4.37}$$

$$\begin{aligned}\rho_1\int_0^1 \varphi_t w_t dx &\leq \sigma\int_0^1 \varphi_t^2 dx + \frac{\rho_1^2}{4\sigma}\int_0^1 w_t^2 dx \\ &\leq \sigma\int_0^1 \varphi_t^2 dx + \frac{\rho_1^2}{4\sigma}\int_0^1 \psi_t^2 dx, \end{aligned} \tag{4.38}$$

$$\beta\int_0^1 \theta_t\psi\, dx \leq \sigma\int_0^1 \psi_x^2 dx + \frac{\beta^2}{4\sigma}\int_0^1 \theta_{tx}^2 dx, \tag{4.39}$$

$$\begin{aligned}&\int_0^1 \left(\int_0^\infty g(s)\psi_x(x,t-s)ds\right)\psi_x dx \\ &= \int_0^1 \left(\int_0^\infty g(s)(\psi_x(x,t-s) - \psi_x(t) + \psi_x(t))ds\right)\psi_x dx \\ &= \int_0^1 \left(\int_0^\infty g(s)(\psi_x(x,t-s) - \psi_x(t))ds\right)\psi_x dx + \left(\int_0^\infty g(s)ds\right)\int_0^1 \psi_x^2 dx \\ &\leq \sigma\int_0^1 \psi_x^2(t)dx + \frac{1}{4\sigma}\int_0^1 \left(\int_0^\infty g(s)(\psi_x(x,t-s) - \psi_x(t))ds\right)^2 dx \\ &+ \left(\int_0^\infty g(s)ds\right)\int_0^1 \psi_x^2 dx \\ &\leq \sigma\int_0^1 \psi_x^2 dx + \frac{g_0}{4\sigma}(g\circ\psi_x)(t) + g_0\int_0^1 \psi_x^2 dx. \end{aligned} \tag{4.40}$$

$$-\mu_1\int_0^1 \varphi_t w dx \leq \sigma\int_0^1 w_x^2 dx + \frac{\mu_1^2}{4\sigma}\int_0^1 \varphi_t^2 dx \leq \sigma\int_0^1 \psi_x^2 dx + \frac{\mu_1^2}{4\sigma}\int_0^1 \varphi_t^2 dx \tag{4.41}$$

$$\begin{aligned}-\mu_2\int_0^1 \bar{z}\,(x,1,t)\, w dx &\leq \sigma\int_0^1 w_x^2 dx + \frac{\mu_2^2}{4\sigma}\int_0^1 \bar{z}^2\,(x,1,t)\, dx \\ &\leq \sigma\int_0^1 \psi_x^2 dx + \frac{\mu_2^2}{4\sigma}\int_0^1 \bar{z}^2\,(x,1,t)\, dx. \end{aligned} \tag{4.42}$$

By substituting (4.37)-(4.42) in (4.36) and letting $\sigma = \frac{l}{10}$, we have (4.35). □

Lemma 4.8. *Let (φ,ψ,θ) be the solution of (2.1). Then, the functional*

$$J_5(t) := \rho_2 \int_0^1 \psi_t(\varphi_x+\psi)dx + \frac{b\rho_1}{k}\int_0^1 \varphi_t\psi_x dx - \frac{\rho_1}{k}\int_0^1 \varphi_t\left(\int_0^\infty g(s)\psi_x(t-s)ds\right)dx,$$

satisfies

$$\begin{aligned}\frac{dJ_5(t)}{dt} \le &\left[\varphi_x(b\psi_x - \int_0^\infty g(s)\psi_x(t-s)ds)\right]_{x=0}^{x=1} + \rho_2\int_0^1 \psi_t^2 dx \\ &- \frac{k}{2}\int_0^1 (\varphi_x+\psi)^2dx + c\int_0^1 \theta_{tx}^2 dx + c(g\circ\psi_x)(t) \\ &+ c\left(g\circ\psi_x\right)(t) - c(g'\circ\psi_x)(t) \\ &+ c\int_0^1 \varphi_t^2dx - \int_0^1 \widehat{f}(\psi)dx + \left(\frac{b\rho_1}{k}-\rho_2\right)\int_0^1 \varphi_t\psi_{xt}dx \\ &+ c\int_0^1 \psi_x^2dx + c\int_0^1 \bar{z}^2\left(x,1,t\right)dx. \end{aligned} \tag{4.43}$$

Proof: We note that

$$\begin{aligned}&\frac{d}{dt}\left(\int_0^\infty g(s)\psi_x(t-s)ds\right) \\ &= \frac{d}{dt}\left(\int_{-\infty}^t g(t-s)\psi_x(s)ds\right) \\ &= g(0)\psi_x(t) + \int_{-\infty}^t g'(t-s)\psi_x(s)ds \\ &= g(0)\psi_x(t) + \int_0^\infty g'(s)\left(\psi_x(t-s)-\psi_x(t)+\psi_x(t)\right)ds \\ &= g(0)\psi_x(t) + \int_0^\infty g'(s)\left(\psi_x(t-s)-\psi_x(t)\right)ds + \int_0^\infty g'(s)ds\psi_x(t) \\ &= \int_0^\infty g'(s)\left(\psi_x(t-s)-\psi_x(t)\right)ds. \end{aligned} \tag{4.44}$$

Integration by parts, and combination with (2.1) we obtain

$$\begin{aligned}\frac{dJ_5(t)}{dt} = &\left[\varphi_x(b\psi_x - \int_0^\infty g(s)\psi_x(t-s)ds)\right]_{x=0}^{x=1} + \rho_2\int_0^1 \psi_t^2dx - k\int_0^1(\varphi_x+\psi)^2dx \\ &+ \beta\int_0^1 \theta_t(\varphi_x+\psi)dx + \frac{\beta}{k}\int_0^1 \theta_{tx}\left(\int_0^\infty g(s)\left(\psi_x(t-s)-\psi_x(t)\right)\,ds\right)dx \\ &+ \frac{\rho_1}{k}\int_0^1 \varphi_t\left(\int_0^\infty g'(s)\left(\psi_x(t)-\psi_x(t-s)\right)\,ds\right)dx + \left(\frac{b\rho_1}{k}-\rho_2\right)\int_0^1 \varphi_t\psi_{xt}dx \\ &- \int_0^1 f(\psi)\varphi_x\,dx + \frac{\mu_2}{k}\int_0^1 \bar{z}\left(x,1,t\right)\left(\int_0^\infty g(s)\psi_x(t-s)\,ds\right)dx \\ &- \int_0^1 \psi f(\psi)\,dx - \frac{\mu_1 b}{k}\int_0^1 \varphi_t\psi_x\,dx - \frac{\mu_2 b}{k}\int_0^1 \bar{z}\left(x,1,t\right)\psi_x\,dx \\ &+ \frac{\mu_1}{k}\int_0^1 \varphi_t\left(\int_0^\infty g(s)\psi_x(t-s)\,ds\right)dx + \left(\frac{\beta}{k}g_0 - \frac{b\beta}{k}\right)\int_0^1 \theta_{xt}\psi_x\,dx. \end{aligned} \tag{4.45}$$

Using Young's and Poincaré inequalities, Lemma 4.2, we obtain

$$\beta\int_0^1\theta_t(\varphi_x+\psi)dx\leq\frac{k}{4}\int_0^1(\varphi_x+\psi)^2dx+\frac{\beta^2}{k}\int_0^1\theta_{tx}^2dx, \tag{4.46}$$

$$\begin{aligned}&\frac{\beta}{k}\int_0^1\theta_{tx}\left(\int_0^\infty g(s)\left(\psi_x(t-s)-\psi_x(t)\right)ds\right)dx\\&\leq\frac{\beta}{2k}(g\circ\psi_x)(t)+\frac{\beta}{2k}\int_0^1\theta_{tx}^2dx,\end{aligned} \tag{4.47}$$

$$\begin{aligned}&\frac{\rho_1}{k}\int_0^1\varphi_t\int_0^\infty g'(s)\left(\psi_x(t)-\psi_x(t-s)\right)ds\,dx\\&\leq-\frac{g\left(0\right)\rho_1}{2k}(g'\circ\psi_x)(t)+\frac{\rho_1}{2k}\int_0^1\varphi_t^2dx,\end{aligned} \tag{4.48}$$

$$\left(\frac{\beta}{k}g_0-\frac{b\beta}{k}\right)\int_0^1\theta_{xt}\psi_xdx\leq c\int_0^1\left(\theta_{xt}^2+\psi_x^2\right)dx, \tag{4.49}$$

$$\begin{aligned}\int_0^1|\varphi_xf(\psi)|dx&\leq\|\varphi_x\|\|\psi\|_{2(\varrho+1)}^{\varrho}\|\psi\|_{2(\varrho+1)}\\&\leq\frac{k}{8}\int_0^1\varphi_x^2dx+\frac{2}{k}\int_0^1\psi_x^2dx\\&\leq\frac{k}{4}\int_0^1(\varphi_x+\psi)^2dx+(\frac{2}{k}+\frac{k}{4})\int_0^1\psi_x^2dx.\end{aligned} \tag{4.50}$$

$$-\frac{\mu_1b}{k}\int_0^1\varphi_t\psi_x\,dx\leq\frac{\mu_1b}{2k}\int_0^1\varphi_t^2\,dx+\frac{\mu_1b}{2k}\int_0^1\psi_x^2dx,$$

$$-\frac{\mu_2b}{k}\int_0^1\bar{z}\left(x,1,t\right)\psi_x\,dx\leq\frac{\mu_2b}{k}\int_0^1\psi_x^2\,dx+\frac{\mu_2b}{2k}\int_0^1\bar{z}^2\left(x,1,t\right)dx, \tag{4.51}$$

$$\begin{aligned}&+\frac{\mu_1}{k}\int_0^1\varphi_t\left(\int_0^\infty g(s)\psi_x(t-s)\,ds\right)dx\\&+\frac{\mu_2}{k}\int_0^1\bar{z}\left(x,1,t\right)\left(\int_0^\infty g(s)\psi_x(t-s)\,ds\right)dx\\&\leq\frac{\mu_1}{2k}\int_0^1\varphi_t^2\,dx+\frac{\mu_2}{2k}\int_0^1\bar{z}^2\left(x,1,t\right)\,dx\\&+\frac{2\left(\mu_2+\mu_1\right)g_0}{k}(g\circ\psi_x)(t)+\frac{2\left(\mu_2+\mu_1\right)g_0}{k}\int_0^1\psi_x^2dx.\end{aligned} \tag{4.52}$$

Substituting (4.46)-(4.52) into $\frac{dJ_5(t)}{dt}$, gives (4.43). □

For the boundary terms that appears in (4.43), we define the function

$$q(x)=2-4x,\quad x\in[0,1].$$

Lemma 4.9. *Let (φ, ψ, θ) be the solution of (2.1). Then we have,*

$$\begin{aligned}
&\left[\varphi_x\left(b\psi_x - \int_0^\infty g(s)\psi_x(t-s)ds\right)\right]_{x=0}^{x=1} \\
&\leq -\frac{\rho_1}{32}\frac{d}{dt}\int_0^1 q(x)\varphi_t\varphi_x\,dx + c\int_0^1 \psi_x^2 dx + \frac{k}{32}\int_0^1 (\varphi_x+\psi)^2 dx \\
&- \frac{8\rho_2}{k}\frac{d}{dt}\int_0^1 q(x)\psi_t\left(b\psi_x - \int_0^\infty g(s)\psi_x(t-s)ds\right)dx + c(g\circ\psi_x)(t) \\
&+ c\int_0^1 \psi_t^2 dx - c(g'\circ\psi_x)(t) + c\int_0^1 \theta_{tx}^2 dx + \frac{\rho_1}{16}\int_0^1 \varphi_t^2 dx.
\end{aligned} \tag{4.53}$$

Proof: Using Young's and Poincaré inequalities, then for any $\varepsilon_6 > 0$, we have

$$\begin{aligned}
&\left[\varphi_x\left(b\psi_x - \int_0^\infty g(s)\psi_x(t-s)ds\right)\right]_{x=0}^{x=1} \\
&= \varphi_x(1)\left(b\psi_x(1) - \int_0^\infty g(s)\psi_x(1,t-s)ds\right) \\
&- \varphi_x(0)\left(b\psi_x(0) - \int_0^\infty g(s)\psi_x(0,t-s)ds\right) \\
&\leq \frac{k}{32}[\varphi_x(1)^2 + \varphi_x(0)^2] \\
&+ \frac{8}{k}\left[\left(b\psi_x(1) - \int_0^\infty g(s)\psi_x(1,t-s)ds\right)^2 + \left(b\psi_x(0) - \int_0^\infty g(s)\psi_x(0,t-s)ds\right)^2\right].
\end{aligned}$$

By simple computation we have

$$\begin{aligned}
&\frac{d}{dt}\int_0^1 \rho_2 q(x)\psi_t\left(b\psi_x - \int_0^\infty g(s)\psi_x(t-s)ds\right)dx \\
&= -\left[(b\psi_x(1) - \int_0^\infty g(s)\psi_x(1,t-s)ds)^2 + \left(b\psi_x(0) - \int_0^\infty g(s)\psi_x(0,t-s)ds\right)^2\right] \\
&+ 2\int_0^1 \left(b\psi_x - \int_0^\infty g(s)\psi_x(t-s)ds\right)^2 dx \\
&+ \beta\int_0^1 q(x)\theta_t\left(b\psi_x - \int_0^\infty g(s)\psi_x(t-s)ds\right)dx \\
&- k\int_0^1 q(x)(\varphi_x+\psi)\left(b\psi_x - \int_0^\infty g(s)\psi_x(t-s)ds\right)dx + 2\rho_2 b\int_0^1 \psi_t^2 dx \\
&- \int_0^1 q(x)f(\psi)(b\psi_x - \int_0^\infty g(s)\psi_x(t-s)ds)dx \\
&+ \rho_2\int_0^1 q(x)\psi_t\int_0^\infty g'(s)(\psi_x(t) - \psi_x(t-s))\,ds\,dx,
\end{aligned}$$

which gives

$$
\begin{aligned}
&\frac{8}{k}\left[\left(b\psi_x(1)-\int_0^\infty g(s)\psi_x(1,t-s)ds\right)^2+\left(b\psi_x(0)-\int_0^\infty g(s)\psi_x(0,t-s)ds\right)^2\right]\\
&=-\frac{8}{k}\frac{d}{dt}\int_0^1 \rho_2 q(x)\psi_t\left(b\psi_x-\int_0^\infty g(s)\psi_x(t-s)ds\right)dx\\
&+\frac{16}{k}\int_0^1\left(b\psi_x-\int_0^\infty g(s)\psi_x(t-s)ds\right)^2dx\\
&+\frac{8\beta}{k}\int_0^1 q(x)\theta_t\left(b\psi_x-\int_0^\infty g(s)\psi_x(t-s)ds\right)dx\\
&-8\int_0^1 q(x)(\varphi_x+\psi)\left(b\psi_x-\int_0^\infty g(s)\psi_x(t-s)ds\right)dx+\frac{16\rho_2 b}{k}\int_0^1\psi_t^2dx\\
&-\frac{8}{k}\int_0^1 q(x)f(\psi)\left(b\psi_x-\int_0^\infty g(s)\psi_x(t-s)ds\right)dx\\
&+\frac{8\rho_2}{k}\int_0^1 q(x)\psi_t\int_0^\infty g'(s)\left(\psi_x(t)-\psi_x(t-s)\right)ds\,dx.
\end{aligned}
$$

A combination of Young's and Poincaré inequalities, Lemma 4.2, and the fact that $0\le q(x)\le 4$, for $x\in[0,1]$, we get,

$$
\begin{aligned}
&\frac{16}{k}\int_0^1\left(b\psi_x-\int_0^\infty g(s)\psi_x(t-s)ds\right)^2dx\\
&\le\frac{32}{k}\int_0^1 b^2\psi_x^2\,dx+\frac{32}{k}\int_0^1\left(\int_0^\infty g(s)\psi_x(t-s)ds\right)^2dx\\
&\le\left(\frac{32b^2}{k}+2g_0\right)\int_0^1\psi_x^2\,dx+\frac{64g_0}{k}(g\circ\psi_x)(t),
\end{aligned}
$$

$$
\begin{aligned}
&\frac{8\beta}{k}\int_0^1 q(x)\theta_t(b\psi_x-\int_0^\infty g(s)\psi_x(t-s)ds)dx\\
&\le\frac{k}{256}\int_0^1 q^2(x)\theta_t^2dx+\frac{2^{12}\beta^2}{k^3}\int_0^1(b\psi_x-\int_0^\infty g(s)\psi_x(t-s)ds)^2dx\\
&\le\frac{ck}{32}\int_0^1\theta_{tx}^2dx+\frac{32^3c}{k^3}(g\circ\psi_x)(t)+\frac{32^3c}{k^3}\int_0^1\psi_x^2\,dx,
\end{aligned}
$$

$$
\begin{aligned}
&-8\int_0^1 q(x)(\varphi_x+\psi)(b\psi_x-\int_0^\infty g(s)\psi_x(t-s)ds)dx\\
&\le\frac{k}{32}\int_0^1(\varphi_x+\psi)^2dx+\frac{64}{k}\int_0^1 q^2(x)(b\psi_x-\int_0^\infty g(s)\psi_x(t-s)ds)^2dx\\
&\le\frac{k}{32}\int_0^1(\varphi_x+\psi)^2dx+c\int_0^1\psi_x^2\,dx+c(g\circ\psi_x)(t),
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{8}{k}\int_0^1 q(x)f(\psi)\left(b\psi_x-\int_0^\infty g(s)\psi_x(t-s)ds\right)dx\\
&\le\frac{kc}{32}\int_0^1\psi_x^2\,dx+\frac{kc}{32}(g\circ\psi_x)(t),
\end{aligned}
$$

$$\begin{aligned}
&\frac{8\rho_2}{k}\int_0^1 q(x)\psi_t \int_0^\infty g'(s)(\psi_x(t)-\psi_x(t-s))\,ds\,dx \\
&\le \frac{kc}{32}\int_0^1 \psi_t^2 dx - \frac{kc}{32}(g'\circ\psi_x)(t).
\end{aligned}$$

We remark that

$$\begin{aligned}
\frac{d}{dt}\int_0^1 \frac{\rho_1}{k} q(x)\varphi_t\varphi_x\,dx = &-[\varphi_x^2(1)+\varphi_x^2(0)] + 2\int_0^1 \varphi_x^2 dx + \int_0^1 q(x)\varphi_x\psi_x\,dx \\
&- \frac{\beta}{k}\int_0^1 q(x)\theta_{tx}\varphi_x\,dx + \frac{2\rho_1}{k}\int_0^1 \varphi_t^2 dx,
\end{aligned}$$

which gives

$$\begin{aligned}
\frac{k}{32}[\varphi_x^2(1)+\varphi_x^2(0)] = &-\frac{d}{dt}\frac{1}{32}\int_0^1 \rho_1 q(x)\varphi_t\varphi_x\,dx + \frac{k}{16}\int_0^1 \varphi_x^2 dx \\
&+ \frac{k}{32}\int_0^1 q(x)\varphi_x\psi_x\,dx - \frac{\beta}{32}\int_0^1 q(x)\theta_{tx}\varphi_x\,dx \\
&+ \frac{\rho_1 k}{16}\int_0^1 \varphi_t^2 dx.
\end{aligned}$$

By using Young's and Poincaré inequalities we have

$$\begin{aligned}
\frac{k}{16}\int_0^1 \varphi_x^2 dx &\le \frac{k}{8}\int_0^1 (\varphi_x+\psi)^2 dx + \frac{k}{8}\int_0^1 \psi_x^2 dx, \\
\frac{k}{32}\int_0^1 q(x)\varphi_x\psi_x\,dx &\le \frac{k}{32}\int_0^1 (\varphi_x+\psi)^2 dx + c\frac{k}{32}\int_0^1 \psi_x^2 dx. \\
-\frac{\beta}{k}\int_0^1 q(x)\theta_{tx}\varphi_x\,dx &\le \frac{k}{32}\int_0^1 (\varphi_x+\psi)^2 dx + \frac{k}{32}\int_0^1 \psi_x^2 dx + c\frac{k}{32}\int_0^1 \theta_{tx}^2 dx.
\end{aligned}$$

Thus, we obtain (4.53). □

Lemma 4.10. *Let (φ,ψ,θ) be the solution of (2.1). Then for some positive constant α_1 the functional*

$$J_6(t) := \int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \varsigma e^{-\varsigma\rho}|\mu(\varsigma)|z^2(x,\rho,\varsigma,t)\,d\varsigma\,d\rho\,dx,$$

satisfies

$$\begin{aligned}
\frac{dJ_6(t)}{dt} \le &-\alpha_1\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \varsigma|\mu(\varsigma)|z^2(x,\rho,\varsigma,t)\,d\varsigma\,d\rho\,dx + \ell\int_0^1 \theta_{tx}^2 dx \\
&- \alpha_1\int_0^1\int_{\tau_1}^{\tau_2} |\mu(\varsigma)|z^2(x,1,\varsigma,t)d\varsigma\,dx. \qquad (4.54)
\end{aligned}$$

Proof: Differentiating $J_6(t)$ and using (2.1), we obtain

$$\begin{aligned}\frac{dJ_6(t)}{dt} &= 2\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \varsigma e^{-\varsigma\rho}|\mu(\varsigma)|z(x,\rho,\varsigma,t)z_t(x,\rho,\varsigma,t)\,d\varsigma\,d\rho\,dx\\
&= -2\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} e^{-\varsigma\rho}|\mu(\varsigma)|z(x,\rho,\varsigma,t)z_\rho(x,\rho,\varsigma,t)\,d\varsigma\,d\rho\,dx\\
&= -\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \frac{\partial}{\partial\rho}\left(e^{-\varsigma\rho}|\mu(\varsigma)|z^2(x,\rho,\varsigma,t)\right)\,d\varsigma\,d\rho\,dx\\
&\quad -\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \varsigma e^{-\varsigma\rho}|\mu(\varsigma)|z^2(x,\rho,\varsigma,t)\,d\varsigma\,d\rho\,dx\\
&= -\int_0^1\int_{\tau_1}^{\tau_2} |\mu(\varsigma)|\left(e^{-\varsigma}z^2(x,1,\varsigma,t)-z^2(x,0,\varsigma,t)\right)d\varsigma\,dx\\
&\quad -\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \varsigma e^{-\varsigma\rho}|\mu(\varsigma)|z^2(x,\rho,\varsigma,t)\,d\varsigma\,d\rho\,dx.\end{aligned}$$

We know that $z(x,0,\varsigma,t)=\theta_{tx}(x,t)$ and $e^{-\varsigma}\le e^{-\varsigma\rho}\le 1$, then we obtain

$$\begin{aligned}\frac{dJ_6'(t)}{dt} &\le -\int_0^1\int_{\tau_1}^{\tau_2} |\mu(\varsigma)|e^{-\varsigma}z^2(x,1,\varsigma,t)d\varsigma\,dx + \int_{\tau_1}^{\tau_2}|\mu(\varsigma)|d\varsigma\int_0^1\theta_{tx}^2(x,t)dx\\
&\quad -\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \varsigma e^{-\varsigma}|\mu(\varsigma)|z^2(x,\rho,\varsigma,t)\,d\varsigma\,d\rho\,dx.\end{aligned}$$

Because $-e^{-\varsigma}$ is an increasing function, we have $-e^{-\varsigma}\le -e^{-\tau_2}$ for all $\varsigma\in[\tau_1,\tau_2]$. Finally, setting $\alpha_1=e^{-\tau_2}$ and recall (2.2), we obtain (4.54). □

Lemma 4.11. *Let (φ,ψ,θ) be the solution of (2.1). Then the functional*

$$J_7(t) := \int_0^1\int_0^1 e^{-2\tau\rho}\overline{z}^2(x,\rho,t)d\rho dx, \tag{4.55}$$

satisfies for some positive constants α_0 and α_2

$$\begin{aligned}\frac{dJ_7(t)}{dt} &\le -\alpha_0\int_0^1\int_0^1 \overline{z}^2(x,\rho,t)d\rho dx - \frac{\alpha_2}{\tau}\int_0^1\overline{z}^2(x,1,t)\,dx + \frac{1}{\tau}\int_0^1\varphi_t^2dx\\
&\le -\alpha_0\int_0^1\int_0^1 \overline{z}^2(x,\rho,t)d\rho dx + \frac{1}{\tau}\int_0^1\varphi_t^2dx.\end{aligned}\tag{4.56}$$

Proof: By differential of (4.55) we get

$$\begin{aligned}\frac{dJ_7(t)}{dt} &:= -\frac{1}{\tau}\int_0^1\int_0^1 e^{-2\tau\rho}\frac{\partial}{\partial\rho}\overline{z}^2(x,\rho,t)\,d\rho\,dx\\
\frac{dJ_7(t)}{dt} &\le -2\int_0^1\int_0^1 e^{-2\tau\rho}\overline{z}^2(x,\rho,t)d\rho dx - \frac{e^{-2\tau}}{\tau}\int_0^1\overline{z}^2(x,1,t)dx + \frac{1}{\tau}\int_0^1\varphi_t^2dx.\end{aligned}$$

Finally we get the estimate (4.56). □

Let the Lyapunov functional $L(t)$ defined by

$$\begin{aligned}L(t) &= NE(t)+\frac{1}{8}J_1(t)+J_2(t)+N_1J_3(t)+N_2J_4(t)+J_5(t)+N_3J_6(t)\\
&\quad + J_7(t)+\frac{\rho_1}{32}\int_0^1 q(x)\varphi_t\varphi_x dx\\
&\quad + \frac{8\rho_2}{k}\int_0^1 q(x)\psi_t(b\psi_x-\int_0^\infty g(s)ds)dx.\end{aligned}\tag{4.57}$$

where N, N_1, N_2, N_3, N_4 are positive constants to be chosen properly later.

Lemma 4.12. *Let $(\varphi,\ \psi,\ \theta)$ be the solution of (2.1). For N large enough, there exist two positive constants γ_1 and γ_2 satisfies*

$$\gamma_1 E(t) \leq L(t) \leq \gamma_2 E(t). \tag{4.58}$$

$$\begin{aligned} E(t) &= \frac{\gamma}{2}\int_0^1 \left[\rho_1\varphi_t^2 + k(\varphi_x+\psi)^2 + \rho_2\psi_t^2 + \left(b - \int_0^\infty g(s)ds\right)\psi_x^2\right]dx \\ &+ \frac{\gamma}{2}(g\circ\psi_x) + \gamma\int_0^1 \widehat{f}(\psi)dx + \frac{\beta}{2}\int_0^1 \left(\rho_3\theta_t^2 + \delta\theta_x^2\right)dx \\ &+ \frac{\xi}{2}\int_0^1\int_0^1 \bar{z}^2(x,\rho,t)\, d\rho\, dx + \frac{\beta}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \varsigma|\mu(\varsigma)|z^2(x,\rho,\varsigma,t)\, d\varsigma\, d\rho\, dx \end{aligned}$$

Proof: By the same arguments as in [4], using $\int_0^1 \varphi_x^2(t) \leq 2\int_0^1(\varphi_x+\psi)^2dx + 2\int_0^1 \psi_x^2(t)dx$ and Lemma 2, we can deduce

$$\begin{aligned} |L(t) - NE(t)| &\leq \delta_1\int_0^1 \varphi_t^2dx + \delta_2\int_0^1 \psi_t^2dx + \delta_3\int_0^1(\varphi_x+\psi)^2dx \\ &+ \delta_4\int_0^1 \psi_x^2dx + \delta_5\int_0^1 \theta_t^2dx + \delta_6\int_0^1\theta_x^2dx \\ &+\delta_7(g\circ\psi_x)(t) + \delta_8\int_0^1 \widehat{f}(\psi)dx \\ &+ \delta_9\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \varsigma e^{-\varsigma\rho}|\mu(\varsigma)|z^2(x,\rho,\varsigma,t)d\varsigma\, d\rho dx \\ &+ \delta_{10}\int_0^1\int_0^1 e^{-2\tau\rho}\overline{z}^2(x,\rho,t)d\rho dx \\ &\leq CE(t), \end{aligned}$$

in which δ_i $(i = 1,\dots,10)$ are positive constants as in [4]. □

Lemma 4.13. *Assume that (H1)–(H5) hold, then, there exist two positive constants β_1 and β_2 such that for some $c_0 \geq 0$*

$$\int_0^1 \psi_{0x}^2(x,s)dx \leq c_0,\ \ \forall s > 0, \tag{4.59}$$

we have for all $t > 0$

$$\begin{aligned} &\xi(t)L'(t) + \beta_1 E'(t) \\ &\leq -\alpha_4\xi(t)E(t) + \beta_2\xi(t)\int_t^\infty g(s)ds + \xi(t)\left(\frac{\rho_1 b}{k} - \rho_2\right)\int_0^1 \varphi_t(x,t)\psi_{xt}(x,t)dx, \end{aligned} \tag{4.60}$$

with $\beta_1 = \dfrac{2\alpha_5}{\gamma}$ and $\beta_2 = \alpha_5\left(\dfrac{8E(0)}{\gamma(b-g_0)} + 2c_0\right)$.

Remark 4.14. *In what follows, we note that all positive constants are bounded by some positive constant c.*

Proof: By differentiating $L(t)$, using Lemma 4.3-Lemma 4.10, we obtain by letting $\varepsilon_1 = \dfrac{1}{32N_1}$ the following

$$
\begin{aligned}
\frac{dL(t)}{dt} \leq & \left(N\frac{\gamma}{2} - N_1\frac{\rho_2 d_2}{4\varepsilon_1} - c\right)(g' \circ \psi_x) \\
& - \left(N\left(\mu_1\gamma - \frac{\xi}{2\tau} - \frac{\mu_2\gamma}{2}\right) - N_2\left(\frac{l}{10} + \frac{5\mu_1^2}{2l}\right) - c\right)\int_0^1 \varphi_t^2 dx \\
& - \frac{k}{32}\int_0^1 (\varphi_x + \psi)^2 dx - (N_2+1)\int_0^1 \widehat{f}(\psi)dx - \frac{l}{20}\int_0^1 \theta_x^2 dx \\
& - \left(N_1\rho_2 g_0 - N_2\left(\frac{5\rho_1^2}{2l} + \rho_2\right) - \left(\frac{\rho_2}{32} + c\rho_2 + c\right)\right)\int_0^1 \psi_t^2 dx \\
& - \left(N_2\frac{l}{2} - N_1\varepsilon_1 c - \frac{k}{32} - c\right)\int_0^1 \psi_x^2 dx + \left(\frac{b\rho_1}{k} - \rho_2\right)\int_0^1 \varphi_t\psi_{xt} dx \\
& - N_3\alpha_1\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \varsigma|\mu(\varsigma)|z^2(x,\rho,\varsigma,t)\,d\varsigma\,d\rho\,dx - \alpha_0\int_0^1\int_0^1 \overline{z}^2(x,\rho,t)d\rho dx \\
& - \left(N_3\alpha_1 - \frac{32c}{k}\right)\int_0^1\int_{\tau_1}^{\tau_2} |\mu(\varsigma)|z^2(x,1,\varsigma,t)d\varsigma\,dx \\
& - (N\beta m - c(1 + N_2 + N_3))\int_0^1 \theta_{tx}^2 dx \\
& - \left(N\left(\frac{\xi}{2\tau} - \frac{\mu_2\gamma}{2}\right) - c(N_2+1)\right)\int_0^1 \overline{z}^2(x,1,t)\,dx \\
& + c\left[N_1(1 + \varepsilon_1 + \frac{1}{\varepsilon_1}) + N_2 + 1\right](g \circ \psi_x)(t)
\end{aligned}
$$

We choose N_2 and N_3 large enough such that

$$N_3\alpha_1 - \frac{32c}{k} > 0,\ \ N_2\frac{l}{2} - N_1\varepsilon_1 c - \frac{k}{32} - c > 0.$$

Next, we pick N_1 large enough such that

$$N_1\rho_2 g_0 - N_2\left(\frac{5\rho_1^2}{2l} + \rho_2\right) - \left(\frac{\rho_2}{32} + c\rho_2 + c\right) > 0,$$

Finally, we choose N large (such that (4.58) remains valid) so that

$$
\begin{aligned}
N\frac{\gamma}{2} - N_1\frac{\rho_2 d_2}{4\varepsilon_1} - c > 0, \\
N\beta m - c(1 + N_2 + N_3) > 0, \\
N\left(\frac{\xi}{2\tau} - \frac{\mu_2\gamma}{2}\right) - c(N_2+1) > 0,
\end{aligned}
$$

and

$$N\left(\mu_1\gamma - \frac{\xi}{2\tau} - \frac{\mu_2\gamma}{2}\right) - N_2\left(\frac{l}{10} + \frac{5\mu_1^2}{2l}\right) - c > 0.$$

Consequently, by using Poincaré inequality and (2.8), we obtain

$$
\begin{aligned}
L'(t) \leq & -\alpha_4 E(t) + \alpha_5\int_0^1\int_0^\infty g(s)(\psi_x(x,t) - \psi_x(x,t-s))^2\,ds\,dx \\
& + \left(\frac{\rho_1 b}{k} - \rho_2\right)\int_0^1 \varphi_t(x,t)\psi_{xt}(x,t)dx, \qquad (4.61)
\end{aligned}
$$

where α_4 and α_5 are positive constants.

Using (H3) and (4.1), we obtain that for all $t \in R_+$,

$$\begin{aligned}
&\xi(t)\int_0^1\int_0^t g(s)(\psi_x(x,t)-\psi_x(x,t-s))^2\,ds\,dx\\
&\le \int_0^1\int_0^t \xi(s)g(s)(\psi_x(x,t)-\psi_x(x,t-s))^2\,ds\,dx\\
&\le -\int_0^1\int_0^t g'(s)(\psi_x(x,t)-\psi_x(x,t-s))^2\,ds\,dx\\
&\le -\int_0^1\int_0^\infty g'(s)(\psi_x(x,t)-\psi_x(x,t-s))^2\,ds\,dx\\
&\le -\frac{2}{\gamma}E'(t).
\end{aligned}$$

On the other hand, using the definition of $E(t)$, the fact that $E(t)$ is nonincreasing and (4.59), we ask for $t, s \in R_+$,

$$\begin{aligned}
\int_0^1 (\psi_x(x,t)-\psi_x(x,t-s))^2dx &\le 2\int_0^1 \psi_x^2(x,t)dx + 2\int_0^1 \psi_x^2(x,t-s)dx\\
&\le 4\sup_{s>0}\int_0^1 \psi_x^2(x,s)dx + 2\sup_{\tau<0}\int_0^1 \psi_x^2(x,\tau)dx\\
&\le 4\sup_{s>0}\int_0^1 \psi_x^2(x,s)dx + 2\sup_{\tau>0}\int_0^1 \psi_{0x}^2(x,\tau)dx\\
&\le \frac{8E(0)}{\gamma(b-g_0)} + 2c_0
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
&\xi(t)\int_0^1\int_t^\infty g(s)(\psi_x(x,t)-\psi_x(x,t-s))^2\,ds\,dx\\
&\le \left(\frac{8E(0)}{\gamma(b-g_0)} + 2c_0\right)\xi(t)\int_t^\infty g(s)ds.
\end{aligned}$$

Then, we deduce that, for all $t \in R_+$,

$$\begin{aligned}
&\xi(t)\int_0^1\int_0^\infty g(s)(\psi_x(x,t)-\psi_x(x,t-s))^2\,ds\,dx\\
&\le -\frac{2}{\gamma}E'(t)+\left(\frac{8E(0)}{\gamma(b-g_0)} + 2c_0\right)\xi(t)\int_t^\infty g(s)ds.
\end{aligned}$$

The proof is complete. □

Theorem 4.15. *Assume that (H1)–(H5) hold.*

1. If $\frac{k}{\rho_1} = \frac{b}{\rho_2}$ holds, then, for any $((\varphi_0,\varphi_1),(\psi_0,\psi_1),(\theta_0,\theta_1)) \in (H_0^1(0,1)\times L^2(0,1))^3$ there exist constants ϵ_1,$\epsilon_2 > 0$ such that, for all $\epsilon_0 \in [0,\epsilon_1]$, we have

$$E(t) \le \epsilon_2\left(1+\int_0^t (g(s))^{1-\epsilon_0}ds\right)e^{-\epsilon_0\int_0^t \xi(s)ds} + \epsilon_2\int_t^\infty g(s)ds,\ t\ge 0. \tag{4.62}$$

2. If $\frac{k}{\rho_1} \ne \frac{b}{\rho_2}$, then, for any $((\varphi_0,\varphi_1),(\psi_0,\psi_1),(\theta_0,\theta_1)) \in (H_0^1(0,1)\times L^2(0,1))^3$ satisfying, for some $c_0 \ge 0$

$$\max\left\{\int_0^1 \psi_{0x}^2(x,s)dx, \int_0^1 \psi_{0xs}^2(x,s)dx\right\} \le c_0, \quad s>0, \tag{4.63}$$

there exists a constant $\epsilon_2 > 0$ *such that, for all* $t \in R_+$*,*

$$E(t) \leq \frac{\epsilon_2(1 + \int_0^t \xi(s) \int_s^\infty g(\tau)d\tau ds)}{\int_0^t \xi(s)ds}. \tag{4.64}$$

Proof: First, we define

$$L_1(t) = \xi(t)L(t) + \beta_1 E(t), \quad r(t) = \xi(t) \int_t^\infty g(s)ds.$$

Clearly, $L_1(t)$ and $E(t)$ are equivalent, that is, exist positive constants α_6 and α_7, such that

$$\alpha_6 E(t) \leq L_1(t) \leq \alpha_7 E(t).$$

Then using Lemma 4.13 we have

$$L_1'(t) \leq -\epsilon_1 \xi(t) L_1(t) + \beta_2 r(t) + \left(\frac{\rho_1 b}{k} - \rho_2\right) \xi(t) \int_0^1 \varphi_t(x,t)\psi_{xt}(x,t)dx, \tag{4.65}$$

with $\epsilon_1 = \frac{\alpha_4}{\alpha_7}$. This inequality still holds, for any $\epsilon_0 \in [0, \epsilon_1]$, that is

$$L_1'(t) \leq -\epsilon_0 \xi(t) L_1(t) + \beta_2 r(t) + \left(\frac{\rho_1 b}{k} - \rho_2\right) \xi(t) \int_0^1 \varphi_t(x,t)\psi_{xt}(x,t)dx. \tag{4.66}$$

Now, we distinguish two cases.

Case 1: If $\dfrac{k}{\rho_1} = \dfrac{b}{\rho_2}$ holds. Because the last term in (4.66) vanishes, then (4.66) implies that, for all $t \in R_+$,

$$\frac{d}{dt}\left(e^{\epsilon_0 \int_0^t \xi(s)ds} L_1(t)\right) \leq \beta_2 e^{\epsilon_0 \int_0^t \xi(s)ds} r(t).$$

Therefore, by integrating over $[0, T]$ with $T \geq 0$, we have

$$L_1(T) \leq e^{-\epsilon_0 \int_0^T \xi(s)ds} \left(L_1(0) + \beta_2 \int_0^T e^{\epsilon_0 \int_0^t \xi(s)ds} r(t)dt\right),$$

which implies

$$E(T) \leq \frac{1}{\alpha_6} e^{-\epsilon_0 \int_0^T \xi(s)ds} \left(L_1(0) + \beta_2 \int_0^T e^{\epsilon_0 \int_0^t \xi(s)ds} r(t)dt\right). \tag{4.67}$$

Because

$$\begin{aligned} e^{\epsilon_0 \int_0^t \xi(s)ds} r(t)dt &= \frac{1}{\epsilon_0} \frac{d}{dt}\left(e^{\epsilon_0 \int_0^t \xi(s)ds}\right) \int_t^\infty g(s)ds \\ &= \frac{1}{\epsilon_0} \frac{d}{dt}\left(e^{\epsilon_0 \int_0^t \xi(s)ds} \int_t^\infty g(s)ds\right) + g(t) e^{\epsilon_0 \int_0^t \xi(s)ds}, \end{aligned}$$

by integration we obtain

$$\begin{aligned} &\int_0^T e^{\epsilon_0 \int_0^t \xi(s)ds} r(t)dt \\ &= \frac{1}{\epsilon_0}\left(e^{\epsilon_0 \int_0^T \xi(s)ds} \int_T^\infty g(s)ds - \int_0^\infty g(s)ds + \int_0^T e^{\epsilon_0 \int_0^t \xi(s)ds} g(t)dt\right). \end{aligned}$$

Consequently

$$\begin{aligned} E(T) \;&\leq\; \frac{1}{\alpha_6}\left(L_1(0)e^{-\epsilon_0\int_0^T\xi(s)ds}+\frac{\beta_2}{\epsilon_0}\int_T^\infty g(s)ds\right) \\ &\quad+\frac{\beta_2}{\alpha_6\epsilon_0}e^{-\epsilon_0\int_0^T\xi(s)ds}\int_0^T e^{\epsilon_0\int_0^t\xi(s)ds}g(t)dt. \end{aligned} \tag{4.68}$$

On the other hand, using (2.4), multiplying by $\epsilon_0>0$ and integrating over $(0,t)$, for all $t\in R_+$,

$$(g(t))^{\epsilon_0}\leq (g(0))^{\epsilon_0}e^{-\epsilon_0\int_0^t\xi(s)ds},\ t\geq 0.$$

Now, integrating over $(0,T)$, we get

$$\int_0^T e^{\epsilon_0\int_0^t\xi(s)ds}g(t)dt\leq (g(0))^{\epsilon_0}\int_0^T (g(t))^{1-\epsilon_0}dt. \tag{4.69}$$

Inserting (4.69) into (4.68) and letting

$$\epsilon_1=\frac{1}{\alpha_6}\max\left\{L_1(0),\frac{\beta_2}{\epsilon_0},\frac{\beta_2}{\epsilon_0}(g(0))^{\epsilon_0}\right\},$$

we obtain(4.62).

Case 2: $\frac{k}{\rho_1}\neq\frac{b}{\rho_2}$. We estimate the last term of (4.66) as follows:

$$\begin{aligned} &\left(\frac{\rho_1 b}{k}-\rho_2\right)\int_0^1\varphi_t(x,t)\psi_{xt}(x,t)dx \\ &=\frac{\rho_1 b-\rho_2 k}{g_0 k}\int_0^1\varphi_t(x,t)\int_0^\infty g(s)(\psi_{xt}(x,t)-\psi_{xt}(x,t-s))\,ds\,dx \\ &\quad+\frac{\rho_1 b-\rho_2 k}{g_0 k}\int_0^1\varphi_t(x,t)\int_0^\infty g(s)\psi_{xt}(x,t-s)\,ds\,dx. \end{aligned}$$

By using Young's inequality, for any $\varepsilon>0$, there exists a positive constant c_ε (depending on ε) such that

$$\begin{aligned} &\frac{\rho_1 b-\rho_2 k}{g_0 k}\int_0^1\varphi_t(x,t)\int_0^\infty g(s)(\psi_{xt}(x,t)-\psi_{xt}(x,t-s))\,ds\,dx \\ &\leq c\int_0^1|\varphi_t(x,t)|\int_0^\infty g(s)|\psi_{xt}(x,t)-\psi_{xt}(x,t-s)|\,ds\,dx \\ &\leq\frac{\varepsilon}{2}E(t)+c_\varepsilon\int_0^1\int_0^\infty g(s)(\psi_{xt}(x,t)-\psi_{xt}(x,t-s))^2\,ds\,dx. \end{aligned}$$

At the same time we have

$$\int_0^\infty g(s)\psi_{xt}(x,t-s)ds=\int_0^\infty g'(s)\left(\psi_{xt}(t-s)-\psi_x(t)\right)ds,$$

$$\begin{aligned} &\frac{\rho_1 b-\rho_2 k}{g_0 k}\int_0^1\varphi_t(x,t)\int_0^\infty g(s)\psi_{xt}(x,t-s)\,ds\,dx \\ =\;&\frac{\rho_1 b-\rho_2 k}{g_0 k}\int_0^1\varphi_t(x,t)\int_0^\infty g'(s)\left(\psi_{xt}(t-s)-\psi_x(t)\right)\,ds\,dx \\ \leq\;&\frac{\varepsilon}{2}E(t)-c_\varepsilon(g'\circ\psi_{xt})(t) \\ \leq\;&\frac{\varepsilon}{2}E(t)-\frac{2c_\varepsilon}{\gamma}E'(t), \end{aligned}$$

$$
\begin{aligned}
&\left(\frac{\rho_1 b}{k}-\rho_2\right)\int_0^1 \varphi_t(x,t)\psi_{xt}(x,t)dx\\
\leq\ & \varepsilon E(t)+c_\varepsilon\int_0^1\int_0^\infty g(s)(\psi_{xt}(x,t)-\psi_{xt}(x,t-s))^2\,ds\,dx-\frac{2c_\varepsilon}{\gamma}E'(t).\\
\leq\ & \varepsilon E(t)+c_\varepsilon\int_0^1\int_0^\infty g(s)(\psi_{xt}(x,t)-\psi_{xt}(x,t-s))^2\,ds\,dx-\frac{2c_\varepsilon}{\gamma}E'(t).
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&\xi(t)\left(\frac{\rho_1 b}{k}-\rho_2\right)\int_0^1 \varphi_t(x,t)\psi_{xt}(x,t)dx\\
\leq\ & \varepsilon\xi(t)E(t)+c_\varepsilon\xi(t)\int_0^1\int_0^\infty g(s)(\psi_{xt}(x,t)-\psi_{xt}(x,t-s))^2\,ds\,dx\\
&-\frac{2c_\varepsilon}{\gamma}\xi(t)E'(t).
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
L_1'(t)\ \leq\ & -\alpha_8\xi(t)E(t)+\beta_2 r(t)-\frac{2c_\epsilon}{\gamma}\xi(t)E'(t)\\
&+c_\epsilon\xi(t)\int_0^1\int_0^\infty g(s)(\psi_{xt}(x,t)-\psi_{xt}(x,t-s))^2\,ds\,dx, \qquad (4.70)
\end{aligned}
$$

where $\alpha_8=\epsilon_0\alpha_6-\varepsilon$. By using the definition of $E_2(t)$, $E_2'(t)$ and (4.63), we have

$$
\xi(t)\int_0^1\int_0^t g(s)(\psi_{xt}(x,t)-\psi_{xt}(x,t-s))^2\,ds\,dx\leq-\frac{2}{\gamma}E_2'(t), \qquad (4.71)
$$

$$
\begin{aligned}
&\xi(t)\int_0^1\int_t^\infty g(s)(\psi_{xt}(x,t)-\psi_{xt}(x,t-s))^2\,ds\,dx\\
\leq\ & \left(\frac{8E_2(0)}{\gamma(b-g_0)}+2c_0\right)r(t). \qquad (4.72)
\end{aligned}
$$

Hence, combining (4.70), (4.71) and (4.72) we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left(L_1(t)+\frac{2c_\varepsilon}{\gamma}E_2(t)+\frac{2c_\varepsilon}{\gamma}\xi(t)E(t)\right)\\
\leq\ & -\alpha_8\xi(t)E(t)+\beta_3 r(t)+\frac{2c_\varepsilon}{\gamma}\xi'(t)E(t), \qquad (4.73)
\end{aligned}
$$

where $\beta_3=\beta_2+(\frac{8E_2(0)}{\gamma(b-g_0)}+2c_0)c_\varepsilon$. Because ξ is nonincreasing, the last term of (4.73) is nonpositive, therefore, by integration on $[0,T]$ and using the fact $E(t)$ is nonincreasing, we obtain

$$
\alpha_8 E(T)\int_0^T\xi(t)dt\leq L_1(0)+\frac{2c_\varepsilon}{\gamma}E_2(0)+\frac{2c_\varepsilon}{\xi(0)}E(0)+\beta_3\int_0^T r(t)dt,
$$

which gives (4.64) with

$$
\epsilon_2=\frac{1}{\alpha_8}\max\{L_1(0)+\frac{2c_\varepsilon}{\gamma}E_2(0)+\frac{2c_\varepsilon}{\xi}(0)E(0),\beta_3\}.
$$

This completes the proof. $\square$

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