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Stochastic Optimal Control for Dynamics of Forward Backward Doubly SDEs of mean-field Type *

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ABSTRACT: In this paper we establish in first the existence of strong optimal solutions of a control problem for dynamics driven by a linear forward-backward doubly stochastic differential equations of mean-field type (MF-FBDSDEs), with random coefficients and non linear functional cost. Moreover, we establish necessary as well as sufficient optimality conditions for this kind of control problem. In the second part of this paper, we establish necessary as well as sufficient optimality conditions for existence of both optimal relaxed control and optimal strict control for dynamics of nonlinear forward-backward doubly SDEs of mean-field type.

Key Words: Mean-field, forward backward doubly SDEs, strict control, relaxed control, existence, optimality conditions, adjoint equations, variational equation.

Contents

1. Introduction

The problem of existence of optimal controls for various control systems is a fundamental problem in stochastic optimal control theory. Also to establish the existence conditions of an optimal control, which labeled necessary and sufficient conditions for optimality, is one of the important subjects which has attracted comprehensive attention in the past years. Stochastic optimal control of mean-field type recently are extensively studied, due to their applications in economics and mathematical finance. In 2009, Buckdahn et al. [6] established the theory of mean-field backward stochastic differential equations which were derived as a limit of some highly dimensional system of FBSDEs, corresponding to a large number of particles. Since that, many authors treated the system of this kind of Mckean-Vlasov type (see [15] and [1]). As it is well-knew that the adjoint equation of a controlled SDEs of mean-field type is a backward-SDEs of mean-field type, the maximum principle for optimal control systems of mean-field type (MF-SDEs, MF-BSDEs and MF-FBSDEs) has becomes popular topic. In this regard, Carmona and Dularue proved in [7] the existence of solution for mean-field FBSDEs systems. One can refer to [[2],[5], [16], [12] and [14]] for more result on the maximum principles for different types of mean-field systems.

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Mathematical mean-field approaches play a crucial role in diverse areas, such as physics, economics, finance and games theory, see Lasry and Lions [15], Dawson [8] and Huang et al. [13]. In the other hand, the existence of optimal relaxed controls and optimal strict controls for systems of mean-field forward backward stochastic differential equations has been proved by Benbrahim and Gherbal [4], where the diffusion is controlled. The existence of relaxed solutions to mean field games with singular controls has been proved by Fu and Horst in [10]. The authors proved approximations of solutions for a particular class of mean field games with singular controls and relaxed controls by solutions for mean field games with purely regular controls, on the space of càdlàg functions equipped with the Skorokhod M1 topology. Wu and Liu [17] proved existence and uniqueness of solutions for systems of backward doubly SDEs driven by Itô-Levy processes of mean field type and they established necessary and sufficient optimality conditions for partial information optimal control problems of BDSDEs driven by Itô-Levy processes of mean field type. See also in Xu [18] an existence and uniqueness result of the solutions to mean field BDSDEs with locally monotone coefficients and globally monotone coefficients is established and gives the probabilistic representation of the solutions for a class of stochastic partial differential equations by virtue mean field BDSDEs.

Recently, Al-Hussein and Gherbal, [3], established the existence and uniqueness of the solutions of multidimensional forward-backward doubly SDEs with random jumps. For systems of forward-backward doubly SDEs of mean field type, Zhu and Shi [19] proved an existence and uniqueness result for measurable solutions by means of a method of continuation. They given also the probabilistic interpretation for the solutions to a class of nonlocal stochastic partial differential equations (SPDEs) combined with algebra equations.

In this work, we consider a control problem for systems governed by the following FBDSDE of mean field type

$$\begin{cases} dy_t^u = b(t, y_t^u, \mathbb{E}[y_t^u], u_t)dt + \sigma(t, y_t^u, \mathbb{E}[y_t^u], u_t)dW_t \\ dY_t^u = -f(t, y_t^u, \mathbb{E}[y_t^u], Y_t^u, \mathbb{E}[Y_t^u], Z_t^u, \mathbb{E}[Z_t^u], u_t)dt \\ \qquad\qquad -g(t, y_t^u, \mathbb{E}[y_t^u], Y_t^u, \mathbb{E}[Y_t^u], Z_t^u, \mathbb{E}[Z_t^u], u_t)\overleftarrow{dB}_t + Z_t^u dW_t \\ y_0^u = x, Y_T^u = h(y_T^u, \mathbb{E}[y_T^u]), \quad t \in [0, T], \end{cases} \tag{1.1}$$

where b, σ, f, g and h are given functions, $(W_s)_{s\geq 0}$ and $(B_s)_{s\geq 0}$ be two mutually independent standard Brownian motions, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, taking their values respectively in $\mathbb{R}^d$ and in $\mathbb{R}^l$, and $u.$ represents a strict control.

The integral with respect to B_s is a backward Itô integral, while the integral with respect to W_s is a standard forward Itô integral.

We consider a functional cost to be minimized, over the set of strict controls, as the following:

$$\mathbb{J}(u.) := \mathbb{E}\big[\alpha\left(y_T^u, \mathbb{E}\left[y_T^u\right]\right) + \beta\left(Y_0^u, \mathbb{E}\left[Y_0^u\right]\right) + \textstyle\int_0^T \ell\big(t, y_t^u, \mathbb{E}\left[y_t^u\right], Y_t^u, \mathbb{E}\left[Y_t^u\right], Z_t^u, \mathbb{E}\left[Z_t^u\right], u_t\big)dt\big], \tag{1.2}$$

where α, β and ℓ are appropriate functions.

The considered system and the cost functional, depend on the state process, and also on the distribution of the state process.

One of our main aims in this paper is to prove existence of strong optimal control (that is adapted to the initial σ-algebra) for systems governed by a linear FBDSDEs of mean-field type. Also we establish necessary as well as sufficient optimality conditions for a strict control problem. In the second part of this paper, we establish necessary as well as sufficient optimality conditions for both relaxed and strict control problems for systems driven by nonlinear mean-field forward-backward doubly stochastic differential equations.

The paper is organized as follows. In Section 2, we present and prove the first main result concerning the existence of strong optimal strict controls for linear MF-FBDSDEs. Section 3, is devoted to derive necessary and sufficient conditions of optimality for this kind of control problem of linear MF-FBDSDEs. In the last section, we establish necessary as well as sufficient optimality conditions for both relaxed and strict control problems governed by systems of nonlinear MF-FBDSDEs.

2. Existence of a strong optimal control for a linear MF-FBDSDEs

In this section, we prove the existence of a strong optimal strict control which is adapted to the initial σ-algebra, under the convexity of the cost functions and the action space U .

2.1. Formulation of the problem and assumptions

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Let $(W_t)_{t\in[0,T]}$ and $(B_t)_{t\in[0,T]}$ be two Brownian motions valued in $\mathbb{R}^d$ and $\mathbb{R}^l$ respectively, defined on this space.

Let $\mathcal{N}$ denote the class of $\mathbb{P}$-null sets of $\mathcal{F}$. For each $t \in [0,T]$, we define $\mathcal{F}_t \triangleq \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B$, where for any process $\{\delta_t\}$, we set $\mathcal{F}_{s,t}^\delta = \sigma(\delta_r - \delta_s; s \le r \le t) \vee \mathcal{N}, \mathcal{F}_t^\delta = \mathcal{F}_{0,t}^\delta$.

Note that the collection $\{\mathcal{F}_t, t \in [0,T]\}$ is neither increasing nor decreasing, then it does not constitute a classical filtration.

Given ξ a square integrable and $\mathcal{F}_T$-measurable process, x a square integrable and $\mathcal{F}_0$-measurable process and for any admissible control u, we consider a control problem governed by the following controlled linear MF-FBDSDE

$$\begin{cases} dy_t^u = b(t, y_t^u, \mathbb{E}[y_t^u], u_t)dt + \sigma(t, y_t^u, \mathbb{E}[y_t^u], u_t)dW_t \\ dY_t^u = -f(t, y_t^u, \mathbb{E}[y_t^u], Y_t^u, \mathbb{E}[Y_t^u], Z_t^u, \mathbb{E}[Z_t^u], u_t)dt \\ \qquad\qquad -g(t, y_t^u, \mathbb{E}[y_t^u], Y_t^u, \mathbb{E}[Y_t^u], Z_t^u, \mathbb{E}[Z_t^u], u_t)\overleftarrow{dB}_t + Z_t^u dW_t, \\ y_0^u = x, Y_T = h(y_T^u, \mathbb{E}[y_T^u]), \end{cases} \tag{2.1}$$

with

$$b(t, y_t^u, \mathbb{E}[y_t^u], u_t) = a_t y_t^u + \widehat{a}_t \mathbb{E}[y_t^u] + b_t u_t,$$

$$\sigma(t, y_t^u, \mathbb{E}[y_t^u], u_t) = c_t . y_t^u + \widehat{c}_t \mathbb{E}[y_t^u] + \widehat{b}_t u_t,$$

$$f(t, y_t^u, \mathbb{E}[y_t^u], Y_t^u, \mathbb{E}[Y_t^u], Z_t^u, \mathbb{E}[Z_t^u], u_t) = d_t y_t^u + \widehat{d}_t \mathbb{E}[y_t^u] + e_t Y_t^u + \widehat{e}_t \mathbb{E}[Y_t^u] + f_t Z_t^u + \widehat{f}_t \mathbb{E}[Z_t^u] + g_t u_t,$$

$$g(t, y_t^u, \mathbb{E}[y_t^u], Y_t^u, \mathbb{E}[Y_t^u], Z_t^u, \mathbb{E}[Z_t^u], u_t) = h_t y_t^u + \widehat{h}_t \mathbb{E}[y_t^u] + k_t Y_t^u + \widehat{k}_t \mathbb{E}[Y_t^u] + m_t Z_t^u + \widehat{m}_t \mathbb{E}[Z_t^u] + \widehat{g}_t u_t,$$

$$h(y_T^u, \mathbb{E}[y_T^u]) = \xi,$$

and a cost functional:

$$\mathbb{J}(u.) := \mathbb{E}\big[\alpha(y_T^u, \mathbb{E}[y_T^u]) + \beta(Y_0^u, \mathbb{E}[Y_0^u]) + \textstyle\int_0^T \ell\big(t, y_t^u, \mathbb{E}[y_t^u], Y_t^u, \mathbb{E}[Y_t^u], Z_t^u, \mathbb{E}[Z_t^u], u_t\big)dt\big], \tag{2.2}$$

where $a., \widehat{a}., b., \widehat{b}., c., \widehat{c}., d., \widehat{d}., e., \widehat{e}., f., \widehat{f}., g., \widehat{g}., h., \widehat{h}., k., \widehat{k}., m.$ and $\widehat{m}.$ are matrix-valued functions of suitable sizes. The solution $(y., Y., Z.)$ takes values in $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m+d}$ and $u.$ is the control variable values in subset U of $\mathbb{R}^k$. α, β, ℓ are a given functions define by

$$\begin{aligned} &\ell : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathbb{R}^{m\times d} \times U \to \mathbb{R}, \\ &\alpha : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \\ &\beta : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}. \end{aligned}$$

Definition 2.1. *An admissible control $u.$ is a square integrable, $\mathcal{F}_t$-measurable process with values in some subset $U \subseteq \mathbb{R}^k$. We denote by $\mathcal{U}_L$ the set of all admissible controls.*

Note that we have an additional constraint that a control must be square-integrable just to ensure the existence of solutions of (2.1) under $u.$. We say that an admissible control $u^*_. \in \mathcal{U}_L$ is an optimal control if

$$\mathbb{J}(u^*_.) = \inf_{v. \in \mathcal{U}_L} \mathbb{J}(v.). \tag{2.3}$$

The following notations are needed

$\mathcal{S}_{\mathcal{F}}^{2}\left(0,T;\mathbb{R}^{m}\right)$: the set of process $\pi.$, $\mathcal{F}_t$-adapted with values in $\mathbb{R}^m$ such that

$$\mathbb{E}\Big[\int_0^T \left|\pi_t\right|^2 dt\Big] < \infty,$$

$\mathcal{M}_{\mathcal{F}}^{2}\left(0,T;\mathbb{R}^{n}\right)$: the set of process $\eta.$, $\mathcal{F}_t$-adapted and $\mathbb{R}^n$-valued continuous processes such that

$$\mathbb{E}\big[\sup_{0\leq t\leq T}\left|\eta_t\right|^2\big] < \infty,$$

$\mathcal{U}_L \triangleq \left\{v. \in \mathcal{S}_{\mathcal{F}}^{2}\left(0,T;\mathbb{R}^{k}\right)/v_t \in U, a.e.t \in [0,T], \mathbb{P}-a.s.\right\}.$

We shall consider in the first part of this paper, the following assumptions

$$(\mathbf{H1}): \quad \text{the set } U \subseteq \mathbb{R}^k \text{ is convex and compact and the functions } \ell, \alpha \text{ and } \beta \\ \text{are continuous, bounded and convex,}$$

$(\mathbf{H2})$: $a_t, \widehat{a}_t, b_t, \widehat{b}_t, c_t, \widehat{c}_t, d_t, \widehat{d}_t, e_t, \widehat{e}_t, f_t, \widehat{f}_t, g_t, \widehat{g}_t, h_t, \widehat{h}_t, k_t$ and $\widehat{k}_t$ are bounded by $\lambda > 0$ and $m_t, \widehat{m}_t$ are bounded by $\gamma \in \left]0, \frac{1}{2}\right[$. That is:

$$\lambda \triangleq \sup_{t,\omega}\left|\varphi_t\left(\omega\right)\right| \text{ and } \gamma \triangleq \sup_{t,\omega}\left|\sigma_t\left(\omega\right)\right|,$$

where $\varphi_t\left(\omega\right) = a_t, \widehat{a}_t, b_t, \widehat{b}_t, c_t, \widehat{c}_t, d_t, \widehat{d}_t, e_t, \widehat{e}_t, f_t, \widehat{f}_t, g_t, \widehat{g}_t, h_t, \widehat{h}_t, k_t, \widehat{k}_t$ and $\sigma_t = m_t, \widehat{m}_t$.

Proposition 2.2. *Under assumptions* $(\mathbf{H2})$ *the system of linear FBDSDE of mean-field type (2.1), has a unique strong solution.*

Proof. The proof of this proposition is established in Zhu and Shi [19], by using a method of continuation, and the fact that our system (2.1) is a special case of the one given in [19]. □

Remark 2.3. *A special case is that in which both α, β and ℓ are convex quadratic functions. The control problem* $\{(2.1), (2.2), (2.3)\}$ *is then reduced to a stochastic linear quadratic optimal control problem.*

2.2. Existence of a strong optimal control

The following theorem confirms the existence of a strong optimal solutions for the control problem $\{(2.1), (2.2), (2.3)\}$.

Theorem 2.4. *Under either* $(\mathbf{H1})-(\mathbf{H2})$, *if the strict control problem* $\{(2.1), (2.2), (2.3)\}$ *is finite, then it admits an optimal strong solution.*

Proof. Assume that $(\mathbf{H1})$-$(\mathbf{H2})$ holds. Let $(u_\cdot^n)$ be a minimizing sequence, i.e.,

$$\lim_{n\to\infty}\mathbb{J}\left(u_\cdot^n\right) = \inf_{v.\in\mathcal{U}_L}\mathbb{J}\left(v.\right).$$

With associated trajectories $\left(y_\cdot^{u_\cdot^n}, Y_\cdot^{u_\cdot^n}, Z_\cdot^{u_\cdot^n}\right)$ satisfies the linear FBDSDE of mean-field type (2.1).

From the fact that U is a compact set, there exists a subsequence (which is still labeled by $(u_\cdot^n)_{n\geq 0}$) such that

$$u_\cdot^n \longrightarrow \overline{u}_\cdot, \text{ weakly in } \mathcal{S}_{\mathcal{F}}^{2}\left([0,T];\mathbb{R}^{k}\right).$$

Applying Mazur's theorem, there is a sequence of convex combinations

$$\widetilde{U}_\cdot^n = \sum_{\jmath\geq 0}\theta_{\jmath n}u_\cdot^{\jmath+n} \quad (\text{with } \theta_{\jmath n}\geq 0, \quad \text{and} \quad \sum_{\jmath\geq 0}\theta_{\jmath n} = 1),$$

such that

$$\widetilde{U}^n_{\cdot} \to \overline{u}_{\cdot} \ \text{ strongly in } \mathcal{S}^2_{\mathcal{F}}\left([0,T];\mathbb{R}^k\right). \tag{2.4}$$

Since the set $U \subseteq \mathbb{R}^k$ is convex and compact, it follows that $\overline{u}_{\cdot} \in \mathcal{U}_L$.

Let $(y^{\widetilde{U}^n_{\cdot}}_{\cdot}, Y^{\widetilde{U}^n_{\cdot}}_{\cdot}, Z^{\widetilde{U}^n_{\cdot}}_{\cdot})$ and $(y^{\overline{u}_{\cdot}}_{\cdot}, Y^{\overline{u}_{\cdot}}_{\cdot}, Z^{\overline{u}_{\cdot}}_{\cdot})$ be the solutions of the linear MF-FBDSDE (2.1), associated with $\widetilde{U}^n_{\cdot}$ and $\overline{u}_{\cdot}$ respectively i.e.,

$$\begin{cases} dy^{\widetilde{U}^n_{\cdot}}_t = \big(a_t y^{\widetilde{U}^n_{\cdot}}_t + \widehat{a}_t \mathbb{E}[y^{\widetilde{U}^n_{\cdot}}_t] + b_t \widetilde{U}^n_t\big)dt + \big(c_t y^{\widetilde{U}^n_{\cdot}}_t + \widehat{c}_t \mathbb{E}[y^{\widetilde{U}^n_{\cdot}}_t] + \widehat{b}_t \widetilde{U}^n_t\big)dW_t \\ dY^{\widetilde{U}^n_{\cdot}}_t = -\big(d_t y^{\widetilde{U}^n_{\cdot}}_t + \widehat{d}_t \mathbb{E}[y^{\widetilde{U}^n_{\cdot}}_t] + e_t Y^{\widetilde{U}^n_{\cdot}}_t + \widehat{e}_t \mathbb{E}[Y^{\widetilde{U}^n_{\cdot}}_t] + f_t Z^{\widetilde{U}^n_{\cdot}}_t + \widehat{f}_t \mathbb{E}[Z^{\widetilde{U}^n_{\cdot}}_t] + g_t \widetilde{U}^n_t\big)dt \\ \qquad -\big(h_t y^{\widetilde{U}^n_{\cdot}}_t + \widehat{h}_t \mathbb{E}[y^{\widetilde{U}^n_{\cdot}}_t] + k_t Y^{\widetilde{U}^n_{\cdot}}_t + \widehat{k}_t \mathbb{E}[Y^{\widetilde{U}^n_{\cdot}}_t] + m_t Z^{\widetilde{U}^n_{\cdot}}_t + \widehat{m}_t \mathbb{E}[Z^{\widetilde{U}^n_{\cdot}}_t] \\ \qquad\qquad + \widehat{g}_t \widetilde{U}^n_t\big)\overleftarrow{dB}_t + Z^{\widetilde{U}^n_{\cdot}}_t dW_t, \\ y^{\widetilde{U}^n_{\cdot}}_0 = x, Y^{\widetilde{U}^n_{\cdot}}_T = \xi, \end{cases} \tag{2.5}$$

and

$$\begin{cases} dy^{\overline{u}_{\cdot}}_t = \big(a_t y^{\overline{u}_{\cdot}}_t + \widehat{a}_t \mathbb{E}[y^{\overline{u}_{\cdot}}_t] + b_t \overline{u}_t\big)dt + \big(c_t y^{\overline{u}_{\cdot}}_t + \widehat{c}_t \mathbb{E}[y^{\overline{u}_{\cdot}}_t] + \widehat{b}_t \overline{u}_t\big)dW_t \\ dY^{\overline{u}_{\cdot}}_t = -\big(d_t y^{\overline{u}_{\cdot}}_t + \widehat{d}_t \mathbb{E}[y^{\overline{u}_{\cdot}}_t] + e_t Y^{\overline{u}_{\cdot}}_t + \widehat{e}_t \mathbb{E}[Y^{\overline{u}_{\cdot}}_t] + f_t Z^{\overline{u}_{\cdot}}_t + \widehat{f}_t \mathbb{E}[Z^{\overline{u}_{\cdot}}_t] + g_t \overline{u}_t\big)dt \\ \qquad -\big(h_t y^{\overline{u}_{\cdot}}_t + \widehat{h}_t \mathbb{E}[y^{\overline{u}_{\cdot}}_t] + k_t Y^{\overline{u}_{\cdot}}_t + \widehat{k}_t \mathbb{E}[Y^{\overline{u}_{\cdot}}_t] + m_t Z^{\overline{u}_{\cdot}}_t + \widehat{m}_t \mathbb{E}[Z^{\overline{u}_{\cdot}}_t] \\ \qquad\qquad + \widehat{g}_t \overline{u}_t\big)\overleftarrow{dB}_t + Z^{\overline{u}_{\cdot}}_t dW_t, \\ y^{\overline{u}_{\cdot}}_0 = x, Y^{\overline{u}_{\cdot}}_T = \xi. \end{cases} \tag{2.6}$$

Then let us prove

$$(y^{\widetilde{U}^n_{\cdot}}_t, Y^{\widetilde{U}^n_{\cdot}}_t, \int_0^T Z^{\widetilde{U}^n_{\cdot}}_s dW_s) \textit{ converges strongly to } (y^{\overline{u}_{\cdot}}_t, Y^{\overline{u}_{\cdot}}_t, \int_0^T Z^{\overline{u}_{\cdot}}_s dW_s), \tag{2.7}$$

in $\mathcal{M}^2_{\mathcal{F}}\left([0,T];\mathbb{R}^{n+m}\right) \times \mathcal{S}^2_{\mathcal{F}}\left([0,T];\mathbb{R}^{m\times d}\right)$.

Firstly, we have

$$\begin{aligned} (\sup_{0\le s\le t} |y^{\widetilde{U}^n_{\cdot}}_s - y^{\overline{u}_{\cdot}}_s|^2) \le \int_0^t &\big(|a_s|^2 (\sup_{0\le r\le s} |y^{\widetilde{U}^n_{\cdot}}_r - y^{\overline{u}_{\cdot}}_r|^2) + |\widehat{a}_s|^2 \mathbb{E}[\sup_{0\le r\le s} |y^{\widetilde{U}^n_{\cdot}}_r - y^{\overline{u}_{\cdot}}_r|^2] \\ &+ |b_s|^2 |\widetilde{U}^n_s - \overline{u}_s|^2\big)ds + \sup_{0\le s\le t} (|\int_0^t \big(c_s (y^{\widetilde{U}^n_{\cdot}}_s - y^{\overline{u}_{\cdot}}_s) \\ &+ \overline{c}_s (\mathbb{E}[y^{\widetilde{U}^n_{\cdot}}_s - y^{\overline{u}_{\cdot}}_s]) + \widehat{b}_s (\widetilde{U}^n_s - \overline{u}_s)\big)dW_s|)^2, \end{aligned}$$

using the Burkholder-Davis-Gundy inequality to the martingale part, we can show

$$\mathbb{E}\big[\sup_{0\le s\le T} |y^{\widetilde{U}^n_{\cdot}}_s - y^{\overline{u}_{\cdot}}_s|^2\big] \le K \int_0^t \mathbb{E}\big[\sup_{0\le r\le s} |y^{\widetilde{U}^n_{\cdot}}_r - y^{\overline{u}_{\cdot}}_r|^2\big]ds + K' \mathbb{E}\big[\int_0^t |\widetilde{U}^n_s - \overline{u}_s|^2 ds\big].$$

Applying Gronwall's lemma and using (2.4), we get

$$\lim_{n\to\infty} \mathbb{E}\big[\sup_{0\le s\le T} |y^{\widetilde{U}^n_{\cdot}}_s - y^{\overline{u}_{\cdot}}_s|^2\big] = 0. \tag{2.8}$$

Secondly, applying Itô's formula to $\left|Y_t^{\widetilde{U}^n_\cdot} - Y_t^{\overline{u}_\cdot}\right|^2$ and taking expectation, we get

$$\begin{aligned}
&\mathbb{E}\big[\sup_{0\le t\le T}|Y_t^{\widetilde{U}^n_\cdot} - Y_t^{\overline{u}_\cdot}|^2\big] + \mathbb{E}\big[\int_0^T \|Z_s^{\widetilde{U}^n_\cdot} - Z_s^{\overline{u}_\cdot}\|^2 ds\big] \le \\
&2\mathbb{E}\big[\int_t^T \langle Y_s^{\widetilde{U}^n_\cdot} - Y_s^{\overline{u}_\cdot}, d_s(y_s^{\widetilde{U}^n_\cdot} - y_s^{\overline{u}_\cdot}) + \widehat{d}_s\mathbb{E}[y_s^{\widetilde{U}^n_\cdot} - y_s^{\overline{u}_\cdot}] + e_s(Y_s^{\widetilde{U}^n_\cdot} - Y_s^{\overline{u}_\cdot}) \\
&+\widehat{e}_s\mathbb{E}[Y_s^{\widetilde{U}^n_\cdot} - Y_s^{\overline{u}_\cdot}] + f_s(Z_s^{\widetilde{U}^n_\cdot} - Z_s^{\overline{u}_\cdot}) + \widehat{f}_s\mathbb{E}[Z_s^{\widetilde{U}^n_\cdot} - Z_s^{\overline{u}_\cdot}] + g_s(\widetilde{U}_s^n - \overline{u}_s)\rangle ds\big] \\
&+\mathbb{E}\big[\int_0^T |h_s(y_s^{\widetilde{U}^n_\cdot} - y_s^{\overline{u}_\cdot}) + \widehat{h}_s\mathbb{E}[y_s^{\widetilde{U}^n_\cdot} - y_s^{\overline{u}_\cdot}] + k_s(Y_s^{\widetilde{U}^n_\cdot} - Y_s^{\overline{u}_\cdot}) \\
&+\widehat{k}_s\mathbb{E}[Y_s^{\widetilde{U}^n_\cdot} - Y_s^{\overline{u}_\cdot}] + m_s(Z_s^{\widetilde{U}^n_\cdot} - Z_s^{\overline{u}_\cdot}) + \widehat{m}_s\mathbb{E}[Z_s^{\widetilde{U}^n_\cdot} - Z_s^{\overline{u}_\cdot}] + \widehat{g}_s(\widetilde{U}_s^n - \overline{u}_s)|^2 ds\big].
\end{aligned}$$

According to the assumption (H2) and by using the Young's formula, we obtain

$$\begin{aligned}
&\mathbb{E}\big[\sup_{0\le t\le T}|Y_t^{\widetilde{U}^n_\cdot} - Y_t^{\overline{u}_\cdot}|^2\big] + \mathbb{E}\big[\int_0^T \|Z_s^{\widetilde{U}^n_\cdot} - Z_s^{\overline{u}_\cdot}\|^2 ds\big] \\
&\le \frac{1}{\rho_1}\mathbb{E}\big[\int_0^T |Y_s^{\widetilde{U}^n_\cdot} - Y_s^{\overline{u}_\cdot}|^2 ds\big] + 14\rho_1\lambda^2\mathbb{E}\big[\int_0^T \big(|y_s^{\widetilde{U}^n_\cdot} - y_s^{\overline{u}_\cdot}|^2 + |Y_s^{\widetilde{U}^n_\cdot} - Y_s^{\overline{u}_\cdot}|^2 \\
&+\|Z_s^{\widetilde{U}^n_\cdot} - Z_s^{\overline{u}_\cdot}\|^2 + \frac{1}{2}|\widetilde{U}_s^n - \overline{u}_s|^2\big) ds\big] + 10\lambda^2\mathbb{E}\big[\int_0^T \big(|y_s^{\widetilde{U}^n_\cdot} - y_s^{\overline{u}_\cdot}|^2 \\
&+|Y_s^{\widetilde{U}^n_\cdot} - Y_s^{\overline{u}_\cdot}|^2 + \frac{1}{2}|\widetilde{U}_s^n - \overline{u}_s|^2\big) ds\big] + 4\gamma^2\mathbb{E}\big[\int_0^T \|Z_s^{\widetilde{U}^n_\cdot} - Z_s^{\overline{u}_\cdot}\|^2 ds\big] \\
&+\frac{5\lambda\gamma}{\rho_2}\mathbb{E}\big[\int_0^T \big(|y_s^{\widetilde{U}^n_\cdot} - y_s^{\overline{u}_\cdot}|^2 + \mathbb{E}[|y_s^{\widetilde{U}^n_\cdot} - y_s^{\overline{u}_\cdot}|^2] + |Y_s^{\widetilde{U}^n_\cdot} - Y_s^{\overline{u}_\cdot}|^2 \\
&+\mathbb{E}[|Y_s^{\widetilde{U}^n_\cdot} - Y_s^{\overline{u}_\cdot}|^2] + |\widetilde{U}_s^n - \overline{u}_s|^2\big) ds\big] \\
&+2\rho_2\lambda\gamma\mathbb{E}\big[\int_0^T \big(\|Z_s^{\widetilde{U}^n_\cdot} - Z_s^{\overline{u}_\cdot}\|^2 + \mathbb{E}[\|Z_s^{\widetilde{U}^n_\cdot} - Z_s^{\overline{u}_\cdot}\|^2]\big) ds\big],
\end{aligned}$$

and therefore

$$\begin{aligned}
&\mathbb{E}\big[\sup_{0\le t\le T}|Y_t^{\widetilde{U}^n_\cdot} - Y_t^{\overline{u}_\cdot}|^2\big] + \mathbb{E}\big[\int_0^T \|Z_s^{\widetilde{U}^n_\cdot} - Z_s^{\overline{u}_\cdot}\|^2 ds\big] \\
&\le (\frac{1}{\rho_1} + 14\rho_1\lambda^2 + 10\lambda^2 + \frac{10\lambda\gamma}{\rho_2})\mathbb{E}\big[\int_0^T |Y_s^{\widetilde{U}^n_\cdot} - Y_s^{\overline{u}_\cdot}|^2 ds\big] \\
&+(14\rho_1\lambda^2 + 4\gamma^2 + 4\rho_2\lambda\gamma)\mathbb{E}\big[\int_0^T \|Z_s^{\widetilde{U}^n_\cdot} - Z_s^{\overline{u}_\cdot}\|^2 ds\big] \\
&+(14\rho_1\lambda^2 + 10\lambda^2 + \frac{10\lambda\gamma}{\rho_2})\mathbb{E}\big[\int_0^T |y_s^{\widetilde{U}^n_\cdot} - y_s^{\overline{u}_\cdot}|^2 ds\big] \\
&+(7\rho_1\lambda^2 + 5\lambda^2 + \frac{5\lambda\gamma}{\rho_2})\mathbb{E}\big[\int_0^T |\widetilde{U}_s^n - \overline{u}_s|^2 ds\big].
\end{aligned}$$

Choosing

$$\rho_1 = \frac{1-4\gamma^2}{28\lambda^2} > 0 \text{ and } \rho_2 = \frac{1-4\gamma^2}{12\lambda\gamma} > 0 \text{ because } 0 < \gamma < \frac{1}{2},$$

the previous inequality becomes

$$\mathbb{E}\big[\sup_{0\leq t\leq T}|Y_t^{\widetilde{U}_\cdot^n}-Y_t^{\overline{u}_\cdot}|^2\big]+\mu_1\mathbb{E}\big[\int_0^T\|Z_s^{\widetilde{U}_\cdot^n}-Z_s^{\overline{u}_\cdot}\|^2ds\big]\leq\mu_2\mathbb{E}\big[\int_0^T|Y_s^{\widetilde{U}_\cdot^n}-Y_s^{\overline{u}_\cdot}|^2ds\big]$$
$$+\mu_3\mathbb{E}\big[\int_0^T|y_s^{\widetilde{U}_\cdot^n}-y_s^{\overline{u}_\cdot}|^2ds\big]+\mu_4\mathbb{E}\big[\int_0^T|\widetilde{U}_s^n-\overline{u}_s|^2ds\big],\tag{2.9}$$

where

$$\mu_1=\frac{1-4\gamma^2}{6}>0,$$
$$\mu_2=\frac{28\lambda^2}{1-4\gamma^2}+\frac{1-4\gamma^2}{2}+10\lambda^2+\frac{120\,(\lambda\gamma)^2}{1-4\gamma^2}>0,$$
$$\mu_3=\frac{1-4\gamma^2}{2}+10\lambda^2+\frac{120\,(\lambda\gamma)^2}{1-4\gamma^2}>0,$$
$$\mu_4=\frac{1-4\gamma^2}{4}+5\lambda^2+\frac{60\,(\lambda\gamma)^2}{1-4\gamma^2}>0.$$

We derive two inequalities from (2.9),

$$\mathbb{E}\big[\sup_{0\leq t\leq T}|Y_t^{\widetilde{U}_\cdot^n}-Y_t^{\overline{u}_\cdot}|^2\big]\leq\mu_2\mathbb{E}\big[\int_0^T|Y_s^{\widetilde{U}_\cdot^n}-Y_s^{\overline{u}_\cdot}|^2ds\big]$$
$$+\mu_3\mathbb{E}\big[\int_0^T|y_s^{\widetilde{U}_\cdot^n}-y_s^{\overline{u}_\cdot}|^2ds\big]+\mu_4\mathbb{E}\big[\int_0^T|\widetilde{U}_s^n-\overline{u}_s|^2ds\big],\tag{2.10}$$

and

$$\mu_1\mathbb{E}\big[\int_0^T\|Z_s^{\widetilde{U}_\cdot^n}-Z_s^{\overline{u}_\cdot}\|^2ds\big]\leq\mu_2\mathbb{E}\big[\int_0^T|Y_s^{\widetilde{U}_\cdot^n}-Y_s^{\overline{u}_\cdot}|^2ds\big]$$
$$+\mu_3\mathbb{E}\big[\int_0^T|y_s^{\widetilde{U}_\cdot^n}-y_s^{\overline{u}_\cdot}|^2ds\big]+\mu_4\mathbb{E}\big[\int_0^T|\widetilde{U}_s^n-\overline{u}_s|^2ds\big].\tag{2.11}$$

Using Burkholder-Davis-Gundy's inequality, applying Gronwall's lemma to (2.10) and passing to the limit as $n\to\infty$, and using the convergence (2.4) and (3.5), we obtain

$$\lim_{n\to\infty}\mathbb{E}\big[\sup_{0\leq s\leq T}|Y_s^{\widetilde{U}_\cdot^n}-Y_s^{\overline{u}_\cdot}|^2\big]=0.\tag{2.12}$$

Then, one can shows directly from (2.4),(2.8) and (2.11) that

$$\mathbb{E}\big[\int_0^T\|Z_s^{\widetilde{U}_\cdot^n}-Z_s^{\overline{u}_\cdot}\|^2ds\big]\longrightarrow 0,\ \text{as } n\to\infty,$$

which gives the result by applying the isometry of Itô.

Finally, let us prove that $\overline{u}_\cdot$ is an optimal control. Using the continuity of functions α,β and ℓ, we get

$$\begin{aligned}\mathbb{J}\left(\overline{u}_\cdot\right)&=\mathbb{E}\big[\alpha\left(y_T^{\overline{u}_\cdot},\mathbb{E}\left[y_T^{\overline{u}_\cdot}\right]\right)+\beta\left(Y_0^{\overline{u}_\cdot},\mathbb{E}\left[Y_0^{\overline{u}_\cdot}\right]\right)\\&\qquad+\textstyle\int_0^T\ell\big(t,y_t^{\overline{u}_\cdot},\mathbb{E}\left[y_t^{\overline{u}_\cdot}\right],Y_t^{\overline{u}_\cdot},\mathbb{E}\left[Y_t^{\overline{u}_\cdot}\right],Z_t^{\overline{u}_\cdot},\mathbb{E}\left[Z_t^{\overline{u}_\cdot}\right],\overline{u}_t\big)dt\big]\\&=\lim_{n\to\infty}\mathbb{E}\big[\alpha\left(y_T^{\widetilde{U}_\cdot^n},\mathbb{E}\left[y_T^{\widetilde{U}_\cdot^n}\right]\right)+\beta\left(Y_0^{\widetilde{U}_\cdot^n},\mathbb{E}\left[Y_0^{\widetilde{U}_\cdot^n}\right]\right)\\&\qquad+\textstyle\int_0^T\ell\big(t,y_t^{\widetilde{U}_\cdot^n},\mathbb{E}\left[y_t^{\widetilde{U}_\cdot^n}\right],Y_t^{\widetilde{U}_\cdot^n},\mathbb{E}\left[Y_t^{\widetilde{U}_\cdot^n}\right],Z_t^{\widetilde{U}_\cdot^n},\mathbb{E}\left[Z_t^{\widetilde{U}_\cdot^n}\right],\widetilde{U}_t^n\big)dt\big].\end{aligned}$$

By the convexity of α, β and ℓ, it follows that

$$\begin{aligned}\mathbb{J}\left(\overline{u}_\cdot\right) \leq &\lim_{n\to\infty}\sum_{\jmath\geq 0}\theta_{\jmath n}\mathbb{E}\big[\alpha\left(y_T^{u_\cdot^{\jmath+n}}, \mathbb{E}\left[y_T^{u_\cdot^{\jmath+n}}\right]\right)+\beta\left(Y_0^{u_\cdot^{\jmath+n}}, \mathbb{E}\left[Y_0^{u_\cdot^{\jmath+n}}\right]\right)\\
&+\int_0^T \ell\big(t, y_t^{u_\cdot^{\jmath+n}}, \mathbb{E}\left[y_t^{u_\cdot^{\jmath+n}}\right], Y_t^{u_\cdot^{\jmath+n}}, \mathbb{E}\left[Y_t^{u_\cdot^{\jmath+n}}\right], Z_t^{u_\cdot^{\jmath+n}}, \mathbb{E}\left[Z_t^{u_\cdot^{\jmath+n}}\right], u_t^{\jmath+n}\big)dt\big]\\
&=\lim_{n\to\infty}\sum_{k\geq 0}\theta_{\jmath n}\mathbb{J}\left(u_\cdot^{\jmath+n}\right)\leq \lim_{n\to\infty}\underset{1\leq \jmath\leq i_n}{Max}\,\mathbb{J}\left(u_\cdot^{\jmath+n}\right)\sum_{\jmath\geq 1}\theta_{\jmath n}=\inf_{v_\cdot\in\mathcal{U}_L}\mathbb{J}\left(v_\cdot\right).\end{aligned}$$

This completes the proof. □

3. Necessary and sufficient conditions for optimality

In this section, we establish necessary as well as sufficient optimality conditions for a strict control problem driven by a linear MF-FBDSDE. In this end, we use the convex perturbation method because the domain of control U is convex.

Let $(\overline{u}_\cdot, y_t^{\overline{u}_\cdot}, Y_t^{\overline{u}_\cdot}, Z_t^{\overline{u}_\cdot})$ be the optimal solution of the control problem $\{(2.1),(2.2),(2.3)\}$ obtained in section 2. Let us define the perturbed control as follow: for each admissible control $v_\cdot$

$$u_t^\varepsilon = \overline{u}_t + \varepsilon\left(v_t - \overline{u}_t\right),$$

where, $\varepsilon > 0$ is sufficiently small.

It's clear that $u_\cdot^\varepsilon$ is admissible control and let $\left(y_t^{u_\cdot^\varepsilon}, Y_t^{u_\cdot^\varepsilon}, Z_t^{u_\cdot^\varepsilon}\right)$ be the solution of (2.1) corresponding to $u_\cdot^\varepsilon$.

The necessary conditions for optimality will be derived by using the optimality of $\overline{u}_\cdot$ and the following inequality,

$$\begin{aligned}0 &\leq \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(\mathbb{J}\left(u_\cdot^\varepsilon\right)-\mathbb{J}\left(\overline{u}_\cdot\right)\right)\\
&=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(\mathbb{J}\left(\overline{u}_\cdot+\varepsilon\left(v_\cdot-\overline{u}_\cdot\right)\right)-\mathbb{J}\left(\overline{u}_\cdot\right)\right)\\
&=\left\langle \mathbb{J}'\left(\overline{u}_\cdot\right), v_\cdot-\overline{u}_\cdot\right\rangle.\end{aligned}$$

Considering in this section the following assumptions

(**H3**) (Regularity conditions)

$$\begin{cases}(i)\ \text{the function } \ell \text{ is continuously differentiable with respect to}\\ \quad (y,y',Y,Y',Z,Z',v), \text{ and the mappings } \alpha \text{ and } \beta \text{ are continuously}\\ \quad\ \text{differentiable with respect to } (y,y') \text{ and } (Y,Y'), \text{ respectively,}\\ (ii)\ \text{the derivatives of } \ell,\alpha,\beta \text{ with respect to their arguments are bounded.}\end{cases}$$

The second main result in this paper, is the following

Theorem 3.1. *(Necessary and sufficient conditions for optimality). Let $\overline{u}_\cdot$ be an admissible control (candidate to be optimal) with associated trajectories $(y_\cdot^{\overline{u}_\cdot}, Y_\cdot^{\overline{u}_\cdot}, Z_\cdot^{\overline{u}_\cdot})$. Then $\overline{u}_\cdot$ is an optimal control for the strict control problem $\{(2.1),(2.2),(2.3)\}$, if and only if, there exists a unique solution $\left(\Phi_\cdot^{\overline{u}_\cdot}, \Psi_\cdot^{\overline{u}_\cdot}, \Sigma_\cdot^{\overline{u}_\cdot}, \Pi_\cdot^{\overline{u}_\cdot}\right)$ of the following adjoint equations of the MF-FBDSDE (2.1),*

$$\begin{cases}-d\Phi_t^{\overline{u}_\cdot} = \left(\mathcal{H}_y\left(t,\zeta_t^{\overline{u}_\cdot},\overline{u}_t,\chi_t^{\overline{u}_\cdot}\right)+\mathbb{E}[\mathcal{H}_{y'}\left(t,\zeta_t^{\overline{u}_\cdot},\overline{u}_t,\chi_t^{\overline{u}_\cdot}\right)]\right)dt-\Sigma_t^{\overline{u}}dW_t,\\[2mm]
d\Psi_t^{\overline{u}_\cdot} = \left(\mathcal{H}_Y\left(t,\zeta_t^{\overline{u}_\cdot},\overline{u}_t,\chi_t^{\overline{u}_\cdot}\right)+\mathbb{E}[\mathcal{H}_{Y'}\left(t,\zeta_t^{\overline{u}_\cdot},\overline{u}_t,\chi_t^{\overline{u}_\cdot}\right)]\right)dt\\
\qquad\qquad +\left(\mathcal{H}_Z\left(t,\zeta_t^{\overline{u}_\cdot},\overline{u}_t,\chi_t^{\overline{u}_\cdot}\right)+\mathbb{E}[\mathcal{H}_{Z'}\left(t,\zeta_t^{\overline{u}_\cdot},\overline{u}_t,\chi_t^{\overline{u}_\cdot}\right)]\right)dW_t-\Pi_t^{\overline{u}}\overleftarrow{dB}_t,\\[2mm]
\Phi_T^{\overline{u}_\cdot} = \alpha_y\left(y_T^{\overline{u}_\cdot},\mathbb{E}[y_T^{\overline{u}_\cdot}]\right)+\mathbb{E}[\alpha_{y'}\left(y_T^{\overline{u}_\cdot},\mathbb{E}[y_T^{\overline{u}_\cdot}]\right)],\\
\Psi_0^{\overline{u}_\cdot} = \beta_Y\left(Y_0^{\overline{u}_\cdot},\mathbb{E}[Y_0^{\overline{u}_\cdot}]\right)+\mathbb{E}[\beta_{Y'}\left(Y_0^{\overline{u}_\cdot},\mathbb{E}[Y_0^{\overline{u}_\cdot}]\right)],\end{cases} \tag{3.1}$$

such that

$$\langle \mathcal{H}_v(t,\zeta_t^{\overline{u}.},\overline{u}_t,\chi_t^{\overline{u}.}),v_t-\overline{u}_t\rangle \geq 0,\ \forall v.\in \mathcal{U}_L,\ a.e,\ as, \tag{3.2}$$

where $\mathcal{H}_\varpi(t,\zeta_t^{\overline{u}.},\overline{u}_t,\chi_t^{\overline{u}.})$ *with* $\varpi := y,y',Y,Y',Z,Z'$, *is the gradient*

$$\nabla_\varpi \mathcal{H}(t,y_t^{\overline{u}.},\mathbb{E}[y_t^{\overline{u}.}],Y_t^{\overline{u}.},\mathbb{E}[Y_t^{\overline{u}.}],Z_t^{\overline{u}.},\mathbb{E}[Z_t^{\overline{u}.}],\overline{u}_t,\Phi_t^{\overline{u}.},\Psi_t^{\overline{u}.},\Sigma_t^{\overline{u}.},\Pi_t^{\overline{u}.}),$$

$$(t,\zeta_t^{\overline{u}.},\overline{u}_t,\chi_t^{\overline{u}.}) := (t,y_t^{\overline{u}.},\mathbb{E}[y_t^{\overline{u}.}],Y_t^{\overline{u}.},\mathbb{E}[Y_t^{\overline{u}.}],Z_t^{\overline{u}.},\mathbb{E}[Z_t^{\overline{u}.}],\overline{u}_t,\Phi_t^{\overline{u}.},\Psi_t^{\overline{u}.},\Sigma_t^{\overline{u}.},\Pi_t^{\overline{u}.}),$$

and the Hamiltonian function is given by

$$\begin{aligned}
&\mathcal{H}\left(t,y,y',Y,Y',Z,Z',v,\Phi,\Psi,\Sigma,\Pi\right) = \left\langle \Psi, dy+\widehat{d}y'+eY+\widehat{e}Y'+fZ+\widehat{f}Z'+gv\right\rangle \\
&\quad +\langle \Phi, ay+\widehat{a}y'+bv\rangle + \left\langle \Pi, hy+\widehat{h}y'+kY+\widehat{k}Y'+mZ+\widehat{m}Z'+\widehat{g}v\right\rangle \\
&\quad +\left\langle \Sigma, cy+\widehat{c}y'+\widehat{b}v\right\rangle + \ell\left(t,y,y',Y,Y',Z,Z',v\right).
\end{aligned}$$

Proof. Our control problem is governed by a linear system, so to establish a necessary and sufficient optimality conditions, we use the following principle: *The convex optimization principle* (see Ekeland-Temam ([9], prop 2.1, p 35). Since the domain of control U is convex, the functional $\mathbb{J}$ is convex in $\overline{u}.$, continuous and Gâteaux-differentiable with continuous derivative $\mathbb{J}'$, thus, we have

$$(\overline{u}.\text{ minimize } \mathbb{J}) \Leftrightarrow \langle \mathbb{J}'\left(\overline{u}.\right), v.-\overline{u}.\rangle \geq 0;\forall v.\in \mathcal{U}_L. \tag{3.3}$$

Firstly, let us calculate the Gâteaux derivative of $\mathbb{J}$ at a point $\overline{u}.$ and in the direction $(v.-\overline{u}.)$, we obtain

$$\begin{aligned}
\langle \mathbb{J}'\left(\overline{u}.\right), v.-\overline{u}.\rangle &= \mathbb{E}\big[\langle \alpha_y(y_T^{\overline{u}.},\mathbb{E}[y_T^{\overline{u}.}])+\mathbb{E}[\alpha_{y'}(y_T^{\overline{u}.},\mathbb{E}[y_T^{\overline{u}.}])], y_T^{v.}-y_T^{\overline{u}.}\rangle\big] \\
&+\mathbb{E}\big[\langle \beta_Y(Y_0^{\overline{u}.},\mathbb{E}[Y_0^{\overline{u}.}])+\mathbb{E}[\beta_{Y'}(Y_0^{\overline{u}.},\mathbb{E}[Y_0^{\overline{u}.}])], Y_0^{v.}-Y_0^{\overline{u}.}\rangle\big] \\
&+\mathbb{E}\big[\int_0^T \langle \ell_y(t,\zeta_t^{\overline{u}.},\overline{u}_t)+\mathbb{E}[\ell_{y'}(t,\zeta_t^{\overline{u}.},\overline{u}_t)], y_t^{v.}-y_t^{\overline{u}.}\rangle dt\big] \\
&+\mathbb{E}\big[\int_0^T \langle \ell_Y(t,\zeta_t^{\overline{u}.},\overline{u}_t)+\mathbb{E}[\ell_{Y'}(t,\zeta_t^{\overline{u}.},\overline{u}_t)], Y_t^{v.}-Y_t^{\overline{u}.}\rangle dt\big] \\
&+\mathbb{E}\big[\int_0^T \langle \ell_Z(t,\zeta_t^{\overline{u}.},\overline{u}_t)+\mathbb{E}[\ell_{Z'}(t,\zeta_t^{\overline{u}.},\overline{u}_t)], Z_t^{v.}-Z_t^{\overline{u}.}\rangle dt\big] \\
&+\mathbb{E}\big[\int_0^T \langle \ell_v(t,\zeta_t^{\overline{u}.},\overline{u}_t), v_t-\overline{u}_t\rangle dt\big].
\end{aligned} \tag{3.4}$$

The adjoint equations (3.1) can be rewritten as follows

$$\begin{cases}
-d\Phi_t^{\overline{u}} = \big(\Psi_t^{\overline{u}}d_t+\Phi_t^{\overline{u}}a_t+\Pi_t^{\overline{u}}h_t+\Sigma_t^{\overline{u}}c_t+\ell_y(t,\zeta_t^{\overline{u}},\overline{u}_t) \\
\qquad +\mathbb{E}[\Psi_t^{\overline{u}}\widehat{d}_t+\Phi_t^{\overline{u}}\widehat{a}_t+\Pi_t^{\overline{u}}\widehat{h}_t+\Sigma_t^{\overline{u}}\widehat{c}_t+\ell_{y'}(t,\zeta_t^{\overline{u}},\overline{u}_t)]\big)dt-\Sigma_t^{\overline{u}}dW_t, \\
\\
d\Psi_t^{\overline{u}} = \big(\Psi_t^{\overline{u}}e_t+\Pi_t^{\overline{u}}k_t+\ell_Y(t,\zeta_t^{\overline{u}},\overline{u}_t)+\mathbb{E}[\big(\Psi_t^{\overline{u}}\widehat{e}_t+\Pi_t^{\overline{u}}\widehat{k}_t+\ell_{Y'}(t,\zeta_t^{\overline{u}},\overline{u}_t)]\big)dt \\
\qquad +\big(\Psi_t^{\overline{u}}f_t+\Pi_t^{\overline{u}}m_t+\ell_Z(t,\zeta_t^{\overline{u}},\overline{u}_t)+\mathbb{E}[\big(\Psi_t^{\overline{u}}\widehat{f}_t+\Pi_t^{\overline{u}}\widehat{m}_t+\ell_{Z'}(t,\zeta_t^{\overline{u}},\overline{u}_t)]\big)dW_t, \\
\qquad -\Pi_t^{\overline{u}}\overleftarrow{dB}_t \\
\\
\Phi_T^{\overline{u}} = \alpha_y\left(y_T^{\overline{u}},\mathbb{E}[y_T^{\overline{u}}]\right)+\mathbb{E}[\alpha_{y'}\left(y_T^{\overline{u}},\mathbb{E}[y_T^{\overline{u}}]\right)], \\
\Psi_0^{\overline{u}} = \beta_Y\left(Y_0^{\overline{u}},\mathbb{E}[Y_0^{\overline{u}}]\right)+\mathbb{E}[\beta_{Y'}\left(Y_0^{\overline{u}},\mathbb{E}[Y_0^{\overline{u}}]\right)].
\end{cases}$$

From (3.1), the equality (3.4) becomes

$$\begin{aligned}\langle \mathbb{J}'(\overline{u}.), v. - \overline{u}.\rangle &= \mathbb{E}\big[\langle \Phi_T^{\overline{u}}, y_T^{v.} - y_T^{\overline{u}.}\rangle\big] + \mathbb{E}\big[\langle \Psi_0^{\overline{u}}, Y_0^{v.} - Y_0^{\overline{u}.}\rangle\big] \\ &+\mathbb{E}\big[\int_0^T \langle \ell_y(t,\zeta_t^{\overline{u}.},\overline{u}_t) + \mathbb{E}[\ell_{y'}(t,\zeta_t^{\overline{u}.},\overline{u}_t)], y_t^{v.} - y_t^{\overline{u}.}\rangle dt\big] \\ &+\mathbb{E}\big[\int_0^T \langle \ell_Y(t,\zeta_t^{\overline{u}.},\overline{u}_t) + \mathbb{E}[\ell_{Y'}(t,\zeta_t^{\overline{u}.},\overline{u}_t)], Y_t^{v.} - Y_t^{\overline{u}.}\rangle dt\big] \\ &+\mathbb{E}\big[\int_0^T \langle \ell_Z(t,\zeta_t^{\overline{u}.},\overline{u}_t) + \mathbb{E}[\ell_{Z'}(t,\zeta_t^{\overline{u}.},\overline{u}_t)], Z_t^{v.} - Z_t^{\overline{u}.}\rangle dt\big] \\ &+\mathbb{E}\big[\int_0^T \langle \ell_v(t,\zeta_t^{\overline{u}.},\overline{u}_t), v_t - \overline{u}_t\rangle dt\big].\end{aligned} \tag{3.5}$$

Applying integration by part to $\langle \Psi_t^{\overline{u}}, Y_t^{v.} - Y_t^{\overline{u}.}\rangle$ and $\langle \Phi_t^{\overline{u}}, y_t^{v.} - y_t^{\overline{u}.}\rangle$, passing to integral on $[0,T]$ and taking the expectations to deduce

$$\begin{aligned}\mathbb{E}\big[\langle \Phi_T^{\overline{u}.}, y_T^{v.} - y_T^{\overline{u}.}\rangle\big] &= -\mathbb{E}\big[\int_0^T \langle \Psi_t^{\overline{u}.} d_t + \Phi_t^{\overline{u}.} a_t + \Pi_t^{\overline{u}.} h_t + \Sigma_t^{\overline{u}.} c_t + \ell_y(t,\zeta_t^{\overline{u}.},\overline{u}_t) \\ &+\mathbb{E}[\Psi_t^{\overline{u}.}\widehat{d}_t + \Phi_t^{\overline{u}.}\widehat{a}_t + \Pi_t^{\overline{u}.}\widehat{h}_t + \Sigma_t^{\overline{u}.}\widehat{c}_t + \ell_{y'}(t,\zeta_t^{\overline{u}.},\overline{u}_t)], y_t^{v.} - y_t^{\overline{u}.}\rangle dt\big] \\ &+\mathbb{E}\big[\int_0^T \langle \Phi_t^{\overline{u}.}, a_t(y_t^{v.} - y_t^{\overline{u}.}) + \widehat{a}_t\mathbb{E}[y_t^{v.} - y_t^{\overline{u}.}] + b_t(v_t - \overline{u}_t)\rangle dt\big] \\ &+\mathbb{E}\big[\int_0^T \langle \Sigma_t^{\overline{u}.}, c_t(y_t^{v.} - y_t^{\overline{u}.}) + \widehat{c}_t\mathbb{E}[y_t^{v.} - y_t^{\overline{u}.}] + \widehat{b}_t(v_t - \overline{u}_t)\rangle dt\big],\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}\mathbb{E}\big[\langle \Psi_0^{\overline{u}.}, Y_0^{v.} - Y_0^{\overline{u}.}\rangle\big] &= -\mathbb{E}\big[\int_0^T \langle \Psi_t^{\overline{u}.} e_t + \Pi_t^{\overline{u}.} k_t + \ell_Y(t,\zeta_t^{\overline{u}.},\overline{u}_t) \\ &+\mathbb{E}[\Psi_t^{\overline{u}.}\widehat{e}_t + \Pi_t^{\overline{u}.}\widehat{k}_t + \ell_{Y'}(t,\zeta_t^{\overline{u}.},\overline{u}_t)], Y_t^{v.} - Y_t^{\overline{u}.}\rangle dt\big] \\ &+\mathbb{E}\big[\int_0^T \langle \Psi_t^{\overline{u}.}, d_t(y_t^{v.} - y_t^{\overline{u}.}) + \widehat{d}_t\mathbb{E}[y_t^{v.} - y_t^{\overline{u}.}] + e_t(Y_t^{v.} - Y_t^{\overline{u}.}) \\ &+\widehat{e}_t\mathbb{E}[Y_t^{v.} - Y_t^{\overline{u}.}] + f_t(Z_t^{v.} - Z_t^{\overline{u}.}) + \widehat{f}_t\mathbb{E}[Z_t^{v.} - Z_t^{\overline{u}.}] + g_t(v_t - \overline{u}_t)\rangle dt\big] \\ &+\mathbb{E}\big[\int_0^T \langle \Pi_t^{\overline{u}.}, h_t(y_t^{v.} - y_t^{\overline{u}.}) + \widehat{h}_t\mathbb{E}[y_t^{v.} - y_t^{\overline{u}.}] + k_t(Y_t^{v.} - Y_t^{\overline{u}.}) \\ &+\widehat{k}_t\mathbb{E}[Y_t^{v.} - Y_t^{\overline{u}.}] + m_t(Z_t^{v.} - Z_t^{\overline{u}.}) + \widehat{m}_t\mathbb{E}[Z_t^{v.} - Z_t^{\overline{u}.}] + \widehat{g}_t(v_t - \overline{u}_t)\rangle dt\big] \\ &-\mathbb{E}\big[\int_0^T \langle \Psi_t^{\overline{u}.} f_t + \Pi_t^{\overline{u}.} m_t + \ell_Z(t,\zeta_t^{\overline{u}.},\overline{u}_t) \\ &+\mathbb{E}[\Psi_t^{\overline{u}.}\widehat{f}_t + \Pi_t^{\overline{u}.}\widehat{m}_t + \ell_{Z'}(t,\zeta_t^{\overline{u}.},\overline{u}_t)], Z_t^{v.} - Z_t^{\overline{u}.}\rangle dt\big].\end{aligned} \tag{3.7}$$

Combining (3.5), (3.6) and (3.7), we obtain

$$\langle \mathbb{J}'(\overline{u}), v. - \overline{u}.\rangle = \mathbb{E}\big[\int_0^T \langle \Phi_t^{\overline{u}.} b_t + \Sigma_t^{\overline{u}.}\widehat{b}_t + \Psi_t^{\overline{u}.} g_t + \Pi_t^{\overline{u}.}\widehat{g}_t + \ell_v(t,\zeta_t^{\overline{u}.},\overline{u}_t), v_t - \overline{u}_t)\rangle dt\big].$$

On the other hand, we calculate the Gâteaux derivative of $\mathcal{H}$ at a point $\overline{u}.$ in the direction $(v. - \overline{u}.)$, we have

$$\begin{aligned}\mathbb{E}\big[\int_0^T \langle \mathcal{H}_v(t,\zeta_t^{\overline{u}.},\overline{u}_t,\chi_t^{\overline{u}.}), v_t - \overline{u}_t\rangle dt\big] &= \mathbb{E}\big[\int_0^T \langle \Phi_t^{\overline{u}.} b_t + \Sigma_t^{\overline{u}.}\widehat{b}_t + \Psi_t^{\overline{u}.} g_t + \Pi_t^{\overline{u}.}\widehat{g}_t \\ &+\ell_v(t,\zeta_t^{\overline{u}.},\overline{u}_t), v_t - \overline{u}_t)\rangle dt\big] \\ &= \langle \mathbb{J}'(\overline{u}), v. - \overline{u}.\rangle.\end{aligned} \tag{3.8}$$

Combines (3.3) and (3.8), we get

$$(\overline{u} \text{ minimize } \mathbb{J}) \Leftrightarrow \mathbb{E}\Big[\int_0^T \langle \mathcal{H}_v(t, \zeta_t^{\overline{u}.}, \overline{u}_t, \chi_t^{\overline{u}.}), v_t - \overline{u}_t \rangle dt\Big] \geq 0, \forall v. \in \mathcal{U}_L.$$

By a standard argument we get the result. □

4. Necessary and sufficient optimality conditions for both relaxed and strict control problems for nonlinear MF-FBDSDE

In this section, we establish necessary as well as sufficient optimality conditions for both relaxed and strict control problems driven by systems of nonlinear MF-FBDSDEs, where the action space U is not necessary convex.

4.1. Necessary and sufficient optimality conditions for relaxed control problems

We start by establish necessary and sufficient optimality conditions for existence of optimal relaxed control. Let $P(U)$ denote the space of probability measures on $\mathcal{B}(U)$ equipped with the topology of weak convergence, where U is a nonempty Borel compact subset of $\mathbb{R}^k$. In a relaxed control problem, the U-valued process v_t is replaced by an $P(U)$-valued process q_t. Moreover, if $q_t(du) = \delta_{v_t}(du)$ is a Dirac measure charging v_t for each t, then we get a strict control problem as a special case of the relaxed one.

We consider a relaxed control problem governed by the following MF-FBDSDE:

$$\begin{cases} dy_t^{\mu} = \int_U b(t, y_t^{\mu}, \mathbb{E}[y_t^{\mu}], u)\mu_t(du)dt + \sigma(t, y_t^{\mu}, \mathbb{E}[y_t^{\mu}])dW_t \\ dY_t^{\mu} = -\int_U f(t, y_t^{\mu}, \mathbb{E}[y_t^{\mu}], Y_t^{\mu}, \mathbb{E}[Y_t^{\mu}], Z_t^{\mu}, \mathbb{E}[Z_t^{\mu}], u)\mu_t(du)dt \\ \qquad\qquad -g(t, y_t^{\mu}, \mathbb{E}[y_t^{\mu}], Y_t^{\mu}, \mathbb{E}[Y_t^{\mu}], Z_t^{\mu}, \mathbb{E}[Z_t^{\mu}])\overleftarrow{dB}_t + Z_t^{\mu}dW_t \\ y_0^{\mu} = x, Y_T^{\mu} = h(y_T^{\mu}, \mathbb{E}[y_T^{\mu}]), \quad t \in [0,T], \end{cases} \tag{4.1}$$

and the functional cost is given by

$$\begin{aligned} \mathbb{J}(\mu.) := \mathbb{E}\big[&\alpha\left(y_T^{\mu}, \mathbb{E}\left[y_T^{\mu}\right]\right) + \beta\left(Y_0^{\mu}, \mathbb{E}\left[Y_0^{\mu}\right]\right) \\ &+ \int_0^T \int_U \ell\big(t, y_t^{\mu}, \mathbb{E}\left[y_t^{\mu}\right], Y_t^{\mu}, \mathbb{E}\left[Y_t^{\mu}\right], Z_t^{\mu}, \mathbb{E}\left[Z_t^{\mu}\right], u\big)\mu_t(du)dt\big]. \end{aligned} \tag{4.2}$$

We say that a relaxed control $q.$ is an optimal control if

$$J(q.) = \inf_{\mu. \in \mathcal{R}} J(\mu.). \tag{4.3}$$

According to the fact that the set of relaxed controls is convex, then to establish necessary optimality condition we use the convex perturbation method. Let $q.$ be an optimal relaxed control with associated trajectories (y_t^q, Y_t^q, Z_t^q) solution of the MF-FBDSDEs (4.1). Then, we can define a perturbed relaxed control by

$$q_t^{\varepsilon} = q_t + \varepsilon(\mu_t - q_t),$$

where $\varepsilon > 0$ is sufficiently small and $\mu.$ is an arbitrary element of $\mathcal{R}$. Denote by $(y_t^{\varepsilon}, Y_t^{\varepsilon}, Z_t^{\varepsilon})$ the solution of the system (4.1) corresponding to $q_\cdot^{\varepsilon}$.

We shall consider in this section the following assumptions.

- (H4) (Lipschitz condition) $\exists\, C > 0, 0 < \gamma < \frac{1}{2}$ such that $\forall\, y_1, y_1', y_2, y_2', Y_1, Y_1', Y_2, Y_2', Z_1, Z_1', Z_2,$

Z_2', u,

$$|b(t,y_1,y_1',u)-b(t,y_2,y_2',u)|^2 \leq C\,(|y_1-y_2|^2+|y_1'-y_2'|^2),$$

$$|\sigma(t,y_1,y_1')-\sigma(t,y_2,y_2')|^2 \leq C\,(|y_1-y_2|^2+|y_1'-y_2'|^2),$$

$$\begin{aligned}&|f(t,y_1,y_1',Y_1,Y_1',Z_1,Z_1',u)-f(t,y_2,y_2',Y_2,Y_2',Z_2,Z_2',u)|^2\\ &\leq C\,(|y_1-y_2|^2+|y_1'-y_2'|^2+|Y_1-Y_2|^2+|Y_1'-Y_2'|^2\\ &\qquad\qquad +\|Z_1-Z_2\|^2+\|Z_1'-Z_2'\|^2),\end{aligned}$$

$$\begin{aligned}&|\ell(t,y_1,y_1',Y_1,Y_1',Z_1,Z_1',u)-\ell(t,y_2,y_2',Y_2,Y_2',Z_2,Z_2',u)|^2\\ &\leq C\,(|y_1-y_2|^2+|y_1'-y_2'|^2+|Y_1-Y_2|^2+|Y_1'-Y_2'|^2\\ &\qquad\qquad +\|Z_1-Z_2\|^2+\|Z_1'-Z_2'\|^2),\end{aligned}$$

$$\begin{aligned}&|g(t,y_1,y_1',Y_1,Y_1',Z_1,Z_1')-g(t,y_2,y_2',Y_2,Y_2',Z_2,Z_2')|^2\\ &\leq C\,(|y_1-y_2|^2+|y_1'-y_2'|^2+|Y_1-Y_2|^2+|Y_1'-Y_2'|^2)\\ &\qquad\qquad +\gamma(\|Z_1-Z_2\|^2+\|Z_1'-Z_2'\|^2).\end{aligned}$$

- (H5) (Regularity conditions)

$$\begin{cases}(i)\text{ the mappings } b,h,\sigma,\alpha \text{ are bounded and continuously differentiable with}\\ \quad\text{respect to } (x,x'),\text{ and the functions } f,g \text{ and } \beta \text{ are bounded and continuously}\\ \quad\text{differentiable with respect to } (y,y',Y,Y',Z,Z') \text{ and } (y,y'),\text{ respectively,}\\ (ii)\text{ the derivatives of } b,h,g,\sigma,f \text{ with respect to the above arguments are}\\ \quad\text{continuous and bounded,}\\ (iii)\text{ the derivatives of } \ell \text{ are bounded by } C(1+|y|+|y'|+|Y|+|Y'|+|Z|+|Z'|),\\ (iv)\text{ the derivatives of } \alpha \text{ and } \beta \text{ are bounded by } C\,(1+|y|+|y'|) \text{ and}\\ \quad C\,(1+|Y|+|Y'|)\text{ respectively,}\end{cases}$$

for some positive constant C.

4.1.1. The variational inequality. Using the optimality of $q.$, the variational inequality will be derived from the following inequality

$$0 \leq J(q^\varepsilon)-J(q).$$

For this end, we need some results.

Proposition 4.1. *Under assumptions* $(\mathbf{H4})-(\mathbf{H5})$, *we have*

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\sup_{0\leq t\leq T}|y_t^\varepsilon-y_t^q|^2\right]=0, \tag{4.4}$$

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\sup_{0\leq t\leq T}|Y_t^\varepsilon-Y_t^q|^2\right]=0, \tag{4.5}$$

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\int_0^T\|Z_t^\varepsilon-Z_t^q\|^2dt\right]=0. \tag{4.6}$$

Proof. We calculate $\mathbb{E}\left[|y_t^\varepsilon - y_t^q|^2\right]$ and using the definition of q_t^ε to get

$$\begin{aligned}\mathbb{E}\left[|y_t^\varepsilon - y_t^q|^2\right] \leq C\mathbb{E}\Big[\int_0^t \Big|\int_U b\left(s, y_s^\varepsilon, \mathbb{E}\left[y_s^\varepsilon\right], u\right) q_s(du) \\ - \int_U b\left(s, y_s^q, \mathbb{E}\left[y_s^q\right], u\right) q_s(du)\Big|^2 ds\Big] \\ + C\varepsilon^2 \mathbb{E}\Big[\int_0^t \Big|\int_U b\left(s, y_s^\varepsilon, \mathbb{E}\left[y_s^\varepsilon\right], u\right) \mu_s(du) - \int_U b\left(s, y_s^\varepsilon, \mathbb{E}\left[y_s^\varepsilon\right], u\right) q_s(du)\Big|^2 ds\Big] \\ + C\mathbb{E}\Big[\int_0^t \left|\sigma\left(s, y_s^\varepsilon, \mathbb{E}\left[y_s^\varepsilon\right]\right) - \sigma\left(s, y_s^q, \mathbb{E}\left[y_s^q\right]\right)\right|^2 ds\Big].\end{aligned}$$

Since b and σ are uniformly Lipschitz and b is bounded, we can show

$$\mathbb{E}\left[|y_t^\varepsilon - y_t^q|^2\right] \leq C\mathbb{E}\Big[\int_0^t |y_s^\varepsilon - y_s^q|^2 ds\Big] + C\varepsilon^2.$$

Applying Granwall's lemma and Burkholder-Davis-Gundy inequality, we get (4.4).

On the other hand, applying Itô's formula to $(Y_t^\varepsilon - Y_t^q)^2$, taking expectation and applying Young's inequality, to obtain

$$\begin{aligned}\mathbb{E}\left[|Y_t^\varepsilon - Y_t^q|^2\right] + \mathbb{E}\left[\int_t^T \|Z_s^\varepsilon - Z_s^q\|^2 ds\right] \leq \mathbb{E}\left[|h(y_T^\varepsilon, \mathbb{E}[y_T^\varepsilon]) - h(y_T^q, \mathbb{E}[y_T^q])|^2\right] \\ + \frac{1}{\theta}\mathbb{E}\left[\int_t^T |Y_s^\varepsilon - Y_s^q|^2 ds\right] + \theta\mathbb{E}\Big[\int_t^T \Big|\int_U f\left(s, y_s^\varepsilon, \mathbb{E}\left[y_s^\varepsilon\right], Y_s^\varepsilon, \mathbb{E}\left[Y_s^\varepsilon\right], Z_s^\varepsilon, \mathbb{E}\left[Z_s^\varepsilon\right], u\right) q_s^\varepsilon(du) \\ - \int_U f\left(s, y_s^q, \mathbb{E}\left[y_s^q\right], Y_s^q, \mathbb{E}\left[Y_s^q\right], Z_s^q, \mathbb{E}\left[Z_s^q\right], u\right) q_s(du)\Big|^2 ds\Big] \\ + \mathbb{E}\Big[\int_t^T \big|g\left(s, y_s^\varepsilon, \mathbb{E}\left[y_s^\varepsilon\right], Y_s^\varepsilon, \mathbb{E}\left[Y_s^\varepsilon\right], Z_s^\varepsilon, \mathbb{E}\left[Z_s^\varepsilon\right]\right) \\ - g\left(s, y_s^q, \mathbb{E}\left[y_s^q\right], Y_s^q, \mathbb{E}\left[Y_s^q\right], Z_s^q, \mathbb{E}\left[Z_s^q\right]\right)\big|^2 ds\Big].\end{aligned}$$

Using the definition of q_t^ε, we obtain

$$\begin{aligned}\mathbb{E}\left[|Y_t^\varepsilon - Y_t^q|^2\right] + \mathbb{E}\left[\int_t^T \|Z_s^\varepsilon - Z_s^q\|^2 ds\right] \leq \mathbb{E}\left[|h(y_T^\varepsilon, \mathbb{E}[y_T^\varepsilon]) - h(y_T^q, \mathbb{E}[y_T^q])|^2\right] \\ + \frac{1}{\theta}\mathbb{E}\left[\int_t^T |Y_s^\varepsilon - Y_s^q|^2 ds\right] \\ + C\theta\varepsilon^2\mathbb{E}\Big[\int_t^T \Big|\int_U f\left(s, y_s^\varepsilon, \mathbb{E}\left[y_s^\varepsilon\right], Y_s^\varepsilon, \mathbb{E}\left[Y_s^\varepsilon\right], Z_s^\varepsilon, \mathbb{E}\left[Z_s^\varepsilon\right], u\right) \mu_s(du) \\ - \int_U f\left(s, y_s^\varepsilon, \mathbb{E}\left[y_s^\varepsilon\right], Y_s^\varepsilon, \mathbb{E}\left[Y_s^\varepsilon\right], Z_s^\varepsilon, \mathbb{E}\left[Z_s^\varepsilon\right], u\right) q_s(du)\Big|^2 ds\Big] \\ + C\theta\mathbb{E}\Big[\int_t^T \Big|\int_U f\left(s, y_s^\varepsilon, \mathbb{E}\left[y_s^\varepsilon\right], Y_s^\varepsilon, \mathbb{E}\left[Y_s^\varepsilon\right], Z_s^\varepsilon, \mathbb{E}\left[Z_s^\varepsilon\right], u\right) q_s(du) \\ - \int_U f\left(s, y_s^q, \mathbb{E}\left[y_s^q\right], Y_s^q, \mathbb{E}\left[Y_s^q\right], Z_s^q, \mathbb{E}\left[Z_s^q\right], u\right) q_s(du)\Big|^2 ds\Big] \\ + \mathbb{E}\Big[\int_t^T \big|g\left(s, y_s^\varepsilon, \mathbb{E}\left[y_s^\varepsilon\right], Y_s^\varepsilon, \mathbb{E}\left[Y_s^\varepsilon\right], Z_s^\varepsilon, \mathbb{E}\left[Z_s^\varepsilon\right]\right) \\ - g\left(s, y_s^q, \mathbb{E}\left[y_s^q\right], Y_s^q, \mathbb{E}\left[Y_s^q\right], Z_s^q, \mathbb{E}\left[Z_s^q\right]\right)\big|^2 ds\Big].\end{aligned}$$

Since f and h are uniformly Lipschitz with respect to their arguments, we have

$$\mathbb{E}\left[|Y_t^\varepsilon - Y_t^q|^2\right] + \mathbb{E}\left[\int_t^T \|Z_s^\varepsilon - Z_s^q\|^2 ds\right] \leq (\frac{1}{\theta} + 2C\theta + 2C)\mathbb{E}\left[\int_t^T |Y_s^\varepsilon - Y_s^q|^2 ds\right] + (2C\theta + 2\gamma)\mathbb{E}\left[\int_t^T \|Z_s^\varepsilon - Z_s^q\|^2 ds\right] + \phi_t^\varepsilon, \tag{4.7}$$

where

$$\phi_t^\varepsilon = 2C\mathbb{E}\left[|y_T^\varepsilon - y_T^q|^2\right] + (2C\theta + 2C)\mathbb{E}\big[\int_t^T |y_s^\varepsilon - y_s^q|^2 ds\big] + C\varepsilon\theta^2.$$

From (4.4) we can show that

$$\lim_{\varepsilon \to 0} \phi_t^\varepsilon = 0. \tag{4.8}$$

Choose $\theta = \frac{1-2\gamma}{4C} > 0$, thus $2C\theta + 2\gamma = \frac{1-2\gamma}{2} + 2\gamma = \frac{1+2\gamma}{2} < 1$, so the inequality (4.7) becomes

$$\mathbb{E}\left[|Y_t^\varepsilon - Y_t^q|^2\right] + \frac{1-2\gamma}{2}\mathbb{E}\left[\int_t^T \|Z_s^\varepsilon - Z_s^q\|^2 ds\right] \leq C\mathbb{E}\left[\int_t^T |Y_s^\varepsilon - Y_s^q|^2 ds\right] + \phi_t^\varepsilon,$$

we derive from this inequality, two inequalities

$$\mathbb{E}\left[|Y_t^\varepsilon - Y_t^q|^2\right] \leq C\mathbb{E}\left[\int_t^T |Y_s^\varepsilon - Y_s^q|^2 ds\right] + \phi_t^\varepsilon, \tag{4.9}$$

and

$$\mathbb{E}\left[\int_t^T \|Z_s^\varepsilon - Z_s^q\|^2 ds\right] \leq C\mathbb{E}\left[\int_t^T |Y_s^\varepsilon - Y_s^q|^2 ds\right] + \phi_t^\varepsilon. \tag{4.10}$$

Applying Granwall's lemma and Burkholder-Davis-Gundy inequality in (4.9) and using (4.4) and (4.8) to get (4.5). Finally (4.6) derived from (4.5), (4.8) and (4.10). □

Proposition 4.2. *Let* $\left(\widehat{y}_t, \widehat{Y}_t, \widehat{Z}_t\right)$, *be the solution of the following variational equations of MF-FBDSDE (4.1)*

$$\begin{cases} d\widehat{y}_t = \int_U b_y\left(t, y_t^q, \mathbb{E}\left[y_t^q\right], u\right) q_t(du)\widehat{y}_t dt \\ \qquad\qquad + \mathbb{E}\left[\int_U b_{y'}\left(t, y_t^q, \mathbb{E}\left[y_t^q\right], u\right) q_t(du)\mathbb{E}[\widehat{y}_t]\right] dt \\ \quad + \left(\sigma_y\left(t, y_t^q, \mathbb{E}\left[y_t^q\right]\right)\widehat{y}_t + \mathbb{E}\left[\sigma_{y'}\left(t, y_t^q, \mathbb{E}\left[y_t^q\right]\right)\mathbb{E}[\widehat{y}_t]\right]\right) dW_t \\ \quad + \left(\int_U b\left(t, y_t^q, \mathbb{E}\left[y_t^q\right], u\right) q_t(du) - \int_U b\left(t, y_t^q, \mathbb{E}\left[y_t^q\right], u\right) \mu_t(du)\right) dt \\ \\ d\widehat{Y}_t = -(\int_U f_y(t, \pi_t^q, u) q_t(du)\widehat{y}_t + \mathbb{E}\left[\int_U f_{y'}(t, \pi_t^q, u) q_t(du)\mathbb{E}[\widehat{y}_t]\right] \\ \qquad + \int_U f_Y(t, \pi_t^q, u) q_t(du)\widehat{Y}_t + \mathbb{E}\left[\int_U f_{Y'}(t, \pi_t^q, u) q_t(du)\mathbb{E}[\widehat{Y}_t]\right] \\ \qquad + \int_U f_Z(t, \pi_t^q, u) q_t(du)\widehat{Z}_t + \mathbb{E}\left[\int_U f_{Z'}(t, \pi_t^q, u) q_t(du)\mathbb{E}[\widehat{Z}_t]\right] \\ \qquad + (\int_U f\left(t, \pi_t^q, u\right) q_t(du) - \int_U f\left(t, \pi_t^q, u\right) \mu_t(du)))dt \\ \quad -(g_y(t, \pi_t^q)\widehat{y}_t + \mathbb{E}\left[g_{y'}(t, \pi_t^q)\mathbb{E}[\widehat{y}_t]\right] + g_Y(t, \pi_t^q)\widehat{Y}_t + \mathbb{E}\left[g_{Y'}(t, \pi_t^q)\mathbb{E}[\widehat{Y}_t]\right] \\ \qquad + g_Z(t, \pi_t^q)\widehat{Z}_t + \mathbb{E}\left[g_{Z'}(t, \pi_t^q)\mathbb{E}[\widehat{Z}_t]\right])\overleftarrow{dB}_t + \widehat{Z}_t dW_t, \\ \\ \widehat{y}_0 = 0, \widehat{Y}_T = h_y\left(y_T^q, \mathbb{E}[y_T^q]\right)\widehat{y}_T + \mathbb{E}\left[h_{y'}\left(y_T^q, \mathbb{E}[y_T^q]\right)\mathbb{E}[\widehat{y}_T]\right], \end{cases} \tag{4.11}$$

where $(t,\pi_t^q,u) := (t, y_t^q, \mathbb{E}[y_t^q], Y_t^q, \mathbb{E}[Y_t^q], Z_t^q, \mathbb{E}[Z_t^q], u)$. *We have the following estimates*

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\sup_{0\le t\le T}\left|\frac{1}{\varepsilon}(y_t^\varepsilon - y_t^q) - \widehat{y}_t\right|^2\right] = 0, \tag{4.12}$$

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\sup_{0\le t\le T}\left|\frac{1}{\varepsilon}(Y_t^\varepsilon - Y_t^q) - \widehat{Y}_t\right|^2\right] = 0, \tag{4.13}$$

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\int_0^T\left\|\frac{1}{\varepsilon}(Z_t^\varepsilon - Z_t^q) - \widehat{Z}_t\right\|^2 dt\right] = 0. \tag{4.14}$$

Proof. For simplicity, denote by

$$\Upsilon_t^\varepsilon = \frac{1}{\varepsilon}(y_t^\varepsilon - y_t^q) - \widehat{y}_t, \mathbb{Y}_t^\varepsilon = \frac{1}{\varepsilon}(Y_t^\varepsilon - Y_t^q) - \widehat{Y}_t, \mathbb{Z}_t^\varepsilon = \frac{1}{\varepsilon}(Z_t^\varepsilon - Z_t^q) - \widehat{Z}_t. \tag{4.15}$$

i) Let us prove (4.12). From (4.1), (4.11) and notations (4.15), we have

$$\begin{aligned}
\Upsilon_t^\varepsilon = {} & \frac{1}{\varepsilon}\int_0^t\left[\int_U b\left(s,y_s^\varepsilon,\mathbb{E}\left[y_s^\varepsilon\right],u\right)q_s^\varepsilon(du) - \int_U b\left(s,y_s^q,\mathbb{E}\left[y_s^q\right],u\right)q_s^\varepsilon(du)\right]ds \\
& + \frac{1}{\varepsilon}\int_0^t\left[\int_U b\left(s,y_s^q,\mathbb{E}\left[y_s^q\right],u\right)q_s^\varepsilon(du) - \int_U b\left(s,y_s^q,\mathbb{E}\left[y_s^q\right],u\right)q_s(du)\right]ds \\
& + \frac{1}{\varepsilon}\int_0^t\left[\sigma\left(s,y_s^\varepsilon,\mathbb{E}\left[y_s^\varepsilon\right]\right) - \sigma\left(s,y_s^q,\mathbb{E}\left[y_s^q\right]\right)\right]dW_s \\
& - \int_0^t\int_U b_y\left(s,y_s^q,\mathbb{E}\left[y_s^q\right],u\right)q_s(du)\widehat{y}_s ds \\
& \qquad - \int_0^t\mathbb{E}\left[\int_U b_{y'}\left(s,y_s^q,\mathbb{E}\left[y_s^q\right],u\right)q_s(du)\mathbb{E}[\widehat{y}_s]\right]ds \\
& - \int_0^t\left(\sigma_y\left(s,y_s^q,\mathbb{E}\left[y_s^q\right]\right)\widehat{y}_s + \mathbb{E}\left[\sigma_{y'}\left(s,y_s^q,\mathbb{E}\left[y_s^q\right]\right)\mathbb{E}[\widehat{y}_s]\right]\right)dW_s \\
& - \int_0^t\left(\int_U b\left(s,y_s^q,\mathbb{E}\left[y_s^q\right],u\right)q_s(du) - \int_U b\left(s,y_s^q,\mathbb{E}\left[y_s^q\right],u\right)\mu_s(du)\right)ds.
\end{aligned}$$

Using the definition of q_s^ε and taking expectation, we obtain

$$\begin{aligned}
\mathbb{E}\left[|\Upsilon_t^\varepsilon|^2\right] \le {} & C\mathbb{E}\left[\int_0^t\int_0^1\int_U\left|b_y\left(s,\Lambda_s^\varepsilon,u\right)\Upsilon_t^\varepsilon\right|^2 q_s(du)d\lambda ds\right] \\
& + C\mathbb{E}\left[\int_0^t\int_0^1\int_U\left|\mathbb{E}\left[b_{y'}\left(s,\Lambda_s^\varepsilon,u\right)\mathbb{E}[\Upsilon_t^\varepsilon]\right]\right|^2 q_s(du)d\lambda ds\right] \\
& + C\mathbb{E}\left[\int_0^t\int_0^1\left|\sigma_y\left(s,\Lambda_s^\varepsilon\right)\Upsilon_t^\varepsilon\right|^2 d\lambda ds\right] \\
& + C\mathbb{E}\left[\int_0^t\int_0^1\left|\mathbb{E}\left[\sigma_{y'}\left(s,\Lambda_s^\varepsilon\right)\mathbb{E}[\Upsilon_t^\varepsilon]\right]\right|^2 d\lambda ds\right] + C\mathbb{E}\left[|\Gamma_t^\varepsilon|^2\right],
\end{aligned}$$

where $(s,\Lambda_s^\varepsilon,u):=(s,y_s^q+\lambda\varepsilon(\Upsilon_s^\varepsilon+\widehat{y}_s),\mathbb{E}[y_s^q+\lambda\varepsilon(\Upsilon_s^\varepsilon+\widehat{y}_s)],u)$, and

$$\begin{aligned}
\Gamma_t^\varepsilon &= \int_0^t\int_0^1\int_U b_y\left(s,\Lambda_s^\varepsilon,u\right)\left(y_s^\varepsilon-y_s^q\right)\mu_s(du)d\lambda ds \\
&+\int_0^t\int_0^1\int_U \mathbb{E}\left[b_{y'}\left(s,\Lambda_s^\varepsilon,u\right)\mathbb{E}[y_s^\varepsilon-y_s^q]\right]\mu_s(du)d\lambda ds \\
&-\int_0^t\int_0^1\int_U b_y\left(s,\Lambda_s^\varepsilon,u\right)\left(y_s^\varepsilon-y_s^q\right)q_s(du)d\lambda ds \\
&-\int_0^t\int_0^1\int_U \mathbb{E}\left[b_{y'}\left(s,\Lambda_s^\varepsilon,u\right)\mathbb{E}[y_s^\varepsilon-y_s^q]\right]q_s(du)d\lambda ds \\
&+\int_0^t\int_0^1\int_U \left(b_y\left(s,\Lambda_s^\varepsilon,u\right)\widehat{y}_s+\mathbb{E}\left[b_{y'}\left(s,\Lambda_s^\varepsilon,u\right)\mathbb{E}[\widehat{y}_s]\right]\right)q_s(du)d\lambda ds \\
&\quad+\int_0^t\int_0^1\left(\sigma_y\left(s,\Lambda_s^\varepsilon\right)\widehat{y}_t+\mathbb{E}\left[\sigma_{y'}\left(s,\Lambda_s^\varepsilon\right)\mathbb{E}[\widehat{y}_s]\right]\right)d\lambda dW_s \\
&-\int_0^t\int_U b_y\left(s,y_s^q,\mathbb{E}[y_s^q],u\right)\widehat{y}_s q_s(du)ds \\
&\quad-\int_0^t\int_U \mathbb{E}\left[b_{y'}\left(s,y_s^q,\mathbb{E}[y_s^q],u\right)\mathbb{E}[\widehat{y}_s]\right]q_s(du)ds \\
&-\int_0^t\left(\sigma_y\left(s,y_s^q,\mathbb{E}[y_s^q]\right)\widehat{y}_s+\mathbb{E}\left[\sigma_{y'}\left(s,y_s^q,\mathbb{E}[y_s^q]\right)\mathbb{E}[\widehat{y}_s]\right]\right)dW_s,
\end{aligned}$$

since $b_y,b_{y'},\sigma_y,\sigma_{y'}$ are continuous and bounded we have

$$\mathbb{E}\left[|\Upsilon_t^\varepsilon|^2\right]\le C\mathbb{E}\left[\int_0^t|\Upsilon_s^\varepsilon|^2ds\right]+C\mathbb{E}\left[|\Gamma_t^\varepsilon|^2\right], \tag{4.16}$$

and

$$\lim_{\varepsilon\to0}\mathbb{E}\left[|\Gamma_t^\varepsilon|^2\right]=0. \tag{4.17}$$

By using (4.17), Granwall's lemma and Burkholder-Davis-Gundy inequality in (4.16), one can show (4.12).

ii) Let us prove (4.13) and (4.14). We put

$$\begin{aligned}
(s,\Delta_s^\varepsilon,u):=\big(&s,y_s^q+\lambda\varepsilon(\Upsilon_s^\varepsilon+\widehat{y}_s),\mathbb{E}[y_s^q+\lambda\varepsilon(\Upsilon_s^\varepsilon+\widehat{y}_s)],Y_s^q+\lambda\varepsilon(\mathbb{Y}_s^\varepsilon+\widehat{Y}_s)\\
&,\mathbb{E}[Y_s^q+\lambda\varepsilon(\mathbb{Y}_s^\varepsilon+\widehat{Y}_s)],Z_s^q+\lambda\varepsilon(\mathbb{Z}_s^\varepsilon+\widehat{Z}_s),\mathbb{E}[Z_s^q+\lambda\varepsilon(\mathbb{Z}_s^\varepsilon+\widehat{Z}_s)],u\big).
\end{aligned}$$

From (4.1), (4.11) and (4.15) we have

$$\left\{\begin{aligned}
d\mathbb{Y}_t^\varepsilon &= -\big(F_Y^\varepsilon\mathbb{Y}_t^\varepsilon+\mathbb{E}\left[F_{Y'}^\varepsilon\mathbb{E}[\mathbb{Y}_t^\varepsilon]\right]+F_Z^\varepsilon\mathbb{Z}_t^\varepsilon+\mathbb{E}\left[F_{Z'}^\varepsilon\mathbb{E}[\mathbb{Z}_t^\varepsilon]\right]+\Theta_t^\varepsilon\big)dt \\
&\quad-\big(g_Y\left(t,\Delta_t^\varepsilon\right)\mathbb{Y}_t^\varepsilon+\mathbb{E}\left[g_{Y'}\left(t,\Delta_t^\varepsilon\right)\mathbb{E}[\mathbb{Y}_t^\varepsilon]\right]+g_Z\left(t,\Delta_t^\varepsilon\right)\mathbb{Z}_t^\varepsilon \\
&\qquad+\mathbb{E}\left[g_{Z'}\left(t,\Delta_t^\varepsilon\right)\mathbb{E}[\mathbb{Z}_t^\varepsilon]\right]+\Xi_t^\varepsilon\big)\overleftarrow{dB}_t+\mathbb{Z}_t^\varepsilon dW_t \\
\mathbb{Y}_T^\varepsilon &= \tfrac{1}{\varepsilon}\left(h\left(y_T^\varepsilon,\mathbb{E}[y_T^\varepsilon]\right)-h\left(y_T^q,\mathbb{E}[y_T^q]\right)\right) \\
&\qquad-\left(h_{y'}\left(y_T^q,\mathbb{E}[y_T^q]\right)\widehat{y}_T+\mathbb{E}\left[h_{y'}\left(y_T^q,\mathbb{E}[y_T^q]\right)\mathbb{E}[\widehat{y}_T]\right]\right),
\end{aligned}\right. \tag{4.18}$$

where

$$F_\varpi^{\varepsilon,q}=\int_0^1\int_U f_\varpi\big(t,\Delta_t^\varepsilon,u\big)q_t(du)d\lambda,\quad\text{for }\varpi=y,y',Y,Y',Z,Z',$$

$$\begin{aligned}\Theta_t^\varepsilon &= F_y^{\varepsilon,q}\Upsilon_t^\varepsilon + \mathbb{E}\left[F_{y'}^{\varepsilon,q}\mathbb{E}[\Upsilon_t^\varepsilon]\right] + F_y^{\varepsilon,q}\widehat{y}_t + \mathbb{E}\left[F_{y'}^{\varepsilon,q}\mathbb{E}[\widehat{y}_t]\right]\\
&\quad -\int_U f_y\big(t,\pi_t^q,u\big)q_t(du)\widehat{y}_t - \mathbb{E}\left[\int_U f_{y'}\big(t,\pi_t^q,u\big)q_t(du)\mathbb{E}[\widehat{y}_t]\right] + F_Y^{\varepsilon,q}\widehat{Y}_t\\
&\quad +\mathbb{E}\left[F_{Y'}^{\varepsilon,q}\mathbb{E}[\widehat{Y}_t]\right] - \int_U f_Y\big(t,\pi_t^q,u\big)q_t(du)\widehat{Y}_t - \mathbb{E}\left[\int_U f_{Y'}\big(t,\pi_t^q,u\big)q_t(du)\mathbb{E}[\widehat{Y}_t]\right]\\
&\quad +F_Z^{\varepsilon,q}\widehat{Z}_t + \mathbb{E}\left[F_{Z'}^{\varepsilon,q}\mathbb{E}[\widehat{Z}_t]\right] - \int_U f_Z\big(t,\pi_t^q,u\big)q_t(du)\widehat{Z}_t\\
&\qquad -\mathbb{E}\left[\int_U f_{Z'}\big(t,\pi_t^q,u\big)q_t(du)\mathbb{E}[\widehat{Z}_t]\right]\\
&\quad +F_y^{\varepsilon,\mu}\left(y_t^\varepsilon - y_t^q\right) + \mathbb{E}\left[F_{y'}^{\varepsilon,\mu}\mathbb{E}[y_t^\varepsilon - y_t^q]\right] + F_Y^{\varepsilon,\mu}\left(Y_t^\varepsilon - Y_t^q\right)\\
&\qquad +\mathbb{E}\left[F_{Y'}^{\varepsilon,\mu}\mathbb{E}[Y_t^\varepsilon - Y_t^q]\right] + F_Z^{\varepsilon,\mu}\left(Z_t^\varepsilon - Z_t^q\right) + \mathbb{E}\left[F_{Z'}^{\varepsilon,\mu}\mathbb{E}[Z_t^\varepsilon - Z_t^q]\right]\\
&\quad -\big(F_y^{\varepsilon,q}\left(y_t^\varepsilon - y_t^q\right) + \mathbb{E}\left[F_{y'}^{\varepsilon,q}\mathbb{E}[y_t^\varepsilon - y_t^q]\right] + F_Y^{\varepsilon,q}\left(Y_t^\varepsilon - Y_t^q\right)\\
&\qquad +\mathbb{E}\left[F_{Y'}^{\varepsilon,q}\mathbb{E}[Y_t^\varepsilon - Y_t^q]\right] + F_Z^{\varepsilon,q}\left(Z_t^\varepsilon - Z_t^q\right) + \mathbb{E}\left[F_{Z'}^{\varepsilon,q}\mathbb{E}[Z_t^\varepsilon - Z_t^q]\right]\big),\end{aligned}$$

and

$$\begin{aligned}\Xi_t^\varepsilon &= \int_0^1 \big(g_y\big(t,\Delta_t^\varepsilon\big)\widehat{y}_t + \mathbb{E}\left[g_{y'}\big(t,\Delta_t^\varepsilon\big)\mathbb{E}[\widehat{y}_t]\right] - g_y\big(t,\pi_t^q\big)\widehat{y}_t - \mathbb{E}\left[g_{y'}\big(t,\pi_t^q\big)\mathbb{E}[\widehat{y}_t]\right]\big)d\lambda\overleftarrow{dB}_t\\
&\quad +\int_0^1 \big(g_Y\big(t,\Delta_t^\varepsilon\big)\widehat{Y}_t + \mathbb{E}\left[g_{Y'}\big(t,\Delta_t^\varepsilon\big)\mathbb{E}[\widehat{Y}_t]\right] - g_Y\big(t,\pi_t^q\big)\widehat{Y}_t - \mathbb{E}\left[g_{Y'}\big(t,\pi_t^q\big)\mathbb{E}[\widehat{Y}_t]\right]\big)d\lambda\overleftarrow{dB}_t\\
&\quad +\int_0^1 \big(g_Z\big(t,\Delta_t^\varepsilon\big)\widehat{Z}_t + \mathbb{E}\left[g_{Z'}\big(t,\Delta_t^\varepsilon\big)\mathbb{E}[\widehat{Z}_t]\right] - g_Z\big(t,\pi_t^q\big)\widehat{Z}_t - \mathbb{E}\left[g_{Z'}\big(t,\pi_t^q\big)\mathbb{E}[\widehat{Z}_t]\right]\big)d\lambda\overleftarrow{dB}_t.\end{aligned}$$

Using the fact that the derivatives $f_y, f_{y'}, f_Y, f_{Y'}, f_Z, f_{Z'}$ are continuous and bounded and from (4.4), (4.5), (4.6) and (4.12) we show

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\int_t^T |\Theta_s^\varepsilon|^2 ds\right] = 0, \text{ and } \lim_{\varepsilon\to 0}\mathbb{E}\left[\int_t^T |\Xi_s^\varepsilon|^2 ds\right] = 0. \tag{4.19}$$

Applying Itô's formula to $|\mathbb{Y}_t^\varepsilon|^2$ we obtain

$$\begin{aligned}\mathbb{E}\left[|\mathbb{Y}_t^\varepsilon|^2\right] + \mathbb{E}\left[\int_t^T \|\mathbb{Z}_s^\varepsilon\|^2 ds\right] &= \mathbb{E}\left[|\mathbb{Y}_T^\varepsilon|^2\right] + 2\mathbb{E}\left[\int_t^T \langle \mathbb{Y}_s^\varepsilon, +F_s^Y\mathbb{Y}_s^\varepsilon + \mathbb{E}\left[F_s^{Y'}\mathbb{E}[\mathbb{Y}_s^\varepsilon]\right]\right.\\
&\quad \left. +F_s^Z\mathbb{Z}_s^\varepsilon + \mathbb{E}\left[F_s^{Z'}\mathbb{E}[\mathbb{Z}_s^\varepsilon]\right] + \Theta_s^\varepsilon\rangle ds\right] + \mathbb{E}\left[\int_t^T |g_Y\left(t,\Delta_t^\varepsilon\right)\mathbb{Y}_s^\varepsilon\right.\\
&\quad \left. +\mathbb{E}\left[g_{Y'}\left(t,\Delta_t^\varepsilon\right)\mathbb{E}[\mathbb{Y}_s^\varepsilon]\right] + g_Z\left(t,\Delta_t^\varepsilon\right)\mathbb{Z}_s^\varepsilon + \mathbb{E}\left[g_{Z'}\left(t,\Delta_t^\varepsilon\right)\mathbb{E}[\mathbb{Z}_s^\varepsilon]\right] + \Xi_s^\varepsilon|^2 ds\right].\end{aligned}$$

Applying Young's inequality and the boundedness of the derivatives $F_s^Y, F_s^{Y'}, F_s^Z, F_s^{Z'}, g_Y, g_{Y'}, g_Z, g_{Z'}$, we obtain

$$\begin{aligned}
\mathbb{E}\left[|\mathbb{Y}_t^\varepsilon|^2\right]+\mathbb{E}\left[\int_t^T\|\mathbb{Z}_s^\varepsilon\|^2ds\right] &\leq \mathbb{E}\left[|\mathbb{Y}_T^\varepsilon|^2\right]+\frac{1}{\theta_1}\mathbb{E}\left[\int_t^T|\mathbb{Y}_s^\varepsilon|^2ds\right]\\
&+5C\theta_1\mathbb{E}\left[\int_t^T\left(|\mathbb{Y}_s^\varepsilon|^2+\mathbb{E}\left[|\mathbb{Y}_s^\varepsilon|^2\right]+\|\mathbb{Z}_s^\varepsilon\|^2+\mathbb{E}\left[\|\mathbb{Z}_s^\varepsilon\|^2\right]+|\Theta_s^\varepsilon|^2\right)ds\right]\\
&+3C\mathbb{E}\left[\int_t^T\left(|\mathbb{Y}_s^\varepsilon|^2+\mathbb{E}\left[|\mathbb{Y}_s^\varepsilon|^2\right]+|\Xi_s^\varepsilon|^2\right)ds\right]\\
&+2\gamma^2\mathbb{E}\left[\int_t^T\left(\|\mathbb{Z}_s^\varepsilon\|^2+\mathbb{E}\left[\|\mathbb{Z}_s^\varepsilon\|^2\right]\right)ds\right]\\
&+2C\gamma\mathbb{E}\left[\int_t^T\langle\mathbb{Y}_s^\varepsilon+\mathbb{E}\left[\mathbb{Y}_s^\varepsilon\right]+\Xi_s^\varepsilon,\mathbb{Z}_s^\varepsilon+\mathbb{E}\left[\mathbb{Z}_s^\varepsilon\right]\rangle ds\right].
\end{aligned}$$

Applying Young's inequality again

$$\begin{aligned}
\mathbb{E}\left[|\mathbb{Y}_t^\varepsilon|^2\right]+\mathbb{E}\left[\int_t^T\|\mathbb{Z}_s^\varepsilon\|^2ds\right] &\leq \mathbb{E}\left[|\mathbb{Y}_T^\varepsilon|^2\right]+\frac{1}{\theta_1}\mathbb{E}\left[\int_t^T|\mathbb{Y}_s^\varepsilon|^2ds\right]\\
&+5C\theta_1\mathbb{E}\left[\int_t^T\left(|\mathbb{Y}_s^\varepsilon|^2+\mathbb{E}\left[|\mathbb{Y}_s^\varepsilon|^2\right]+\|\mathbb{Z}_s^\varepsilon\|^2+\mathbb{E}\left[\|\mathbb{Z}_s^\varepsilon\|^2\right]+|\Theta_s^\varepsilon|^2\right)ds\right]\\
&+3C\mathbb{E}\left[\int_t^T\left(|\mathbb{Y}_s^\varepsilon|^2+\mathbb{E}\left[|\mathbb{Y}_s^\varepsilon|^2\right]+|\Xi_s^\varepsilon|^2\right)ds\right]+2\gamma^2\mathbb{E}\left[\int_t^T\left(\|\mathbb{Z}_s^\varepsilon\|^2+\mathbb{E}\left[\|\mathbb{Z}_s^\varepsilon\|^2\right]\right)ds\right]\\
&+\frac{6C\gamma}{\theta_2}\mathbb{E}\left[\int_t^T\left(|\mathbb{Y}_s^\varepsilon|^2+\mathbb{E}\left[|\mathbb{Y}_s^\varepsilon|^2\right]+|\Xi_s^\varepsilon|^2\right)ds\right]\\
&+2C\gamma\,\theta_2\mathbb{E}\left[\int_t^T\left(\|\mathbb{Z}_s^\varepsilon\|^2+\mathbb{E}\left[\|\mathbb{Z}_s^\varepsilon\|^2\right]\right)ds\right].
\end{aligned}$$

Hence

$$\begin{aligned}
&\mathbb{E}\left[|\mathbb{Y}_t^\varepsilon|^2\right]+\mathbb{E}\left[\int_t^T\|\mathbb{Z}_s^\varepsilon\|^2ds\right]\\
&\leq \mathbb{E}\left[|\mathbb{Y}_T^\varepsilon|^2\right]+\left(\frac{1}{\theta_1}+10C\,\theta_1+6C+\frac{12C\gamma}{\theta_2}\right)\mathbb{E}\left[\int_t^T|\mathbb{Y}_s^\varepsilon|^2ds\right]\\
&+(10C\,\theta_1+4\gamma^2+8C\gamma\,\theta_2)\mathbb{E}\left[\int_t^T\|\mathbb{Z}_s^\varepsilon\|^2ds\right]\\
&+5C\,\theta_1\mathbb{E}\left[\int_t^T|\Theta_s^\varepsilon|^2ds\right]+\left(3C+\frac{6C\gamma}{\theta_2}\right)\mathbb{E}\left[\int_t^T|\Xi_s^\varepsilon|^2ds\right].
\end{aligned}\tag{4.20}$$

We choose

$$\theta_1=\frac{1-4\gamma^2}{20C}>0,\theta_2=\frac{1-4\gamma^2}{24C\,\gamma}>0,$$

thus

$$10C\,\theta_1+4\gamma^2+8C\gamma\,\theta_2=\frac{1-4\gamma^2}{2}+4\gamma^2+\frac{1-4\gamma^2}{3}=\frac{5+4\gamma^2}{6}<1.$$

Then the inequality (4.19) becomes

$$\mathbb{E}\left[|\mathbb{Y}_t^\varepsilon|^2\right] + K_1\mathbb{E}\left[\int_t^T \|\mathbb{Z}_s^\varepsilon\|^2 ds\right] \leq \mathbb{E}\left[|\mathbb{Y}_T^\varepsilon|^2\right] + K_2\mathbb{E}\left[\int_t^T |\mathbb{Y}_s^\varepsilon|^2 ds\right] + K_3\mathbb{E}\left[\int_t^T |\Theta_s^\varepsilon|^2 ds\right] + K_4\mathbb{E}\left[\int_t^T |\Xi_s^\varepsilon|^2 ds\right], \tag{4.21}$$

with $K_1 = \frac{1-4\gamma^2}{6} > 0, K_2 > 0, K_3 > 0, K_4 > 0$.

We derive from (4.21) two inequality

$$\mathbb{E}\left[|\mathbb{Y}_t^\varepsilon|^2\right] \leq \mathbb{E}\left[|\mathbb{Y}_T^\varepsilon|^2\right] + K_2\mathbb{E}\left[\int_t^T |\mathbb{Y}_s^\varepsilon|^2 ds\right] + K_3\mathbb{E}\left[\int_t^T |\Theta_s^\varepsilon|^2 ds\right] + K_4\mathbb{E}\left[\int_t^T |\Xi_s^\varepsilon|^2 ds\right], \tag{4.22}$$

and

$$\mathbb{E}\left[\int_t^T \|\mathbb{Z}_s^\varepsilon\|^2 ds\right] \leq \frac{1}{K_1}\mathbb{E}\left[|\mathbb{Y}_T^\varepsilon|^2\right] + \frac{K_2}{K_1}\mathbb{E}\left[\int_t^T |\mathbb{Y}_s^\varepsilon|^2 ds\right] + \frac{K_3}{K_1}\mathbb{E}\left[\int_t^T |\Theta_s^\varepsilon|^2 ds\right] + \frac{K_4}{K_1}\mathbb{E}\left[\int_t^T |\Xi_s^\varepsilon|^2 ds\right]. \tag{4.23}$$

On the other hand we have

$$\begin{aligned}
\mathbb{E}\left[|\mathbb{Y}_T^\varepsilon|^2\right] &= \mathbb{E}\Bigg[\Bigg|\frac{1}{\varepsilon}\left(h\left(y_T^\varepsilon, \mathbb{E}[y_T^\varepsilon]\right) - h\left(h_T^q, \mathbb{E}[y_T^q]\right)\right) \\
&\qquad - \left(h_y\left(y_T^q, \mathbb{E}[y_T^q]\right)\widehat{y}_T + \mathbb{E}\left[h_{y'}\left(y_T^q, \mathbb{E}[y_T^q]\right)\mathbb{E}[\widehat{y}_T]\right]\right)\Bigg|^2\Bigg] \\
&\leq 4\mathbb{E}\left[\left|\int_0^1 h_y\left(\Lambda_T^\varepsilon\right)d\lambda - h_y\left(y_T^q, \mathbb{E}[y_T^q]\right)\right|^2 \cdot \left|\widehat{y}_T\right|^2\right] \\
&\qquad + 4\mathbb{E}\left[\mathbb{E}\left[\left|\int_0^1 h_{y'}\left(\Lambda_T^\varepsilon\right)d\lambda - h_{y'}\left(y_T^q, \mathbb{E}[y_T^q]\right)\right|^2\right] \cdot \mathbb{E}\left[\,\left|\widehat{y}_T\right|^2\,\right]\right] \\
&\qquad + 4\mathbb{E}\left[\int_0^1 \left(\left|h_y\left(\Lambda_T^\varepsilon\right)\right|^2 \cdot \left|\Upsilon_T^\varepsilon\right|^2 + \mathbb{E}\left[\left|h_{y'}\left(\Lambda_T^\varepsilon\right)\right|^2\right] \cdot \mathbb{E}\left[\left|\Upsilon_T^\varepsilon\right|^2\right]\right)d\lambda\right].
\end{aligned}$$

Since h_y, h'_y are continuous and bounded, using (4.12) to get

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[|\mathbb{Y}_T^\varepsilon|^2\right] = 0. \tag{4.24}$$

Now, applying Gronwall's lemma in (4.22) and using (4.19) and (4.24) to obtain (4.13) and from (4.13), (4.19) and (4.24) we get (4.14). □

Proposition 4.3 (Variational inequality)**.** *Let* **(H4)** − **(H5)** , *holds. Let* $q.$ *be an optimal relaxed control* *with associated trajectories* $(X_t^{q_\cdot}, Y_t^{q_\cdot}, Z_t^{q_\cdot})$. *Then, for any element* $\mu_\cdot$ *of* $\mathcal{R}$, *we have*

$$\begin{aligned}
0 \le\ & \mathbb{E}\left[\alpha_y(y_T^q, \mathbb{E}[y_T^q])\widehat{y}_T + \mathbb{E}\left[\alpha_{y'}(y_T^q, \mathbb{E}\left[y_T^q\right])\mathbb{E}[\widehat{y}_T]\right]\right] \\
& + \mathbb{E}\left[\beta_Y(Y_0^q, \mathbb{E}[Y_0^q])\widehat{Y}_0 + \mathbb{E}\left[\beta_{Y'}(Y_0^q, \mathbb{E}[Y_0^q])\mathbb{E}[\widehat{Y}_0]\right]\right] \\
& + \mathbb{E}\Big[\int_0^T \int_U \big(\ell_y(t, \pi_t^q, u)\widehat{y}_t + \mathbb{E}\left[\ell_{y'}(t, \pi_t^q, u)\mathbb{E}[\widehat{y}_t]\right] \\
& \qquad + \ell_Y(t, \pi_t^q, u)\widehat{Y}_t + \mathbb{E}\left[\ell_{Y'}(t, \pi_t^q, u)\mathbb{E}[\widehat{Y}_t]\right] \\
& \qquad + \ell_Z(t, \pi_t^q, u)\widehat{Z}_t + \mathbb{E}\left[\ell_{Z'}(t, \pi_t^q, u)\mathbb{E}[\widehat{Z}_t]\right]\big) q_t(du)dt\Big] \\
& + \mathbb{E}\Big[\int_0^T \big(\int_U \ell(t, y_t^q, \mathbb{E}[y_t^q], Y_t^q, \mathbb{E}[Y_t^q], Z_t^q, \mathbb{E}[Z_t^q], u)\mu_t(du) \\
& \qquad - \int_U \ell(t, y_t^q, \mathbb{E}[y_t^q], Y_t^q, \mathbb{E}[Y_t^q], Z_t^q, \mathbb{E}[Z_t^q], u)q_t(du)\big)dt\Big].
\end{aligned} \tag{4.25}$$

Proof. From the optimality of $q_\cdot$ we have

$$\begin{aligned}
0 \le\ & \mathbb{E}\left[\alpha(y_T^\varepsilon, \mathbb{E}[y_T^\varepsilon]) - \alpha(y_T^q, \mathbb{E}[y_T^q])\right] + \mathbb{E}\left[\beta(Y_0^\varepsilon, \mathbb{E}[Y_0^\varepsilon]) - \beta(Y_0^q, \mathbb{E}[Y_0^q])\right] \\
& + \mathbb{E}\Big[\int_0^T \big(\int_U \ell(t, y_t^\varepsilon, \mathbb{E}[y_t^\varepsilon], Y_t^\varepsilon, \mathbb{E}[Y_t^\varepsilon], Z_t^\varepsilon, \mathbb{E}[Z_t^\varepsilon], u)q_t^\varepsilon(du) \\
& \qquad - \int_U \ell(t, y_t^q, \mathbb{E}[y_t^q], Y_t^q, \mathbb{E}[Y_t^q], Z_t^q, \mathbb{E}[Z_t^q], u)q_t^\varepsilon(du)\big)dt\Big] \\
& + \mathbb{E}\Big[\int_0^T \big(\int_U \ell(t, y_t^q, \mathbb{E}[y_t^q], Y_t^q, \mathbb{E}[Y_t^q], Z_t^q, \mathbb{E}[Z_t^q], u)q_t^\varepsilon(du) \\
& \qquad - \int_U \ell(t, y_t^q, \mathbb{E}[y_t^q], Y_t^q, \mathbb{E}[Y_t^q], Z_t^q, \mathbb{E}[Z_t^q], u)q_t(du)\big)dt\Big].
\end{aligned}$$

Let us divide this inequality by ε and using the definition of q_t^ε and from the notation (4.15), we have

$$\begin{aligned}
0 \le\ & \mathbb{E}\left[\int_0^1 \left(\alpha_y(\Lambda_T^\varepsilon)\widehat{y}_T + \mathbb{E}\left[\alpha_{y'}(\Lambda_T^\varepsilon)\mathbb{E}[\widehat{y}_T]\right]\right) d\lambda\right] \\
& + \mathbb{E}\Big[\int_0^1 \Big(\beta_Y(Y_0^q + \lambda\varepsilon(\mathbb{Y}_0^\varepsilon + \widehat{Y}_0), \mathbb{E}[Y_0^q + \lambda\varepsilon(\mathbb{Y}_0^\varepsilon + \widehat{Y}_s)])\widehat{Y}_0 \\
& + \mathbb{E}\left[\beta_{Y'}(Y_0^q + \lambda\varepsilon(\mathbb{Y}_0^\varepsilon + \widehat{Y}_0), \mathbb{E}[Y_0^q + \lambda\varepsilon(\mathbb{Y}_0^\varepsilon + \widehat{Y}_s)])\mathbb{E}[\widehat{Y}_0]\right]\Big) d\lambda\Big] \\
& + \mathbb{E}\Big[\int_0^T \int_0^1 \int_U \big(\ell_y(t, \Delta_t^\varepsilon, u)\widehat{y}_t + \mathbb{E}\left[\ell_{y'}(t, \Delta_t^\varepsilon, u)\mathbb{E}[\widehat{y}_t]\right] \\
& \qquad + \ell_Y(t, \Delta_t^\varepsilon, u)\widehat{Y}_t + \mathbb{E}\left[\ell_{Y'}(t, \Delta_t^\varepsilon, u)\mathbb{E}[\widehat{Y}_t]\right] \\
& \qquad + \ell_Z(t, \Delta_t^\varepsilon, u)\widehat{Y}_t + \mathbb{E}\left[\ell_{Z'}(t, \Delta_t^\varepsilon, u)\mathbb{E}[\widehat{Y}_t]\right]\big) q_t(du)d\lambda dt\Big] \\
& + \mathbb{E}\Big[\int_0^T \big(\int_U \ell(t, y_t^q, \mathbb{E}[y_t^q], Y_t^q, \mathbb{E}[Y_t^q], Z_t^q, \mathbb{E}[Z_t^q], u)\mu_t(du) \\
& \qquad - \int_U \ell(t, y_t^q, \mathbb{E}[y_t^q], Y_t^q, \mathbb{E}[Y_t^q], Z_t^q, \mathbb{E}[Z_t^q], u)q_t(du)\big)dt\Big] + \nabla_t^\varepsilon,
\end{aligned} \tag{4.26}$$

where ∇_t^ε is given by

$$
\begin{aligned}
\nabla_t^\varepsilon = &\mathbb{E}\left[\int_0^1 \left(\alpha_y(\Lambda_T^\varepsilon)\Upsilon_T^\varepsilon + \mathbb{E}\left[\alpha_{y'}(\Lambda_T^\varepsilon)\mathbb{E}[\Upsilon_T^\varepsilon]\right]\right) d\lambda\right] \\
&+\mathbb{E}\left[\int_0^1 \Big(\beta_Y\big(Y_0^q + \lambda\varepsilon(\mathbb{Y}_0^\varepsilon + \widehat{Y}_0), \mathbb{E}[Y_0^q + \lambda\varepsilon(\mathbb{Y}_0^\varepsilon + \widehat{Y}_s)]\big)\mathbb{Y}_0^\varepsilon \right. \\
&+\mathbb{E}\Big[\beta_{Y'}\big(Y_0^q + \lambda\varepsilon(\mathbb{Y}_0^\varepsilon + \widehat{Y}_0), \mathbb{E}[Y_0^q + \lambda\varepsilon(\mathbb{Y}_0^\varepsilon + \widehat{Y}_s)]\big)\mathbb{E}[\mathbb{Y}_0^\varepsilon]\Big]\Big) d\lambda\Big] \\
&+\mathbb{E}\Big[\int_0^T \int_0^1 \int_U \big(\ell_y(t,\Delta_t^\varepsilon,u)(y_t^\varepsilon - y_t^q) + \mathbb{E}\left[\ell_{y'}(t,\Delta_t^\varepsilon,u)\mathbb{E}[y_t^\varepsilon - y_t^q]\right] \\
&\qquad +\ell_Y(t,\Delta_t^\varepsilon,u)(Y_t^\varepsilon - Y_t^q) + \mathbb{E}\left[\ell_{Y'}(t,\Delta_t^\varepsilon,u)\mathbb{E}[Y_t^\varepsilon - Y_t^q]\right] \\
&\qquad +\ell_Z(t,\Delta_t^\varepsilon,u)(Z_t^\varepsilon - Z_t^q) + \mathbb{E}\left[\ell_{Z'}(t,\Delta_t^\varepsilon,u)\mathbb{E}[Z_t^\varepsilon - Z_t^q]\right]\big)\mu_t(du)d\lambda dt\Big] \\
&-\mathbb{E}\Big[\int_0^T \int_0^1 \int_U \big(\ell_y(t,\Delta_t^\varepsilon,u)(y_t^\varepsilon - y_t^q) + \mathbb{E}\left[\ell_{y'}(t,\Delta_t^\varepsilon,u)\mathbb{E}[y_t^\varepsilon - y_t^q]\right] \\
&\qquad +\ell_Y(t,\Delta_t^\varepsilon,u)(Y_t^\varepsilon - Y_t^q) + \mathbb{E}\left[\ell_{Y'}(t,\Delta_t^\varepsilon,u)\mathbb{E}[Y_t^\varepsilon - Y_t^q]\right] \\
&\qquad +\ell_Z(t,\Delta_t^\varepsilon,u)(Z_t^\varepsilon - Z_t^q) + \mathbb{E}\left[\ell_{Z'}(t,\Delta_t^\varepsilon,u)\mathbb{E}[Z_t^\varepsilon - Z_t^q]\right]\big)q_t(du)d\lambda dt\Big] \\
&+\mathbb{E}\Big[\int_0^T \int_0^1 \int_U \big(\ell_y(t,\Delta_t^\varepsilon,u)\Upsilon_t^\varepsilon + \mathbb{E}\left[\ell_{y'}(t,\Delta_t^\varepsilon,u)\mathbb{E}[\Upsilon_t^\varepsilon]\right] \\
&\qquad +\ell_Y(t,\Delta_t^\varepsilon,u)\mathbb{Y}_t^\varepsilon + \mathbb{E}\left[\ell_{Y'}(t,\Delta_t^\varepsilon,u)\mathbb{E}[\mathbb{Y}_t^\varepsilon]\right] \\
&\qquad +\ell_Z(t,\Delta_t^\varepsilon,u)\mathbb{Z}_t^\varepsilon + \mathbb{E}\left[\ell_{Z'}(t,\Delta_t^\varepsilon,u)\mathbb{E}[\mathbb{Z}_t^\varepsilon]\right]\big)q_t(du)d\lambda dt\Big].
\end{aligned}
$$

Since the derivatives $\alpha_y, \alpha_{y'}, \beta_Y, \beta_{Y'}, \ell_y, \ell_{y'}, \ell_Y, \ell_{Y'}, \ell_Z, \ell_{Z'}$ are continuous and bounded, then by using (4.4), (4.5),(4.6), (4.12), (4.13),(4.14) and the Cauchy-Schwartz inequality we show that

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[|\nabla_t^\varepsilon|^2\right] = 0.$$

Then let ε go to 0 in (4.26), we get the variational inequality. □

4.1.2. Necessary optimality conditions for relaxed control. Let us introduce the adjoint equations of the MF-FBDSDE (4.1) and then gives the maximum principle.

Define the Hamiltonian H from

$$[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times\mathbb{R}^{m\times d}\times U\times\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}^{m\times l}\times\mathbb{R}^{n\times d},$$

to $\mathbb{R}$ by

$$
\begin{aligned}
H(t,y,y',Y,Y',Z,Z',\mu,\Phi,\Psi,\Sigma,\Pi) &:= \Phi\int_U b(t,y,y',u)\mu(du) + \Sigma\sigma(t,y,y') \\
&+ \Psi\int_U f(t,y,y',Y,Y',Z,Z',u)\mu(du) + \Pi g(t,y,y',Y,Y',Z,Z') \\
&+ \int_U \ell(t,y,y',Y,Y',Z,Z',u)\mu(du).
\end{aligned}
\tag{4.27}
$$

Theorem 4.4. *(Necessary optimality conditions for relaxed control) Assume that* $(\mathbf{H4})-(\mathbf{H5})$ *, holds. Let* $q_\cdot \in \mathcal{R}$ *an optimal relaxed control. Let* (y^q, Y^q, Z^q) *be the associated solution of MF-FBDSDE (4.1). Then there exists a unique solution* $(\Phi^q, \Psi^q, \Sigma^q, \Pi^q)$ *of the following adjoint equations of MF-FBDSDE (4.1):*

$$
\begin{cases}
d\Phi_t^q = -\big(H_y(t,\zeta_t^q,q_t,\chi_t^q) + \mathbb{E}\big[H_{y'}(t,\zeta_t^q,q_t,\chi_t^q)\big]\big)dt + \Sigma_t^q dW_t, \\
d\Psi_t^q = \big(H_Y(t,\zeta_t^q,q_t,\chi_t^q) + \mathbb{E}\big[H_{Y'}(t,\zeta_t^q,q_t,\chi_t^q)\big]\big)dt \\
\qquad\qquad +\big(H_Z(t,\zeta_t^q,q_t,\chi_t^q) + \mathbb{E}\big[H_{Z'}(t,\zeta_t^q,q_t,\chi_t^q)\big]\big)dW_t - \Pi_t^q \overleftarrow{dB}_t, \\
\\
\Psi_0^q = \beta_Y(Y_0^q,\mathbb{E}[Y_0^q]) + \mathbb{E}\big[k_{y'}(Y_0^q,\mathbb{E}[Y_0^q])\big], \\
\Phi_T^q = \alpha_y(y_T^q,\mathbb{E}[y_T^q]) + \mathbb{E}\big[\alpha_{y'}(y_T^q,\mathbb{E}[y_T^q])\big] \\
\qquad\qquad +h_y(y_T^q,\mathbb{E}[y_T^q])\Psi_T^q + \mathbb{E}\big[h_{y'}(y_T^q,\mathbb{E}[y_T^q])\mathbb{E}[\Psi_T^q]\big],
\end{cases}
\tag{4.28}
$$

such that

$$H(t,y_t^q,\mathbb{E}[y_t^q],Y_t^q,\mathbb{E}[Y_t^q],Z_t^q,\mathbb{E}[Z_t^q],q_t,\Phi_t^q,\Psi_t^q,\Sigma_t^q,\Pi_t^q)$$
$$\leq H(t,y_t^q,\mathbb{E}[y_t^q],Y_t^q,\mathbb{E}[Y_t^q],Z_t^q,\mathbb{E}[Z_t^q],\mu_t,\Phi_t^q,\Psi_t^q,\Sigma_t^q,\Pi_t^q)$$
$$,\ a.e.\,t,\ P-a.s.,\ \forall\mu\in P(U), \tag{4.29}$$

where $(t,\zeta_t^q,q_t,\chi_t^q):=(t,y_t^q,\mathbb{E}[y_t^q],Y_t^q,\mathbb{E}[Y_t^q],Z_t^q,\mathbb{E}[Z_t^q],q_t,\Phi_t^q,\Psi_t^q,\Sigma_t^q,\Pi_t^q)$.

Proof. From (4.28), the inequality variational (4.25) becomes

$$\begin{aligned}0\leq\ &\mathbb{E}\left[\langle\Phi_T^q,\widehat{y}_T\rangle\right]-\mathbb{E}\left[h_y(y_T^q,\mathbb{E}[y_T^q])\Psi_T^q+\mathbb{E}\big[h_{y'}(y_T^q,\mathbb{E}[y_T^q])\mathbb{E}[\Psi_T^q]\big]\right]\\
&+\mathbb{E}\left[\langle\Psi_0^q,\widehat{Y}_0\rangle\right]+\mathbb{E}\Big[\int_0^T\int_U\big(\ell_y(t,\pi_t^q,u)\widehat{y}_t+\mathbb{E}\left[\ell_{y'}(t,\pi_t^q,u)\mathbb{E}[\widehat{y}_t]\right]\\
&\qquad+\ell_Y(t,\pi_t^q,u)\widehat{Y}_t+\mathbb{E}\left[\ell_{Y'}(t,\pi_t^q,u)\mathbb{E}[\widehat{Y}_t]\right]\\
&\qquad+\ell_Z(t,\pi_t^q,u)\widehat{Z}_t+\mathbb{E}\left[\ell_{Z'}(t,\pi_t^q,u)\mathbb{E}[\widehat{Z}_t]\right]\big)q_t(du)dt\Big]\\
&+\mathbb{E}\Big[\int_0^T\Big(\int_U\ell(t,y_t^q,\mathbb{E}[y_t^q],Y_t^q,\mathbb{E}[Y_t^q],Z_t^q,\mathbb{E}[Z_t^q],u)\mu_t(du)\\
&\qquad-\int_U\ell(t,y_t^q,\mathbb{E}[y_t^q],Y_t^q,\mathbb{E}[Y_t^q],Z_t^q,\mathbb{E}[Z_t^q],u)q_t(du)\Big)dt\Big].\end{aligned}\tag{4.30}$$

Now applying Itô's formula to compute $\langle\Phi_t^q,\widehat{y}_t\rangle$ and $\langle\Psi_t^q,\widehat{Y}_t\rangle$ and taking the expectations we derive

$$\begin{aligned}\mathbb{E}\big[\langle\Phi_T^q,\widehat{y}_T\rangle\big]=&-\mathbb{E}\Big[\int_0^T\langle\Psi_t^q\int_U\big(f_y(t,\pi_t^q,u)+\mathbb{E}\big[f_{y'}(t,\pi_t^q,u)\big]\big)q_t(du)\\
&+\Pi_t^q\big(g_y(t,\pi_t^q)+\mathbb{E}\big[g_{y'}(t,\pi_t^q)\big]\big)+\int_U\big(\ell_y(t,\pi_t^q,u)+\mathbb{E}\big[\ell_{y'}(t,\pi_t^q,u)\big]\big)q_t(du),\widehat{y}_t\rangle dt\Big]\\
&+\mathbb{E}\Big[\int_0^T\Phi_t^q\Big(\int_U b(t,y_t^q,\mathbb{E}[y_t^q],u)q_t(du)-\int_U b(t,y_t^q,\mathbb{E}[y_t^q],u)\mu_t(du)\Big)dt\Big],\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}\big[\langle\Psi_0^q,\widehat{Y}_0\rangle\big]=\ &\mathbb{E}\big[\langle\Psi_T^q,\widehat{Y}_T\rangle\big]\\
&+\mathbb{E}\Big[\int_0^T\langle\Psi_t^q,\int_U\big(f_y(t,\pi_t^q,u)\widehat{y}_t+\mathbb{E}\big[f_{y'}(t,\pi_t^q,u)\mathbb{E}[\widehat{y}_t]\big]\big)q_t(du)\rangle dt\Big]\\
&+\mathbb{E}\Big[\int_0^T\langle\Pi_t^q,\big(g_y(t,\pi_t^q)\widehat{y}_t+\mathbb{E}\big[g_{y'}(t,\pi_t^q)\mathbb{E}[\widehat{y}_t]\big]\big)\rangle dt\Big]\\
&-\mathbb{E}\Big[\int_0^T\langle\int_U\big(\ell_Y(t,\pi_t^q,u)+\mathbb{E}\big[\ell_{Y'}(t,\pi_t^q,u)\big]\big)q_t(du),\widehat{Y}_t\rangle dt\Big]\\
&-\mathbb{E}\Big[\int_0^T\langle\int_U\big(\ell_Z(t,\pi_t^q,u)+\mathbb{E}\big[\ell_{Z'}(t,\pi_t^q,u)\big]\big)q_t(du),\widehat{Z}_t\rangle dt\Big]\\
&+\mathbb{E}\Big[\int_0^T\Psi_t^q\Big(\int_U f(t,\pi_t^q,u)q_t(du)-\int_U f(t,\pi_t^q,u)\mu_t(du)\Big)dt\Big].\end{aligned}$$

Substitute the above equalities in inequality (4.30) to get, for every $\mu\in\mathcal{R}$,

$$\begin{aligned}0\leq\ \mathbb{E}\Big[\int_0^T\big(&H(t,y_t^q,\mathbb{E}[y_t^q],Y_t^q,\mathbb{E}[Y_t^q],Z_t^q,\mathbb{E}[Z_t^q],q_t,\Phi_t^q,\Psi_t^q,\Sigma_t^q,\Pi_t^q)\\
&-H(t,y_t^q,\mathbb{E}[y_t^q],Y_t^q,\mathbb{E}[Y_t^q],Z_t^q,\mathbb{E}[Z_t^q],\mu_t,\Phi_t^q,\Psi_t^q,\Sigma_t^q,\Pi_t^q)\big)dt\Big].\end{aligned}$$

Therefore inequality (4.29) follows by a standard arguments. □

4.1.3. Sufficient optimality conditions for relaxed control. In this subsection we study when the necessary conditions for optimality in Theorem 4.4 become sufficient as well.

Theorem 4.5. *(Sufficient optimality conditions for relaxed control) Assume that (H4) hold. Given $q_\cdot \in \mathcal{R}$, let (y^q, Y^q, Z^q) and $(\Phi^q_t, \Psi^q_t, \Sigma^q_t, \Pi^q_t)$ be the corresponding solutions of the MF-FBSDEs (4.1) and (4.28) respectively. Suppose that α, β, ℓ and the function $H(t, \cdot, \cdot, \cdot, \cdot, \cdot, q_t, \Phi^q_t, \Psi^q_t, \Sigma^q_t, \Pi^q_t)$ are convex.*

Then $(y^q_\cdot, Y^q_\cdot, Z^q_\cdot, q_\cdot)$ is an optimal solution of the control problem (4.1)–(4.3) if it satisfies (4.29).

Proof. Let $q_\cdot \in \mathcal{R}$ be arbitrary (candidate to be optimal), and let $(y^q_\cdot, Y^q_\cdot, Z^q_\cdot)$ denote the trajectory associated to $q_\cdot$. For any $\mu_\cdot \in \mathcal{R}$ with associated trajectory $(y^\mu_\cdot, Y^\mu_\cdot, Z^\mu_\cdot)$, we have

$$\begin{aligned}\mathbb{J}(\mu_\cdot) - \mathbb{J}(q_\cdot) &= \mathbb{E}\big[\alpha(y^\mu_T, \mathbb{E}[y^\mu_T]) - \alpha(y^q_T, \mathbb{E}[y^q_T])\big] + \mathbb{E}\big[\beta(Y^\mu_0, \mathbb{E}[Y^\mu_0]) - \beta(Y^q_0, \mathbb{E}[Y^q_0])\big] \\ &\quad + \mathbb{E}\Big[\int_0^T \Big(\int_U \ell(t, y^\mu_t, \mathbb{E}[y^\mu_t], Y^\mu_t, \mathbb{E}[Y^\mu_t], Z^\mu_t, \mathbb{E}[Z^\mu_t], u)\mu_t(du) \\ &\qquad - \int_U \ell(t, y^q_t, \mathbb{E}[y^q_t], Y^q_t, \mathbb{E}[Y^q_t], Z^q_t, \mathbb{E}[Z^q_t], u)q_t(du)\Big)dt\Big].\end{aligned}$$

Since α and β are convex, we get

$$\begin{aligned}\alpha(y^\mu_T, \mathbb{E}[y^\mu_T]) - \alpha(y^q_T, \mathbb{E}[y^q_T]) &\geq \langle \alpha_y(y^q_T, \mathbb{E}[y^q_T]), y^\mu_T - y^q_T\rangle \\ &\quad + \mathbb{E}\big[\langle \alpha_{y'}(y^q_T, \mathbb{E}[y^q_T]), \mathbb{E}[y^\mu_T - y^q_T]\rangle\big], \\ \beta(Y^\mu_0, \mathbb{E}[Y^\mu_0]) - \beta(Y^q_0, \mathbb{E}[Y^q_0]) &\geq \langle \beta_Y(Y^q_0, \mathbb{E}[Y^q_0]), Y^\mu_0 - Y^q_0\rangle \\ &\quad + \mathbb{E}\big[\langle \beta_{Y'}(Y^q_0, \mathbb{E}[Y^q_0]), \mathbb{E}[Y^\mu_0 - Y^q_0]\rangle\big].\end{aligned}$$

Thus

$$\begin{aligned}\mathbb{J}(\mu_\cdot) - \mathbb{J}(q_\cdot) &\geq \langle \alpha_y(y^q_T, \mathbb{E}[y^q_T]), y^\mu_T - y^q_T\rangle + \mathbb{E}\big[\langle \alpha_{y'}(y^q_T, \mathbb{E}[y^q_T]), \mathbb{E}[y^\mu_T - y^q_T]\rangle\big] \\ &\quad + \langle \beta_Y(Y^q_0, \mathbb{E}[Y^q_0]), Y^\mu_0 - Y^q_0\rangle + \mathbb{E}\big[\langle \beta_{Y'}(Y^q_0, \mathbb{E}[Y^q_0]), \mathbb{E}[Y^\mu_0 - Y^q_0]\rangle\big] \\ &\quad + \mathbb{E}\Big[\int_0^T \Big(\int_U \ell(t, y^\mu_t, \mathbb{E}[y^\mu_t], Y^\mu_t, \mathbb{E}[Y^\mu_t], Z^\mu_t, \mathbb{E}[Z^\mu_t], u)\mu_t(du) \\ &\qquad - \int_U \ell(t, y^q_t, \mathbb{E}[y^q_t], Y^q_t, \mathbb{E}[Y^q_t], Z^q_t, \mathbb{E}[Z^q_t], u)q_t(du)\Big)dt\Big].\end{aligned}$$

Therefore after recalling also (4.28) one gets

$$\begin{aligned}\mathbb{J}(\mu_\cdot) - \mathbb{J}(q_\cdot) &\geq \mathbb{E}\big[\langle \Phi^q_T, y^\mu_T - y^q_T\rangle\big] \\ &\quad - \mathbb{E}\big[\langle h_y(y^q_T, \mathbb{E}[y^q_T])\Psi^q_T + \mathbb{E}\big[h_{y'}(y^q_T, \mathbb{E}[y^q_T])\mathbb{E}[\Psi^q_T]\big], y^\mu_T - y^q_T\rangle\big] \\ &\quad + \mathbb{E}\big[\langle \Psi^q_0, Y^\mu_0 - Y^q_0\rangle\big] \\ &\quad + \mathbb{E}\Big[\int_0^T \Big(\int_U \ell(t, y^\mu_t, \mathbb{E}[y^\mu_t], Y^\mu_t, \mathbb{E}[Y^\mu_t], Z^\mu_t, \mathbb{E}[Z^\mu_t], u)\mu_t(du) \\ &\qquad - \int_U \ell(t, y^q_t, \mathbb{E}[y^q_t], Y^q_t, \mathbb{E}[Y^q_t], Z^q_t, \mathbb{E}[Z^q_t], u)q_t(du)\Big)dt\Big]. \end{aligned} \tag{4.31}$$

Applying Itô's formula to $\langle \Phi^q_t, y^\mu_t - y^q_t\rangle$ and $\langle \Psi^q_t, Y^\mu_t - Y^q_t\rangle$, we obtain

$$\begin{aligned}\mathbb{E}\big[\langle \Phi^q_T, y^\mu_T - y^q_T\rangle\big] &= \mathbb{E}\Big[\int_0^T \langle \Phi^q_t, \int_U b(t, y^\mu_t, \mathbb{E}[y^\mu_t], u)\mu_t(du) \\ &\qquad - \int_U b(t, y^q_t, \mathbb{E}[y^q_t], u)q_t(du)\rangle dt\Big] \\ &\quad + \mathbb{E}\Big[\int_0^T \langle \Sigma^q_t, \sigma(t, y^\mu_t, \mathbb{E}[y^\mu_t]) - \sigma(t, y^q_t, \mathbb{E}[y^q_t])\rangle\, dt\Big] \\ &\quad - \mathbb{E}\Big[\int_0^T \langle H_y(t, \zeta^q_t, q_t, \chi^q_t) + \mathbb{E}[H_{y'}(t, \zeta^q_t, q_t, \chi^q_t)], y^\mu_t - y^q_t\rangle dt\Big], \end{aligned} \tag{4.32}$$

and

$$\begin{aligned}\mathbb{E}\big[\,\langle\Psi_0^q, Y_0^\mu - Y_0^q\rangle\,\big] &= \mathbb{E}\big[\,\langle\Psi_T^q, Y_T^\mu - Y_T^q\rangle\,\big]\\ &\quad - \mathbb{E}\Big[\int_0^T \langle H_Y(t,\zeta_t^q,q_t,\chi_t^q) + \mathbb{E}[H_{Y'}(t,\zeta_t^q,q_t,\chi_t^q)], Y_t^\mu - Y_t^q\rangle dt\Big]\\ &\quad + \mathbb{E}\Big[\int_0^T \langle\Psi_t^q, \int_U f(t,y_t^\mu,\mathbb{E}[y_t^\mu],Y_t^\mu,\mathbb{E}[Y_t^\mu],Z_t^\mu,\mathbb{E}[Z_t^\mu],u)\mu_t(du)\\ &\qquad\qquad - \int_U f(t,y_t^q,\mathbb{E}[y_t^q],Y_t^q,\mathbb{E}[Y_t^q],Z_t^q,\mathbb{E}[Z_t^q],u)q_t(du)\rangle dt\Big]\\ &\quad - \mathbb{E}\Big[\int_0^T \langle H_Z(t,\zeta_t^q,q_t,\chi_t^q) + \mathbb{E}[H_{Z'}(t,\zeta_t^q,q_t,\chi_t^q)], Z_t^\mu - Z_t^q\rangle dt\Big]\\ &\quad + \mathbb{E}\Big[\int_0^T \langle\Psi_t^q, g(t,y_t^\mu,\mathbb{E}[y_t^\mu],Y_t^\mu,\mathbb{E}[Y_t^\mu],Z_t^\mu,\mathbb{E}[Z_t^\mu])\\ &\qquad\qquad - g(t,y_t^q,\mathbb{E}[y_t^q],Y_t^q,\mathbb{E}[Y_t^q],Z_t^q,\mathbb{E}[Z_t^q])\rangle dt\Big].\end{aligned}\tag{4.33}$$

From the convexity of h we have

$$\begin{aligned}\mathbb{E}\big[\,\langle\Psi_T^q, Y_T^\mu - Y_T^q\rangle\,\big] &= \mathbb{E}\big[\,\langle\Psi_T^q, h(y_T^\mu,\mathbb{E}[y_T^\mu]) - h(y_T^q,\mathbb{E}[y_T^q])\rangle\,\big]\\ &\ge \mathbb{E}\big[\,\big\langle h_y(y_T^q,\mathbb{E}[y_T^q])\Psi_T^q + \mathbb{E}\big[h_{y'}(y_T^q,\mathbb{E}[y_T^q])\mathbb{E}[\Psi_T^q]\big], y_T^\mu - y_T^q\big\rangle\,\big].\end{aligned}\tag{4.34}$$

Replacing (4.32) and (4.33) in inequality (4.31) and using (4.34), we get

$$\begin{aligned}\mathbb{J}(\mu_\cdot) - \mathbb{J}(q_\cdot) &\ge \mathbb{E}\Big[\int_0^T \big(H(t,\zeta_t^q,\mu_t,\chi_t^q) - H(t,\zeta_t^q,q_t,\chi_t^q)\big)dt\\ &\quad - \mathbb{E}\Big[\int_0^T \langle H_y(t,\zeta_t^q,q_t,\chi_t^q) + \mathbb{E}[H_{y'}(t,\zeta_t^q,q_t,\chi_t^q)], y_t^\mu - y_t^q\rangle dt\Big]\\ &\quad - \mathbb{E}\Big[\int_0^T \langle H_Y(t,\zeta_t^q,q_t,\chi_t^q) + \mathbb{E}[H_{Y'}(t,\zeta_t^q,q_t,\chi_t^q)], Y_t^\mu - Y_t^q\rangle dt\Big]\\ &\quad - \mathbb{E}\Big[\int_0^T \langle H_Z(t,\zeta_t^q,q_t,\chi_t^q) + \mathbb{E}[H_{Z'}(t,\zeta_t^q,q_t,\chi_t^q)], Z_t^\mu - Z_t^q\rangle dt\Big].\end{aligned}\tag{4.35}$$

On the other hand, by the convexity of $H(t,y,y',Y,Y',Z,Z',q,\Phi,\Psi,\Sigma,\Pi)$ in (y,y',Y,Y',Z,Z') and its linearity in q, then by using the clarke generalized gradient of H evaluated at (y,y',Y,Y',Z,Z'), we obtain

$$\begin{aligned}H(t,\zeta_t^q,\mu_t,\chi_t^q) - H(t,\zeta_t^q,q_t,\chi_t^q) &\ge H_y(t,\zeta_t^q,q_t,\chi_t^q)\big(y_t^\mu - y_t^q\big)\\ &\quad + \mathbb{E}\big[H_{y'}(t,\zeta_t^q,q_t,\chi_t^q)\mathbb{E}[y_t^\mu - y_t^q]\big] + H_Y(t,\zeta_t^q,q_t,\chi_t^q)\big(Y_t^\mu - Y_t^q\big)\\ &\quad + \mathbb{E}\big[H_{Y'}(t,\zeta_t^q,q_t,\chi_t^q)\mathbb{E}[Y_t^\mu - Y_t^q]\big] + H_Z(t,\zeta_t^q,q_t,\chi_t^q)\big(Z_t^\mu - Z_t^q\big)\\ &\quad + \mathbb{E}\big[H_{Z'}(t,\zeta_t^q,q_t,\chi_t^q)\mathbb{E}[Z_t^\mu - Z_t^q]\big].\end{aligned}$$

Therefore, applying this inequality in (4.35) gives

$$\mathbb{J}(\mu_\cdot) - \mathbb{J}(q_\cdot) \ge 0, \forall\mu \in \mathfrak{R}.$$

The theorem is proved. □

4.2. Necessary and Sufficient optimality conditions for strict control

In this part, we shall derive necessary and sufficient optimality condition for strict control problem and shows that it follows from the relaxed one. This strict control problem is driven by the following

MF-FBDSDE

$$\begin{cases} y_t^v = x + \int_0^t b(s, y_s^v, \mathbb{E}[y_s^v], v_s)ds + \int_0^t \sigma(s, y_s^v, \mathbb{E}[y_s^v])dW_s \\ Y_t^v = h(y_T^v, \mathbb{E}[y_T^v]) + \int_t^T f(s, y_s^v, \mathbb{E}[y_s^v], Y_s^v, \mathbb{E}[Y_s^v], Z_s^v, \mathbb{E}[Z_s^v], v_s)ds \\ \qquad + \int_t^T g(s, y_s^v, \mathbb{E}[y_s^v], Y_s^v, \mathbb{E}[Y_s^v], Z_s^v, \mathbb{E}[Z_s^v])\overleftarrow{dB}_t - \int_t^T Z_s^v dW_s, \end{cases} \tag{4.36}$$

and the functional cost to be minimize over the set of strict controls $\mathcal{U}$ is given by

$$\begin{aligned} \mathbb{J}(v.) := \mathbb{E}\big[\alpha\left(y_T^v, \mathbb{E}\left[y_T^v\right]\right) + \beta\left(Y_0^v, \mathbb{E}\left[Y_0^v\right]\right) \\ + \int_0^T \ell\big(t, y_t^v, \mathbb{E}\left[y_t^v\right], Y_t^v, \mathbb{E}\left[Y_t^v\right], Z_t^v, \mathbb{E}\left[Z_t^v\right], v_t\big)dt\big]. \end{aligned} \tag{4.37}$$

We say that a strict control $u.$ is an optimal control if

$$\mathbb{J}(u.) = \inf_{v. \in \mathcal{U}} \mathbb{J}(v.). \tag{4.38}$$

We denote by

$$\mathcal{R}^\delta = \{\mu. \in \mathcal{R}/\ \ \mu = \delta_v : v \in \mathcal{U}\},$$

the set of all relaxed controls in the form of Dirac measure charging a strict control. Denote by $P(U^\delta)$ the action set of all relaxed control $\mathcal{R}^\delta$.

4.2.1. Necessary optimality conditions for strict control. Define the Hamiltonian $\mathcal{H}$ in the strict control problem from

$$[0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathbb{R}^{m\times d} \times U \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^{m\times l} \times \mathbb{R}^{n\times d},$$

to $\mathbb{R}$ by

$$\begin{aligned} \mathcal{H}(t, y, y', Y, Y', Z, Z', v, \Phi, \Psi, \Sigma, \Pi) := +\Phi b(t, y, y', v) + \ \Sigma\sigma(t, y, y') \\ +\Psi f(t, y, y', Y, Y', Z, Z', v) + \Pi g(t, y, y', Y, Y', Z, Z', v) \\ +\ell(t, y, y', Y, Y', Z, Z', v). \end{aligned} \tag{4.39}$$

Theorem 4.6. *(Necessary optimality conditions for strict control.) Let $u. \in \mathcal{U}$ an optimal strict control. Let (y^u, Y^u, Z^u) be the associated solution of MF-FBDSDE (4.36). Then there exists a unique solution $(\Phi^u, \Psi^u, \Sigma^u, \Pi^u)$ of the following adjoint equations of MF-FBDSDE (4.36):*

$$\begin{cases} d\Phi_t^u = -\big(\mathcal{H}_y(t, \zeta_t^u, u_t, \chi_t^u) + \mathbb{E}\big[\mathcal{H}_{y'}(t, \zeta_t^u, u_t, \chi_t^u)\big]\big)dt + \Sigma_t^u dW_t, \\ d\Psi_t^u = \big(\mathcal{H}_Y(t, \zeta_t^u, u_t, \chi_t^u) + \mathbb{E}\big[\mathcal{H}_{Y'}(t, \zeta_t^u, u_t, \chi_t^u)\big]\big)dt \\ \qquad + \big(\mathcal{H}_Z(t, \zeta_t^u, u_t, \chi_t^u) + \mathbb{E}\big[\mathcal{H}_{Z'}(t, \zeta_t^u, u_t, \chi_t^u)\big]\big)dW_t - \Pi_t^u \overleftarrow{dB}_t, \\ \Psi_0^u = \beta_Y(Y_0^u, \mathbb{E}[Y_0^u]) + \mathbb{E}\big[\beta_{Y'}(Y_0^u, \mathbb{E}[Y_0^u])\big], \\ \Phi_T^u = \alpha_y(y_T^u, \mathbb{E}[y_T^u]) + \mathbb{E}\big[\alpha_{y'}(y_T^u, \mathbb{E}[y_T^u])\big] \\ \qquad + h_y(y_T^u, \mathbb{E}[y_T^u])\Psi_T^u + \mathbb{E}\big[h_{y'}(y_T^u, \mathbb{E}[y_T^u])\mathbb{E}[\Psi_T^u]\big], \end{cases} \tag{4.40}$$

such that

$$\begin{aligned} &\mathcal{H}(t, y_t^u, \mathbb{E}[y_t^u], Y_t^u, \mathbb{E}[Y_t^u], Z_t^u, \mathbb{E}[Z_t^u], u_t, \Phi_t^u, \Psi_t^u, \Sigma_t^u, \Pi_t^u) \\ &\quad \le \mathcal{H}(t, y_t^u, \mathbb{E}[y_t^u], Y_t^u, \mathbb{E}[Y_t^u], Z_t^u, \mathbb{E}[Z_t^u], v_t, \Phi_t^u, \Psi_t^u, \Sigma_t^u, \Pi_t^u),\ a.e.\, t,\ P\text{-}a.s.,\ \forall v \in \mathcal{U}, \end{aligned} \tag{4.41}$$

where $(t, \zeta_t^u, u_t, \chi_t^u) := (t, y_t^u, \mathbb{E}[y_t^u], Y_t^u, \mathbb{E}[Y_t^u], Z_t^u, \mathbb{E}[Z_t^u], u_t, \Phi_t^u, \Psi_t^u, \Sigma_t^u, \Pi_t^u)$.

Proof. Note that the strict $u_.$ embedded into the space $\mathbb{V}$ in the sense that $u_.$ is corresponding with the Dirac measure $\lambda_{u_.}(dt, da) = \delta_{u_.}(du)$ with the propriety: For any bounded and uniformly continuous function $\hbar(t, y, y', Y, Y', Z, Z', u)$ we have

$$\begin{aligned}\hbar(t, y, y', Y, Y', Z, Z', u_t) &= \int_U \hbar(t, y, y', Y, Y', Z, Z', u)\delta_{u_t}(du)\\ &:= \widehat{\hbar}(t, y, y', Y, Y', Z, Z', \lambda_u).\end{aligned} \tag{4.42}$$

From the necessary optimality condition for relaxed controls (Theorem 4.4), there exist a unique solution $(\Phi_t^q, \Psi_t^q, \Sigma_t^q, \Pi_t^q)$ of (4.28) such that

$$\begin{aligned}&H(t, y_t^q, \mathbb{E}[y_t^q], Y_t^q, \mathbb{E}[Y_t^q], Z_t^q, \mathbb{E}[Z_t^q], q_t, \Phi_t^q, \Psi_t^q, \Sigma_t^q, \Pi_t^q)\\ &\quad\leq H(t, y_t^q, \mathbb{E}[y_t^q], Y_t^q, \mathbb{E}[Y_t^q], Z_t^q, \mathbb{E}[Z_t^q], \mu_t, \Phi_t^q, \Psi_t^q, \Sigma_t^q, \Pi_t^q),\ a.e.\,t,\ P\text{-}a.s.,\ \forall \mu \in \mathcal{R},\end{aligned}$$

and since $\mathcal{R}^\delta \subset \mathcal{R}$ we have

$$\begin{aligned}&H(t, y_t^q, \mathbb{E}[y_t^q], Y_t^q, \mathbb{E}[Y_t^q], Z_t^q, \mathbb{E}[Z_t^q], q_t, \Phi_t^q, \Psi_t^q, \Sigma_t^q, \Pi_t^q)\\ &\quad\leq H(t, y_t^q, \mathbb{E}[y_t^q], Y_t^q, \mathbb{E}[Y_t^q], Z_t^q, \mathbb{E}[Z_t^q], \mu_t, \Phi_t^q, \Psi_t^q, \Sigma_t^q, \Pi_t^q),\ a.e.\,t,\ P\text{-}a.s.,\ \forall \mu \in \mathcal{R}^\delta.\end{aligned} \tag{4.43}$$

Using the fact that if $\mu \in \mathcal{R}^\delta$, then there exist $v_t \in U^\delta \subset U$ such that $\mu = \delta_v$, and if the optimal relaxed control $q_t(du) = \delta_{u_t}(du)$ with u_t an optimal strict control, then we can show that

$$\begin{aligned}&(y_t^q, Y_t^q, Z_t^q) = (y_t^u, Y_t^u, Z_t^u),\ (y_t^\mu, Y_t^\mu, Z_t^\mu) = (y_t^v, Y_t^v, Z_t^v),\\ &(\Phi_t^q, \Psi_t^q, \Sigma_t^q, \Pi_t^q) = (\Phi_t^u, \Psi_t^u, \Sigma_t^u, \Pi_t^u),\ (\Phi_t^\mu, \Psi_t^\mu, \Sigma_t^\mu, \Pi_t^\mu) = (\Phi_t^v, \Psi_t^v, \Sigma_t^v, \Pi_t^v),\\ &H(t, y_t^q, \mathbb{E}[y_t^q], Y_t^q, \mathbb{E}[Y_t^q], Z_t^q, \mathbb{E}[Z_t^q], q_t, \Phi_t^q, \Psi_t^q, \Sigma_t^q, \Pi_t^q)\\ &\qquad\qquad = \mathcal{H}(t, y_t^u, \mathbb{E}[y_t^u], Y_t^u, \mathbb{E}[Y_t^u], Z_t^u, \mathbb{E}[Z_t^u], u_t, \Phi_t^u, \Psi_t^u, \Sigma_t^u, \Pi_t^u),\\ &H(t, y_t^\mu, \mathbb{E}[y_t^\mu], Y_t^\mu, \mathbb{E}[Y_t^\mu], Z_t^\mu, \mathbb{E}[Z_t^\mu], \mu_t, \Phi_t^\mu, \Psi_t^\mu, \Sigma_t^\mu, \Pi_t^\mu)\\ &\qquad\qquad = \mathcal{H}(t, y_t^v, \mathbb{E}[y_t^v], Y_t^v, \mathbb{E}[Y_t^v], Z_t^v, \mathbb{E}[Z_t^v], v_t, \Phi_t^v, \Psi_t^v, \Sigma_t^v, \Pi_t^v).\end{aligned} \tag{4.44}$$

Using (4.42) and (4.43) we get (4.41). The proof is completed. □

4.2.2. Sufficient optimality conditions for strict control. We shall try to shows if the necessary optimality conditions (4.41) for strict control problem $\{(4.36), (4.37), (4.38)\}$ becomes sufficient.

Theorem 4.7. *(Sufficient optimality conditions for strict control.) Assume that the functions* α, β, ℓ *and* $\mathcal{H}(t, \cdot, \cdot, \cdot, \cdot, u_t, \Phi_t^u, \Psi_t^u, \Sigma_t^u, \Pi_t^u)$ *are convex. Then* $(y_.^u, Y_.^u, Z_.^u, u_.)$ *is an optimal solution of the strict control problem* $\{(4.36), (4.37), (4.38)\}$ *if it satisfies (4.41).*

Proof. Let u_t be an arbitrary element of U^δ such that the necessary optimality conditions for strict control (4.41) hold, i.e.

$$\begin{aligned}&\mathcal{H}(t, y_t^u, \mathbb{E}[y_t^u], Y_t^u, \mathbb{E}[Y_t^u], Z_t^u, \mathbb{E}[Z_t^u], u_t, \Phi_t^u, \Psi_t^u, \Sigma_t^u, \Pi_t^u)\\ &\quad\leq \mathcal{H}(t, y_t^u, \mathbb{E}[y_t^u], Y_t^u, \mathbb{E}[Y_t^u], Z_t^u, \mathbb{E}[Z_t^u], v_t, \Phi_t^u, \Psi_t^u, \Sigma_t^u, \Pi_t^u),\ a.e.\,t,\ P\text{-}a.s.,\ \forall v \in \mathcal{U}^\delta,\end{aligned}$$

and by applying the embedding mentioned in (4.42), one can show that

$$\begin{aligned}&H(t, y_t^q, \mathbb{E}[y_t^q], Y_t^q, \mathbb{E}[Y_t^q], Z_t^q, \mathbb{E}[Z_t^q], q_t, \Phi_t^q, \Psi_t^q, \Sigma_t^q, \Pi_t^q)\\ &\quad\leq H(t, y_t^q, \mathbb{E}[y_t^q], Y_t^q, \mathbb{E}[Y_t^q], Z_t^q, \mathbb{E}[Z_t^q], \mu_t, \Phi_t^q, \Psi_t^q, \Sigma_t^q, \Pi_t^q),\ a.e.\,t,\ P\text{-}a.s.,\ \forall \mu \in \mathcal{R}^\delta.\end{aligned}$$

Thus by sufficient optimality conditions for relaxed control (Theorem 4.5) we have

$$J(q_.) = \inf_{\mu_. \in \mathcal{R}^\delta} J(\mu_.),$$

and from the fact that the optimal relaxed control is a Dirac measure charging in optimal strict control $(q_t(du) = \delta_{u_t}(du))$ and by using (4.44), we can show that

$$J(u_\cdot) = \inf_{v_\cdot \in \mathcal{U}^\delta} J(v_\cdot).$$

The prove is completed. □

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