Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 2023 (41)** : 1–12.
ISSN-0037-8712 in press
doi:10.5269/bspm.52035

Stability and Local Attractivity for Non-autonomous Boundary Cauchy Problems

Amine Jerroudi and Mohammed Moussi

ABSTRACT: In this paper we present results concerning the existence, stability and local attractivity for non-autonomous semilinear boundary Cauchy problems. In our method, we assume certain smoothness properties on the linear part and the local Lipschitz continuity on the nonlinear perturbation. We apply our abstract results to population equations.

Key Words: Boundary Cauchy problem, evolution family, population equations, local attractivity, stability.

Contents

1. Introduction

Consider the following non-linear boundary Cauchy problem

$$\begin{cases} \dfrac{d}{dt}u(t) = A_{\max}(t)u(t), \quad t \geq s \geq 0, \\ L(t)u(t) = f(t, u(t)), \quad t \geq s \geq 0, \\ u(s) = x, \end{cases} \tag{1.1}$$

where $A_{\max}(t) \in \mathcal{L}(D, X)$, $L(t) \in \mathcal{L}(D, Y)$, X, Y and D are Banach spaces with D densely and continuously embedded in X and a function f maps from $\mathbb{R}_+ \times X$ to Y. The solution $u : [s, \infty) \to X$ takes the initial value $x \in X$ at time s. The linear boundary Cauchy problem associated with (1.1) is given by

$$\begin{cases} \dfrac{d}{dt}u(t) = A_{\max}(t)u(t), \quad t \geq s \geq 0, \\ L(t)u(t) = 0, \quad t \geq s \geq 0, \\ u(s) = x. \end{cases} \tag{1.2}$$

This type of equation has recently been considered and studied as a model class with various applications like population equations, functional differential equations and boundary control problems (see [3,4,7,8] and the references therein).

A crucial question concerning nonautonomous boundary equations is the existence of solutions. In [4], the authors proved that the solutions of (1.1) in the case that $f(t, x(t)) \equiv f(t)$ are given by a variation of constants formula which, following the same argument as in [10], we extend to a variation of constants formula solution of (1.1) in the present work.

The variation of constants formula has played a very important role in the study of the regularity properties and the long time behavior for this type of evolution equations, see [1,5,9], for example.

2010 *Mathematics Subject Classification:* 35B40, 35L70.
Submitted January 31, 2020. Published September 16, 2020

In the present work, our contribution is concerning with the stability and the local attractivity which are among the most topics in the qualitative theory of infinite dynamical systems attracting in the last years a big interest.

Roughly speaking, our goal is establishing existence of mild solution vanishing at infinity. The second goal is to establish sufficient conditions for guaranteeing the existence of local attractivity. More precisely, by assuming that the solution of the linear problem (1.2) is exponentially stable and the nonlinear perturbation f is locally Lipschitzian, we will prove that the solution of (1.1) is locally attractive.

Our plan in this paper is as follows: in section 2, we give some preliminaries concerning definitions and natural assumptions for well-posedness. Section 3 is devoted to the existence of mild solution in a variation of constants formula form. In section 4 and section 5, we prove a stability and a local attractivity results, respectively, of the problem (1.1). The last section is devoted to an application of structured population equation.

To end this section, we give notations used in this paper. For Banach spaces X, Y, $\mathcal{L}(X,Y)$ denotes the space of all linear bounded operators from X to Y. We denote by id_X the identity map defined on X. By $C(\mathbb{R}_+, X)$ we denote the space of all continuous functions from $\mathbb{R}_+$ into X, $C_b(\mathbb{R}_+, X)$ is the space of all bounded continuous functions from $\mathbb{R}_+$ into X, and by $C_0(\mathbb{R}_+, X)$ we denote the space of the continuous functions on $\mathbb{R}_+$ converging to zero at the infinity.
$\overline{B}(R) := \{x \in X : \|x\| \leq R\}$ is the closed ball centred at zero with radius equal to R .

Let $A : D(A) \subset X \longrightarrow X$ be a closed linear operator, we denote by

$$\rho(A) := \{\lambda \in \mathbb{R} \mid \lambda\, id_X - A : D(A) \longrightarrow X \text{ is bijective}\}$$

the *resolvent set* of A. For $\lambda \in \rho(A)$, the operator $R(\lambda, A) := (\lambda\, id_X - A)^{-1}$ is called the *resolvent* of A.

2. Preliminaries

In this section we recall some definitions and results, and formulate assumptions.

Definition 2.1. *A family of bounded linear operators $\mathcal{U} := (U(t,s))_{t\geq s\in J}$, $J := \mathbb{R}_+$ or $\mathbb{R}$, on a Banach space X is an evolution family if*

1. *$U(t,r)U(r,s) = U(t,s)$ and $U(t,t) = id_X$ for all $t \geq r \geq s \in J$;*
2. *the mapping $\{(t,s) \in J \times J\ : t \geq s\} \ni (t,s) \mapsto U(t,s) \in \mathcal{L}(X,X)$ is strongly continuous.*

An evolution family is called exponentially bounded if, in addition,

3. *There exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that:*

$$\|U(t,s)\| \leq Me^{\omega(t-s)} \quad t \geq s \geq 0.$$

When $\omega < 0$ we say that the evolution family is exponentially stable.

Definition 2.2. *Let $I \subseteq \mathbb{R}$. A family of linear (unbounded) operators $(A(t))_{t\in I}$ on a Banach space X is called a stable family if there are constants $M \geq 1$, $\omega \in \mathbb{R}$ such that $(\omega, +\infty) \subset \rho(A(t))$ for all $t \in I$ and*

$$\left\| \prod_{i=1}^{m} R(\lambda, A(t_i)) \right\| \leq M(\lambda - \omega)^{-m}$$

for $\lambda > \omega$ and any finite sequence $t_1, ..., t_m$ in I such that $t_1 \leq t_2 \leq \cdots \leq t_m$, $m = 1, 2, ...$

Let X, D and Y be Banach spaces such that D is densely and continuously embedded in X. The operators $A_{\max}(t) \in \mathcal{L}(D,X)$, $L(t) \in \mathcal{L}(D,Y)$ are supposed to satisfy the following assumptions.

(H1) There are positive constants c_1, c_2 such that

$$c_1||x||_D \leq ||x|| + ||A_{\max}(t)x|| \leq c_2||x||_D$$

for all $x \in D$ and $t \geq 0$.

(H2) For each $x \in D$, the mapping $t \longmapsto A_{\max}(t)x$ is continuously differentiable.

(H3) The family of operators $(A(t))_{t\geq 0}$, where $A(t) := A_{\max}(t)|_{\ker L(t)}$, is stable with stability constants M and ω_0.

(H4) The operators $L(t) : D \longrightarrow Y$, $t \geq 0$, are surjective.

(H5) For each $x \in D$ the mapping $t \longmapsto L(t)x$ is continuously differentiable.

(H6) There exist constants $\gamma > 0$ and $\omega \in \mathbb{R}$ such that

$$||L(t)x||_Y \geq \frac{\lambda - \omega}{\gamma}||x||_X$$

for all $x \in \ker(\lambda - A_{\max}(t))$, $\lambda > \omega$ and $t \geq 0$.

In the following lemma, we cite consequences of the above assumptions from [6, Lemma 1.2].

Lemma 2.3. *If the conditions (H1)–(H6) are satisfied, then for all λ in $\rho(A(t))$*

1. *$L(t)|_{\ker(\lambda - A_{\max}(t))}$ is an isomorphism from $\ker(\lambda - A_{\max}(t))$ to Y.*
2. *The function $t \longmapsto L_{\lambda,t}y$ is continuously differentiable for all $y \in Y$ and $\|\lambda L_{\lambda,t}\| \leq \dfrac{\lambda\gamma}{\lambda - \omega}$, where $L_{\lambda,t} = \left(L(t)|_{\ker(\lambda - A_{\max}(t))}\right)^{-1}, \forall \lambda > \omega$.*

Under the above assumptions, it was shown in [7] that there exists an evolution family $(U(t,s))_{t\geq s\geq 0}$ generated by $(A(t))_{t\geq s}$ and

$$||U(t,s)|| \leq Me^{\omega_0(t-s)}, \quad \forall t \geq s \geq 0. \tag{2.1}$$

That is $U(t,s)x$ is a solution of the problem (1.2).

3. Existence of mild solution

In this section we are interested in the nonlinear case. More precisely, we discuss the existence of mild solution for the problem (1.1). We start by giving the following definition.

Definition 3.1. *A function $u \in C([s,+\infty[, X)$ is said to be a mild solution of the problem (1.1) if it satisfies the integral equation*

$$u(t) = U(t,s)x + \lim_{\lambda\to+\infty}\int_s^t U(t,\sigma)\lambda L_{\lambda,\sigma} f(\sigma, u(\sigma))\, d\sigma, \quad \forall\, t \geq s. \tag{3.1}$$

Remark 3.2. *Let $x \in X$ and $t \geq s \geq 0$. Define the operator S on the space $C([s;+\infty[, X)$ by*

$$(Su)(t) = U(t,s)x + \lim_{\lambda\to+\infty}\int_s^t U(t,\sigma)\lambda L_{\lambda,\sigma} f(\sigma, u(\sigma))d\sigma. \tag{3.2}$$

It was shown in [8, Proposition 3.2.2] that $Su \in C([s;+\infty[, X)$. Therefore, one can see that a continuous function u is a mild solution of (1.1) if and only if it is a fixed point of the operator S.

In the next theorem, we give a result on the existence and uniqueness of mild solution of the problem (1.1). For that, the nonlinear perturbation f is needed to satisfy $f(t,0) \equiv 0$ and the following local Lipschicity condition:

(C) For every $T > 0$ and $R > 0$, there is a constant $C_{R,T}$ such that:

$$\|f(t,u) - f(t,v)\| \leq C_{R,T}\,\|u - v\|\,,$$

for all u and v in $\overline{B}(R)$ and $t \in [s,T]$.

Theorem 3.3. *Under the conditions (H1)–(H6), (C) and for all $x \in X$, there exists a positive constant $T_{\max} \leq +\infty$ such that the problem (1.1) has a unique mild solution u on $[s, T_{\max}[$ with the initial value $u(s) = x$. Moreover, if $T_{\max} < +\infty$, then*

$$\lim_{t \to T_{\max}} \|u(t)\| = +\infty$$

Proof. We follow the same arguments as in the proof of [10, p 185, Therorem 1.4]. Let $s \geq 0$ and $x \in X$. Denoting by $\delta_{s,x} := \min\left\{1, \dfrac{\|x\|}{\gamma R_s C_{R_s,s+1}}\right\}$, where $R_s := 2M(s)\|x\|$, $M(s) := \max_{s \leq t \leq s+1} \|U(t,s)\|$ and $C_{R_s,s+1}$ the constant provided by (C). Let $\tau := s + \delta_{s,x}$, we will prove that there is a mild solution of (1.1) on the interval $[s, \tau]$ whose length is $\delta_{s,x}$.

Let S be the operator as defined in (3.2) on the space $C([s,\tau], X)$. From [8, Proposition 3.2.2], we have $S(C([s,\tau], X)) \subset C([s,\tau], X)$. Furthermore, S maps the ball $\overline{B}(R_s) \subset C([s,\tau], X)$ into itself. Indeed, supposing that $u \in C([s,\tau]; X)$ such that $\|u\| \leq R_s$. Then, for $t \in [s,\tau]$ we have

$$\begin{aligned}
\|(Su)(t)\| &\leq \|U(t,s)x\| + \lim_{\lambda \to +\infty} \int_s^t \|U(t,\sigma)\| \, \|\lambda L_{\lambda,\sigma}\| \, \|(f(\sigma, u(\sigma))\| \, d\sigma \\
&\leq M(s)\|x\| + M(s)\gamma \int_s^t \|f(\sigma, u(\sigma))\| \, d\sigma \\
&\leq M(s)\|x\| + M(s)(t-s)\gamma C_{R_s,s+1} \|u\| \\
&\leq M(s)\|x\| + M(s)\delta_{s,x}\gamma C_{R_s,s+1} R_s \\
&\leq 2M(s)\|x\| \\
&= R_s.
\end{aligned}$$

Thus, $S\left(\overline{B}(R_s)\right) \subset \overline{B}(R_s)$. Moreover, one can show, by induction, that for all u and v in $\overline{B}(R_s)$

$$\|(S^n u - S^n v)\| \leq \frac{\left(\gamma \tau M(s) C_{R_s,s+1}\right)^n}{n!} \|u - v\|_\infty, \quad \forall n \in \mathbb{N}^*.$$

Hence, for n sufficiently large, S is a contraction on $\overline{B}(R_s)$. Thus, by the fixed point theorem, there exists a unique function u in $\overline{B}(R_s)$ satisfying $Su = u$. This fixed point is the desired mild solution on $[s,\tau]$. The above procedure can be reproduced in the way that the mild solution u on $[s,\tau]$ can be extended to the interval $[s, \tau + \delta]$ for $\delta > 0$ depending on $\|u(\tau)\|$, R_τ and $C_{R_\tau,\tau+1}$ by $u(t) = v(t)$ on $[\tau, \tau+\delta]$, where v is the solution of the integral equation

$$v(t) = U(t,\tau)u(\tau) + \lim_{\lambda \to +\infty} \int_\tau^t U(t,\sigma)\lambda L_{\lambda,\sigma} f(\sigma, v(\sigma)) d\sigma, \text{ for } \tau \leq t \leq \tau + \delta.$$

Hence, one can conclude that there exists $T_{\max} > s$ such that $[s; T_{\max}[$ is the maximal interval of the existence of mild solution of (1.1).

Now, supposing that $T_{\max} < \infty$ and $\lim\limits_{t \to T_{\max}} \|u(t)\| < \infty$. Then, there is a sequence (t_n) and $n_0 \in \mathbb{N}$ such that $\lim\limits_{n \to +\infty} t_n = T_{\max}$ and $\|u(t_n)\| \leq C$ for all $n \geq n_0$. From the above argument, we can prove that, for t_n near enough to $T_{\max}$, the mild solution on $[s, t_n]$ can be extended on $[t_n, t_n + \delta]$ with $\delta > 0$, which contradicts the definition of $T_{\max}$.

For the unicity, let $T, R > 0$ and let u, v be mild solutions of (1.1) on $[s,T]$. Supposing that $u(s) \in \overline{B}(R)$. Then, we have

$$\begin{aligned}
\|u(t)\| &\leq M e^{\omega_0(t-s)} \|u(s)\| + M\gamma C_{R,T} \int_s^t e^{\omega_0(t-\sigma)} \|u(\sigma)\| \, d\sigma \\
&\leq RM e^{\omega_0(t-s)} + M\gamma C_{R,T} \int_s^t e^{\omega_0(t-\sigma)} \|u(\sigma)\| \, d\sigma.
\end{aligned}$$

Therefore,

$$e^{-\omega_0 t} \|u(t)\| \leq RM e^{-\omega_0 s} + M\gamma_0 C_{R,T} \int_s^t e^{-\omega_0 \sigma} \|u(\sigma)\| \, d\sigma.$$

Using the Gronwall's lemma we get

$$\|u(t)\| \leq RMe^{(t-s)(\omega_0+M\gamma C_{R,T})} := R'.$$

Now, let $u(s), v(s) \in \overline{B}(R)$ and $t \in [s,T]$, we have

$$\|u(t)-v(t)\| \quad \leq Me^{\omega_0(t-s)}\|u(s)-v(s)\| + M\gamma C_{R',T}\int_s^t e^{\omega_0(t-\sigma)}\|u(\sigma)-v(\sigma)\|\, d\sigma.$$

By the same argument as above, one can get

$$\|u(t)-v(t)\| \quad \leq Me^{(t-s)(\omega+M\gamma C_{R',T})}\|u(s)-v(s)\|.$$

This implies the unicity of the mild solution. □

4. Stability of mild solution

Our aim, in the present section, is to show the stability of mild solutions associated with the problem (1.1). Let $x \in X$ and $t \geq s \geq 0$. Define the operator Γ on $C_0(\mathbb{R}_+, Y)$ by

$$(\Gamma u)(t) = U(t,s)x + \lim_{\lambda\to+\infty}\int_s^t U(t,\sigma)\lambda L_{\lambda,\sigma}u(\sigma)d\sigma. \tag{4.1}$$

We have the following lemma.

Lemma 4.1. *Under the assumptions (H1)–(H6) and if $(U(t,s))_{t\geq s\geq 0}$ is exponentially stable, then the operator Γ maps $C_0(\mathbb{R}_+, Y)$ into $C_0(\mathbb{R}_+, X)$ and is $\frac{M\gamma}{|\omega_0|}$–Lipschitzian.*

Proof. Let $x \in X$ and $u \in C_0(\mathbb{R}_+, Y)$, by [8, Proposition 3.2.2] we have $\Gamma(C(\mathbb{R}_+, Y)) \subset C(\mathbb{R}_+, X)$. Moreover, for all $t \geq s \geq 0$ we have

$$\|\Gamma u(t)\| \leq \|U(t,s)x\| + \lim_{\lambda\to+\infty}\int_s^t \|U(t,\sigma)\|\,\lambda\,\|L_{\lambda,\sigma}\|\,\|u(\sigma)\|\, d\sigma.$$

Since $(U(t,s))_{t\geq s\geq 0}$ is exponentially stable, then the first claim can be obtained by (2.1) and by applying the dominated convergence theorem.

On the other hand, for u and v in $C_0(\mathbb{R}_+, X)$ and $t \geq s \geq 0$, we have

$$\begin{aligned}\|\Gamma u(t)-\Gamma v(t)\| &= \left\|\lim_{\lambda\to+\infty}\int_s^t U(t,\sigma)\lambda L_{\lambda,\sigma}(u(\sigma)-v(\sigma))d\sigma\right\| \\ &\leq \lim_{\lambda\to+\infty}\frac{M\gamma\lambda}{\lambda-\omega}\int_s^t e^{\omega_0(t-\sigma)}\|u(\sigma)-v(\sigma)\|\, d\sigma \\ &\leq M\gamma\|u-v\|_\infty\int_s^t e^{\omega_0(t-\sigma)}d\sigma \\ &\leq \frac{M\gamma}{|\omega_0|}\|u-v\|_\infty.\end{aligned}$$

This ends the proof. □

Remark 4.2. *For $f \in C(\mathbb{R}_+ \times X, Y)$ define the operator N_f on $C(\mathbb{R}_+, X)$ by $N_f(u) := f(\cdot, u(\cdot))$. One can see that a function $u \in C(\mathbb{R}_+, X)$ is a mild solution of the problem (1.1) if and only if $\Gamma \circ N_f(u) = u$.*

Theorem 4.3. *Let $f \in C(\mathbb{R}_+ \times X, Y)$ be a globally Lipschitz function with a Lipschitz constant $C \geq 0$. Moreover, assume that $\lim_{t\to+\infty} f(t,0) = 0$, $\omega_0 < 0$ and $\frac{|\omega_0|}{M\gamma} > C$. Then, under the assumptions (H1)–(H6), every unique mild solution of (1.1) is vanishing at infinity.*

Proof. Let u_x be the mild solution of (1.1) with an initial value $x \in X$. By the Lemma 4.1, the operator Γ is $\frac{M\gamma}{|\omega_0|}-$Lipschitzian from $C_0(\mathbb{R}_+, Y)$ to $C_0(\mathbb{R}_+, X)$, and by [2, Theorem 3.6], the operator N_f is $C-$Lipschitzian from $C_0(\mathbb{R}_+, X)$ to $C_0(\mathbb{R}_+, Y)$. Thus, for all $u, v \in C_0(\mathbb{R}_+, X)$ and $t \geq s \geq 0$, we have

$$\|\Gamma \circ N_f(u)(t) - \Gamma \circ N_f(v)(t)\| \leq \frac{M\gamma C}{|\omega_0|} \|u(t) - v(t)\|.$$

This implies that

$$\|\Gamma \circ N_f(u) - \Gamma \circ N_f(v)\|_\infty < \|u - v\|_\infty,$$

which means that $\Gamma \circ N_f$ is a contraction from $C_0(\mathbb{R}_+, X)$ to $C_0(\mathbb{R}_+, X)$. Therefore, using the Banach fixed point theorem, we infer that there exists a unique $w \in C_0(\mathbb{R}_+, X)$ such that

$$(\Gamma \circ N_f)(w) = w.$$

In other word, by the above remark, w is a mild solution of (1.1) with the initial value x. By the unicity of the mild solution, one can conclude that $u_x = w$ and then the proof is achieved. □

5. Local attractivity of mild solution

The aim of this section is to show an attractivity result of the problem (1.1). For this purpose, we need the following Lipschitz condition on the nonlinear perturbation:

$(H7)$ There exist constants $R > 0$ and $0 < \alpha \leq 1$ such that

$$\|f(t, \bar{u}(t) + x) - f(t, \bar{u}(t) + y)\| \leq \alpha \|x - y\|, \ \forall t \geq 0, \forall x, y \in \overline{B}(R).$$

Where $\bar{u}$ is a mild solution of (1.1).

Definition 5.1. *For an element $x \in X$, we denote by u_x the mild solution of the problem (1.1) with initial value $u_x(s) = x \in X$. A mild solution u of (1.1) is called locally attractive if there exists a number $R > 0$ such that for all x in the ball $B(u(s), R) \subset X$, we have*

$$\lim_{t \to +\infty} \|u_x(t) - u(t)\| = 0. \tag{5.1}$$

If the limit in (5.1) exists for all $x \in X$, we say that u is globally attractive.

Let us define the mapping $g : \mathbb{R}_+ \times X \longrightarrow Y$ by

$$g(t, x) := f(t, \bar{u}(t) + x) - f(t, \bar{u}(t))$$

and its corresponding boundary Cauchy problem

$$\begin{cases} \frac{d}{dt}u(t) = A_{\max}(t)u(t), \quad t \geq s \geq 0, \\ L(t)u(t) = g(t, u(t)), \quad t \geq s \geq 0, \\ u(s) = x. \end{cases} \tag{5.2}$$

We have the following lemma which is needed below.

Lemma 5.2. *The problem (5.2) has a unique mild solution belonging to the space $C_0(\mathbb{R}_+, \overline{B}(R))$ provided that $\frac{|\omega_0|}{M\gamma} > \alpha$.*

Proof. Let $t \in \mathbb{R}_+$ and x, y in $\overline{B}(R)$, from the assumption (H7), we have

$$\begin{aligned} \|g(t,x)\| &\leq \alpha \|x\| \\ &\leq R \end{aligned} \tag{5.3}$$

and

$$\|g(t,x)-g(t,y)\| \leq \alpha \|x-y\|. \tag{5.4}$$

We define the operator N_g on $C_0(\mathbb{R}_+, \overline{B}(R))$ by

$$N_g(v)(t) := g(t, v(t)).$$

Using (5.3), one can see that N_g maps $C_0(\mathbb{R}_+, \overline{B}(R))$ into itself. Furthermore, using (5.4) we get, for v_1 and $v_2 \in C_0(\mathbb{R}_+, \overline{B}(R))$,

$$\|N_g(v_1)-N_g(v_2)\|_\infty \leq \alpha \|v_1-v_2\|_\infty, \tag{5.5}$$

since $N_g(0)=0$, then one has

$$\|N_g(v)\|_\infty \leq \alpha \|v\|_\infty, \quad \forall v \in C_0(\mathbb{R}_+, \overline{B}(R)).$$

Using the Lemma 4.1, we obtain, for all $v \in C_0(\mathbb{R}_+, \overline{B}(R))$

$$\begin{aligned}\|\Gamma(N_g(v))\|_\infty &\leq \frac{M\gamma}{|\omega_0|}\|N_g(v)\|_\infty \\ &\leq \frac{M\gamma\alpha}{|\omega_0|}\|v\|_\infty \\ &\leq R.\end{aligned}$$

This shows that

$$\Gamma \circ N_g\Big(C_0(\mathbb{R}_+, \overline{B}(R))\Big) \subset C_0(\mathbb{R}_+, \overline{B}(R)).$$

In addition, since Γ is $\frac{M\gamma}{|\omega_0|}$−Lipschitzian and using (5.5), we conclude that for all v_1 and $v_2 \in C_0\left(\mathbb{R}_+, \overline{B}(R)\right)$,

$$\begin{aligned}\|\Gamma \circ N_g(v_1) - \Gamma \circ N_g(v_2)\| &\leq \frac{M\gamma}{|\omega_0|}\|N_g(v_1)-N_g(v_2)\| \\ &\leq \frac{M\gamma\alpha}{|\omega_0|}\|v_1-v_2\|_\infty \\ &< \|v_1-v_2\|_\infty.\end{aligned}$$

Thus, $\Gamma \circ N_g$ is a contraction from $C_0(\mathbb{R}_+, \overline{B}(R))$ to $C_0(\mathbb{R}_+, \overline{B}(R))$. Applying the Banach fixed point theorem we infer that there exists a unique function $u_x \in C_0(\mathbb{R}_+, \overline{B}(R))$ such that $\Gamma \circ N_g(u_x) = u_x$. Then, by Remark 4.2, the proof is achieved. □

We are now ready to state the main result of this section.

Theorem 5.3. *Assume that the problem (1.1) has a unique mild solution u_x with initial value $x \in X$ and let $\bar{u}$ be another mild solution of (1.1) associated with the initial value $\bar{u}(s)$. Under the assumptions (H1)–(H7) and if the evolution family $(U(t,s))_{t\geq s\geq 0}$ is exponentially stable. Then $\bar{u}$ is locally attractive provided that $\frac{|\omega_0|}{M\gamma} > \alpha$.*

Proof. Let $a \in X$ and let u_a be the mild solution of (1.1) with the initial value $u_a(s) = a$. Supposing that $\|a - \bar{u}(s)\| \leq R$. We put $x := a - \bar{u}(s) \in \overline{B}(R)$ and consider the function $v(t) := u_a(t) - \bar{u}(t)$. Then for all $t \geq s \geq 0$ we have

$$\begin{aligned}v(t) &= u_a(t) - \bar{u}(t) \\ &= U(t,s)a - U(t,s)\bar{u}(s) + \lim_{\lambda\to+\infty}\int_s^t U(t,\sigma)\lambda L_{\lambda,\sigma} f(\sigma, u_a(\sigma))d\sigma \\ &\qquad - \lim_{\lambda\to+\infty}\int_s^t U(t,\sigma)\lambda L_{\lambda,\sigma} f(\sigma, \bar{u}(\sigma))d\sigma \\ &= U(t,s)x + \lim_{\lambda\to+\infty}\int_s^t U(t,\sigma)\lambda L_{\lambda,\sigma} g(\sigma, v(\sigma))d\sigma \\ &= \Gamma \circ N_g(v)(t).\end{aligned}$$

Therefore, v is the unique mild solution of the problem (5.2) with the initial value $v(s) = x \in \overline{B}(R)$. Thus, from the above lemma, $v \in C_0(\mathbb{R}_+, \overline{B}(R))$ and

$$\begin{aligned} \lim_{t\to+\infty} \|u_a(t) - \bar{u}(t)\| &= \lim_{t\to+\infty} \|v(t)\| \\ &= 0. \end{aligned}$$

This proves that $\bar{u}$ is locally attractive and then the theorem is shown. □

6. Application: Non-autonomous dynamical population equation

Consider the following non-autonomous population equation:

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x) = -\dfrac{\partial}{\partial x}(g(t,x)u(t,x)) - \mu(t,x)u(t,x), \quad t \geq s \geq 0,\ x \geq 0, \\ g(t,0)u(t,0) = \displaystyle\int_0^{+\infty} \beta(t,x,Pu)u(t,x)dx, \quad t \geq s \geq 0, \\ u(s,x) = \varphi(x),\ x \geq 0. \end{cases} \tag{6.1}$$

Where $u(t,x)$ represents the density of population at time t with size x, the functions g, μ and β represent respectively the growth, the mortality and the fertility rates and $Pu := \int_0^{+\infty} u(x)dx$ is the total population.

The problem (6.1) can be reformulated as an abstract boundary Cauchy problem :

$$\begin{cases} \dfrac{d}{dt}u(t) = A_{\max}(t)u(t), t \geq s \geq 0, \\ L(t)u(t) = f(t,u(t)),\ \ t \geq s \geq 0, \\ u(s) = \varphi, \end{cases}$$

where $A_{\max}(t)$ is defined on $X := L^1(\mathbb{R}_+)$ by

$$(A_{\max}(t)\phi)(x) = -\frac{\partial}{\partial x}(g(t,x)\phi(x)) - \mu(t,x)\phi(x), \text{ for all } x \in \mathbb{R}_+,$$

with the domain

$$D := W^{1,1}(\mathbb{R}_+),$$

$L(t) : D \longrightarrow Y := \mathbb{R}$ is defined by

$$L(t)\phi = g(t,0)\phi(0).$$

The function f is defined on $\mathbb{R}_+ \times X$ by

$$f(t,u) = \int_0^{+\infty} \beta(t,x,Pu)u(x)\,dx. \tag{6.2}$$

To get our purpose, we make the following assumptions:

(i) $0 < \mu' < \mu(t,x)$ and $0 < \mu < \mu(t,x) + \dfrac{\partial}{\partial x}g(t,x)$, $\forall t \in \mathbb{R}_+$, $a.e.$, $x \in \mathbb{R}_+$.

(ii) $0 < \nu < g(t,x)$, $\quad \forall t \in \mathbb{R}_+$, $x \in \mathbb{R}_+$.

(iii) $t \longmapsto \mu(t,.) \in C_b^1(\mathbb{R}_+, L^\infty(\mathbb{R}_+))$.

(iv) $t \longmapsto g(t,.) \in C_b^1(\mathbb{R}_+, W^{1,\infty}(\mathbb{R}_+))$.

(v) $\beta(.,x,.) \in C(\mathbb{R}_+ \times X, \mathbb{R})$, for all $x \in \mathbb{R}_+$. Moreover, $\beta(t,x,y) < \beta$, for all $t, y \in \mathbb{R}_+$ and a.e. $x \in \mathbb{R}_+$.

(vi) For each $T > 0$, there exists a constant C_T such that for all $t \in [s,T]$

$$\|\beta(t,.,y_1) - \beta(t,.,y_2)\| \leq C_T\,|y_1 - y_2|\,, \forall y_1, y_2 \in \mathbb{R}_+.$$

Lemma 6.1. *Under the above assumptions, the hypothesis (H1)–(H6) are fulfilled.*

Proof. Verification of (H1): Let $\phi \in D$, we have

$$\begin{aligned}
\|\phi\|_D &= \int_0^{+\infty} |\phi(x)|dx + \int_0^{+\infty} |\phi'(x)|dx \\
&\leq \int_0^{+\infty} |\phi(x)|dx + \int_0^{+\infty} \left| \frac{1}{g(t,x)} g(t,x)\phi'(x) + \frac{\mu(t,x) + \frac{\partial}{\partial x} g(t,x)}{g(t,x)} \phi(x) \right| dx \\
&\qquad + \int_0^{+\infty} \left| \frac{\mu(t,x) + \frac{\partial}{\partial x} g(t,x)}{g(t,x)} \phi(x) \right| dx \\
&= \int_0^{+\infty} |\phi(x)|dx + \int_0^{+\infty} \left| \frac{1}{g(t,x)} (A_{\max}(t)\phi)(x) \right| dx \\
&\qquad + \int_0^{+\infty} \left| \frac{\mu(t,x) + \frac{\partial}{\partial x} g(t,x)}{g(t,x)} \phi(x) \right| dx \\
&\leq \left(1 + \sup_{t \geq s} \frac{\left\| \mu(t,.) + \frac{\partial}{\partial x} g(t,.) \right\|_\infty}{\nu} \right) \|\phi\|_X + \frac{1}{\nu} \|A_{\max}(t)\phi\|_X \\
&\leq \max \left(1 + \sup_{t \geq s} \frac{\left\| \mu(t,.) + \frac{\partial}{\partial x} g(t,.) \right\|_\infty}{\nu}, \frac{1}{\nu} \right) \left(\|\phi\|_X + \|A_{\max}(t)\phi\|_X \right).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\|\phi\|_X + \|A_{\max}(t)\phi\|_X &= \int_0^{+\infty} |\phi(x)|dx \\
&\quad + \int_0^{+\infty} \left| g(t,x)\phi'(x) + \Big(\mu(t,x) + \frac{\partial}{\partial x} g(t,x) \Big) \phi(x) \right| dx \\
&\leq \left(1 + \sup_{t \geq s} \left\| \mu(t,.) + \frac{\partial}{\partial x} g(t,.) \right\|_\infty \right) \int_0^{+\infty} |\phi(x)|\, dx \\
&\quad + \sup_{t \geq s} \|g(t,.)\|_\infty \int_0^{\infty} |\phi'(x)|\, dx \\
&\leq \max \left(1 + \sup_{t \geq s} \left\| \mu(t,.) + \frac{\partial}{\partial x} g(t,.) \right\|_\infty, \sup_{t \geq s} \|g(t,.)\|_\infty \right) \|\phi\|_D .
\end{aligned}$$

Verification of (H2): Follows immediately from the assumptions (iii) and (iv).

Verification of (H3): It can be shown that the resolvent of $A(t) := A_{\max}(t)|_{\ker L(t)}$ is given by

$$R(\lambda, A(t))\phi = \frac{1}{g(t,\cdot)} \int_0^{\cdot} e^{-\int_\tau^{\cdot} \frac{\lambda + \mu(t,\sigma)}{g(t,\sigma)} d\sigma} \phi(\tau)\, d\tau, \ \forall \phi \in X.$$

Then we have, for $\lambda > -\mu$ and $\phi \in X$,

$$\begin{aligned}
\|R(\lambda, A(t))\phi\|_X &= \int_0^{+\infty} |(R(\lambda, A(t))\phi)(x)|\, dx \\
&\leq \int_0^{+\infty} \frac{1}{g(t,x)} \int_0^x e^{-\int_\tau^x \frac{\lambda+\mu(t,\sigma)}{g(t,\sigma)} d\sigma} |\phi(\tau)|\, d\tau dx \\
&= \int_0^{+\infty} |\phi(\tau)| \int_\tau^{+\infty} \frac{1}{g(t,x)} e^{-\int_\tau^x \frac{\lambda+\mu(t,\sigma)}{g(t,\sigma)} d\sigma} dx d\tau \\
&\leq \int_0^{+\infty} |\phi(\tau)| \int_\tau^{+\infty} \frac{1}{g(t,x)} e^{\int_\tau^x \frac{-(\lambda+\mu)+\frac{\partial}{\partial\sigma} g(t,\sigma)}{g(t,\sigma)} d\sigma} dx d\tau \\
&\leq \frac{1}{\lambda+\mu} \int_0^{\infty} |\phi(\tau)|\, d\tau \\
&= \frac{1}{\lambda+\mu} \|\phi\|_X .
\end{aligned}$$

Hence, the family of operators $(A(t))_{t\geq 0}$ is stable with the constants of stability $\omega_0 = -\mu$ and $M = 1$.

Verification of (H4): $L(t)$ is linear bounded. Indeed, for $\phi \in D$ we have $\phi(0) = -\int_0^{+\infty} \frac{\partial}{\partial x}\phi(x)dx$ and

$$\begin{aligned}
\|L(t)\phi\| &= |\phi(0)g(t,0)| \\
&\leq \sup_{t\geq s} \|g(t,.)\|_\infty \|\phi\|_D .
\end{aligned}$$

For $y \in \mathbb{R}$ we take $\phi(\cdot) = \frac{ye^{-\cdot}}{g(t,\cdot)}$ and we get the surjectivity of $L(t)$.

Verification of (H5): Follows from (iv).

Verification of (H6): For $\lambda > \mu'$, let $\phi \in \ker(\lambda - A_{\max}(t))$. We have

$$\lambda\phi(\cdot) + \frac{\partial}{\partial x}(g(t,\cdot)\phi(\cdot)) + \mu(t,\cdot)\phi(\cdot) = 0.$$

Therefore,

$$\begin{aligned}
\|L(t)\phi\| &= |\phi(0)g(t,0)| \\
&= \int_0^{+\infty} \left| \frac{\partial}{\partial x} (\phi(x)g(t,x)) \right| dx \\
&= \int_0^{+\infty} |(\lambda + \mu(t,x))\phi(x)|\, dx \\
&\geq (\lambda + \mu') \|\phi\| .
\end{aligned}$$

We take $\omega = -\mu'$ and $\gamma = 1$. □

Lemma 6.2. *The function f defined in (6.2) is continuous and locally Lipschitzian.*

Proof. Let $u \in L^1(\mathbb{R}_+)$. For all $x \in \mathbb{R}_+$, the mapping $(t,u) \mapsto \beta(t,x,Pu)u(x)$ is continuous and, for all $t, y \in \mathbb{R}_+$, the mapping $x \mapsto \beta(t,x,Pu)u(x)$ takes values in $L^1(\mathbb{R}_+)$. Thus, by application of continuity of integral theorem we infer that f is continuous. Let $T > 0$, for $u, v \in L^1(\mathbb{R}_+)$ and $t \in [s,T]$, we have

$$\begin{aligned}
\|f(t,u) - f(t,v)\| &= \left| \int_0^{+\infty} \beta(t,x,Pu)u(x) - \beta(t,x,Pv)v(x)dx \right| \\
&\leq \int_0^{+\infty} |\beta(t,x,Pu)(u(x) - v(x))|\, dx + \int_0^{\infty} \left| \Big(\beta(t,x,Pu) - \beta(t,x,Pv)\Big) v(x) \right| dx \\
&\leq \beta \|u - v\| + C_T \|u - v\| \|v\| .
\end{aligned}$$

Thus, for a given $R > 0$ and u, v in X such that $u, v \in \overline{B}(R)$ we obtain

$$\|f(t,u) - f(t,v)\| \leq C_{T,R,\beta} \|u - v\|,$$

with $C_{T,R,\beta} = \beta + RC_T$. This ends the proof. □

We are now ready to state our first application result.

Theorem 6.3. *Under the assumptions (i)–(vi), and for every $\varphi \in L^1(\mathbb{R}_+)$ and $s \geq 0$, there exists a constant $T_{\max} \leq +\infty$ such that the population equation (6.1) has a unique mild solution u on $[s, T_{\max}[$. Moreover, if u is bounded, then $T_{\max} = +\infty$.*

Proof. From Lemma 6.1 and Lemma 6.2, the assumptions (H1)–(H6) and (C) are fulfilled. Thus, the proof is a direct consequence of Theorem 3.3. □

As a second application of the abstract result, we state the following theorem.

Theorem 6.4. *Assume that $\beta < m := \min\{1, \mu\}$. If the assumptions (i)–(vi) are satisfied, then for every $\varphi \in L^1(\mathbb{R}_+)$ and $s \geq 0$, there is $T_{\max} \leq +\infty$ such that every mild solution u of (6.1) on $[s, T_{\max}[$ is locally attractive. Moreover, if u is bounded then u is locally attractive on $[s, +\infty[$.*

Proof. By Lemma 6.1 and Lemma 6.2, the assumptions (H1)–(H6) and (C) are fulfilled and the problem (6.1) has a unique mild solution.
By the proof of Lemma 6.1, especially the proof of assumptions (H3) and (H5), we have $\omega_0 = -\mu < 0$, $M = 1$ and $\gamma = 1$.

Let u be the mild solution of (6.1) and let $x, y \in X$. From (vi) we have

$$\|f(t, u(t) + x) - f(t, u(t) + y)\| \leq \beta \|x - y\| + C_T \|x - y\| \|y\|.$$

Let $R > 0$ such that $\|x\| \leq R$ and $\|y\| \leq R$. Then,

$$\begin{aligned} \|f(t, u(t) + x) - f(t, u(t) + y)\| &\leq \beta \|x - y\| + RC_T \|x - y\| \\ &\leq (\beta + RC_T) \|x - y\|. \end{aligned}$$

Denoting by $\alpha := \beta + RC_T$. If we take $R > 0$ such that $R < \dfrac{m - \beta}{C_T}$, then we get the result by applying Theorem 5.3. □

References

1. T. Akrid, L. Maniar, and A. Ouhinou, *Periodic Solutions of Nondensely Nonautonomous Differential Equations with Delay*, Afr. Diaspora J. Math. (N.S.), Volume 15, Number 1 (2013), 25-42.
2. J. Blot, C. Buse and P. Cieutat, *Local attractivity in nonautonomous semilinear evolution equations*, Nonauton. Dyn. Syst., Volume 1, Number 1 (2014), 72-82.
3. S. Boulite, A. Idrissi and L. Maniar, *Controllability of semi-linear boundary problems with nonlocal initial conditions*, Journal of Mathematical Analysis and Applications, Volume 316, Number 1 (2006), 566-578.
4. S. Boulite, L. Maniar and M. Moussi, *Wellposedness and asymptotic behaviour of nonautonomous boundary Cauchy problems*, Forum Mathematicum, Volume 18, Number 4 (2006), 611-638.
5. T.S.Doan, Moussi and S. Siegmund, *Integral manifolds of nonautonomous boundary Cauchy problem*, Journal of Nonlinear Evolution Equations and Applications, Volume 2012, Number 1 (2012), 1-15.
6. G. Greiner, *Perturbing the boundary conditions of a generator*, Houston Journal of Mathematics, Volume 13, Number 2 (1987), 213-229.
7. N.T.Lan, *Non-autonomous operator matrices*, Ph.D. thesis, Tubingen university, 1998.
8. M.Moussi, *Well-posdness and asymptotic behavior of non-autonomous boundary Cauchy problems*, Ph.D. thesis, Faculty of science, Oujda, 2003.
9. M. Moussi, *Pullback attractors of Nonautonomous Boundary Cauchy Problems*, Nonlinear Dynamics and Systems Theory, Volume 14, Number 4 (2014), 383-394.

10. A. Pazy, *Semi-groups of linear operators and applications to partial differential equations*, Applied Mathematical Sciences 44, Springer-Verlag, New York, 1983.

Amine jerroudi,
Department of Informatics,
Faculty of Science Oujda,
Mohamed first University,
Morocco.
E-mail address: aminejerroudi@gmail.com

and

Mohammed Moussi,
Department of Informatics,
Faculty of Science Oujda,
Mohamed first University,
Morocco.
E-mail address: mohmoussi@hotmail.com