Lacunary Statistical Convergence of Sequences of Complex Uncertain Variables

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ABSTRACT: In this paper, we introduce the notion of lacunary statistical convergence for the sequences of complex uncertain variables for almost sure, mean, measure and distribution. We investigate some of the basic properties of the notion. We have established relation between these notions.

Key Words: Complex uncertain variable, lacunary convergence, statistical convergence, uncertainty theory.

Contents

1. Introduction

Uncertainty theory was introduced by Liu [10] has now become a new branch of mathematics to deal with the randomness and fuzziness of the real world. Uncertainty theory is that branch of mathematics which is based on the axioms of normality, duality, subadditivity and product. Probability theory is based on the frequency of random events whereas uncertainty theory is based on the belief degree of a fact(or an event) to be true. It has been applied to different areas, such as uncertain programming, uncertain risk analysis, uncertain differential equation, uncertain finance, uncertain optimal control, uncertain game, uncertain graph etc. For more detail one may refer to Liu [8,9,10], Tripathy and Dowari [23] and others.

Complex uncertain variables are measurable functions from uncertainty spaces to the set of complex numbers. Convergence of sequences always plays a crucial role in different theory of mathematics. The convergence of complex uncertain sequence was first introduced by Chen,et. al [1]. Studies on convergence of sequences of uncertain variables is due to You [24]. The concept of statistical convergence, which is an extension of usual idea of convergence was introduced by Fast [4] and also independently by Schoenberg [14] for real and complex number sequences. The concept of convergence of sequences of numbers has been extended by several researcher (one may refer to Connor [2], Fridy [6], Salat [13]). Fridy and Orhan [7] extended the notion of statistical convergence to lacunary statistical convergence, and established some basic theorems. Recently lots of interesting developments have occurred in lacunary statistical convergence and related topics (One may refer to Tripathy and Baruah [17], Tripathy and Mahanta [15], Tripathy and Et [16], Tripathy and Dutta [18], Tripathy and Dutta [20,21]). Lacunary convergence has been investigated by Dutta et al. [3], Tripathy and Hazarika [19] and others.

The existing literature on statistical convergence have been studied from different aspects and extended up to sequences of fuzzy numbers [17] and sequences of uncertain variables by many researcher see for instance [11,12,22,25]. The main aim of this paper is to introduce the concept of lacunary statistical convergence concepts of complex uncertain variables. Our approach is a bit different from that of Fridy and Orhan [7].

2. Preliminaries

Some definitions and results of uncertainty theory and statistical convergence are procured in this section.

Definition 2.1. *[10] Let $\mathcal{L}$ be a $\sigma-$algebra on a nonempty set Γ. A set function $\mathcal{M}$ is called an uncertain measure if it satisfies the following axioms:*

Axiom 1 (Normality). $\mathcal{M}\{\Gamma\} = 1$.

Axiom 2 (Duality). $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any $\Lambda \in \mathcal{L}$.

Axiom 3 (Subadditivity). For every countable sequence of $\{\lambda_j\} \in \mathcal{L}$, we have

$$\mathcal{M}\left\{\bigcup_{j=1}^{\infty} \lambda_j\right\} \leq \sum_{j=1}^{\infty} \mathcal{M}\{\lambda_j\}.$$

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space, and each element Λ in $\mathcal{L}$ is called an event.

In order to obtain an uncertain measure of compound event, a product uncertain measure is defined by Liu [10] as follows:

Axiom 4 (Product). Let $(\Gamma_k, \mathcal{L}, \mathcal{M}_k)$ be uncertainty space for $k = 1, 2, 3, ...$ The product uncertain measure $\mathcal{M}$ is an measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\}$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, ...,$ respectively.

Definition 2.2. *[1] A complex uncertain variable is a measurable function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of complex numbers, i.e., for any Borel set $\mathcal{B}$ of complex numbers, the set*

$$\{\xi \in \mathcal{B}\} = \{\gamma \in \Gamma : \xi(\gamma) \in \mathcal{B}\}$$

is an event. When the range is the set of real numbers, we call it as an uncertain variable, introduced and investigated by Liu [10]. As a complex function on uncertainty space, complex uncertain variable is mainly used to model a complex uncertain quantity.

Definition 2.3. *[10] The expected value operator of an uncertain variable was defined by Liu as*

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \geq r\}dr - \int_{-\infty}^0 \mathcal{M}\{\xi \leq r\}dr,$$

provided that at least one of the two integrals is finite.

Definition 2.4. *[1] The complex uncertainty distribution $\Phi(x)$ of a complex uncertain variable ξ is a function from C to $[0,1]$ defined by*

$$\Phi(c) = \mathcal{M}\left\{Re(\xi) \leq Re(c), Im(\xi) \leq Im(c)\right\},$$

for any complex number c.

An uncertain variable is said to be positive, when it maps from $R_+ \bigcup \{0\}$ (non-negative real numbers) to $[0, 1]$. Considering the importance of convergence of sequence in mathematics, some concepts of convergence for complex uncertain sequences were introduced in Chen et.al [1]. Complex uncertain sequences are sequence of complex uncertain variables indexed by integers.

Definition 2.5. *[1] The complex uncertain sequence $\{\xi_n\}$ is said to be convergent almost surely(a.s.) to L if there exists an event Λ with $\mathcal{M}\{\Lambda\} = 1$ such that*

$$\lim_{n \to \infty} \|\xi_n(\gamma) - L(\gamma)\| = 0,$$

for every $\gamma \in \Lambda$. In that case we write $\xi_n \to L$, a.s.

Definition 2.6. *[1] The complex uncertain sequence $\{\xi_n\}$ is said to be convergent in measure to L if for a given $\varepsilon > 0$,*

$$\lim_{n \to \infty} \mathcal{M}\{\|\xi_n(\gamma) - L(\gamma)\| \geq \varepsilon\} = 0.$$

Definition 2.7. *[1] The complex uncertain sequence $\{\xi_n\}$ is said to be convergent in mean to L if*

$$\lim_{n \to \infty} E[\|\xi_n(\gamma) - L(\gamma)\|] = 0.$$

Definition 2.8. *[1] Let $\Phi_1, \Phi_2, \Phi_3, ...$ be the complex uncertainty distributions of complex uncertain variables $\xi_1, \xi_2, \xi_3...$, respectively. We say the complex uncertain sequence $\{\xi_n\}$ converges in distribution to L if*

$$\lim_{n \to \infty} \Phi_n(c) = \Phi(c),$$

for all $c \in C$, at which $\Phi(c)$ is continuous.

Definition 2.9. *[1] The complex uncertain sequence $\{\xi_n\}$ is said to be convergent uniformly almost surely (u.a.s.) to L if there exists a sequence of events $\{E_k'\}$, $\mathcal{M}\{E_k'\} \to 0$ such that $\{\xi_n\}$ converges uniformly to L in $\Gamma - E_k'$, for any fixed $k \in N$.*

The notion of statistical convergence depends on the density of the subsets of the set $\mathbb{N}$ of natural numbers. The following three definitions are well known (one may refer to Tripathy and Et [16].

Definition 2.10. *A subset A of N is said to have density $\delta(A)$ if*

$$\delta(A) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \chi_A(k), \ provided \ the \ limit \ exists,$$

where χ_A is the characteristic function of A.

Equivalently, if A is a subset of N, then A_n denotes the set $\{k \in A : k \leq n\}$ and $|A_n|$ denotes the cardinality of A_n. The natural density of A is given by

$$\delta(A) = \lim_{n \to \infty} \frac{1}{n} |A_n|,$$

i.e. $\delta(A) = \lim_{n \to \infty} \frac{1}{n} |\{k \in A : k \leq n\}|.$

It can be noted that $\delta(A^c) = 1 - \delta(A)$ whenever either side exists and where $A^c = N - A$.

Definition 2.11. *A real sequence $\{x_n\}$ is said to be statistically convergent to L, written as $stat-\lim x_n = L$, if for every $\varepsilon > 0$,*

$$\delta(\{k \in N : |x_k - L| \geq \varepsilon\}) = 0$$

.

Definition 2.12. *A real sequence $\{x_n\}$ is said to be statistically divergent to ∞ if for any real number J,*

$$\delta(\{k \in N : x_k > J\}) = 1.$$

It is said to be statistically divergent to $-\infty$ if for any real number Q,

$$\delta(\{k \in N : x_k < Q\}) = 1.$$

By a lacunary sequence $\theta = \{k_r\}$; where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $k_r - k_{r-1} \to \infty$, as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and $q_r = \frac{k_r}{k_{r-1}}$, $h_r = k_r - k_{r-1}$ for $r = 1, 2, 3, ...$ Lacunary sequences has been introduced by Freedman et.al [5]. Fridy and Orhan [7] and Tripathy and Baruah [17] have studied the notion of lacunary statistical convergent.

3. Definitions

In this section we extend the notions of strongly Cesàro summable sequences with general lacunary θ to the idea of sequences of complex uncertain variables. Different types of new classes of lacunary statistical convergence of sequences of complex uncertain variables are introduced.

Definition 3.1. *The complex uncertain sequence $\xi = \{\xi_k\}$ is said to be lacunary statistically convergent to L if for every $\varepsilon > 0$,*

$$\lim_{n \to \infty} \frac{1}{n} \left| \left\{ r \le n : \left\| \frac{1}{h_r} \sum_{k \in I_r} \xi_k(\gamma) - L(\gamma) \right\| \ge \varepsilon \right\} \right| = 0,$$

where the vertical bars denote the cardinality of the set.

We denote this by $\xi_k \to L(S_\theta^U)$ or $S_\theta^U - \lim \xi_k = L$. The set of all lacunary statistically convergent sequences of complex uncertain variables is denoted by S_θ^U.

Definition 3.2. *The complex uncertain sequence $\{\xi_k\}$ is said to be lacunary statistically strongly convergent almost surely to L if for every $\varepsilon > 0$ there exists an event Λ with $\mathcal{M}\{\Lambda\} = 1$ such that*

$$\lim_{n \to \infty} \frac{1}{n} \left| \left\{ r \le n : \frac{1}{h_r} \sum_{k \in I_r} \|\xi_k(\gamma) - L(\gamma)\| \ge \varepsilon \right\} \right| = 0,$$

for every $\gamma \in \Lambda$.

Definition 3.3. *The complex uncertain sequence $\{\xi_k\}$ is said to be lacunary statistically convergent in measure to L if*

$$\lim_{n \to \infty} \frac{1}{n} \left| \left\{ r \le n : \mathcal{M} \left[\left\{ \gamma \in \Gamma : \frac{1}{h_r} \sum_{k \in I_r} \|\xi_k(\gamma) - L(\gamma)\| \ge \varepsilon \right\} \right] \ge \delta \right\} \right| = 0,$$

for every $\varepsilon, \delta > 0$.

Definition 3.4. *The complex uncertain sequence $\{\xi_k\}$ is said to be lacunary statistically convergent in mean to L if*

$$\lim_{n \to \infty} \frac{1}{n} \left| \left\{ r \le n : E \left[\frac{1}{h_r} \sum_{k \in I_r} \|\xi_k(\gamma) - L(\gamma)\| \right] \ge \varepsilon \right\} \right| = 0,$$

for every $\varepsilon > 0$.

Definition 3.5. *Let* $\Phi_1, \Phi_2, \Phi_3, ...$ *be the complex uncertainty distributions of complex uncertain variables* $\xi_1, \xi_2, \xi_3, ...,$ *respectively. We say the complex uncertain sequence* $\{\xi_k\}$ *lacunary statistically convergent in distribution to L if for every* $\varepsilon > 0$,

$$\lim_{n \to \infty} \frac{1}{n} \left| \left\{ r \leq n : \frac{1}{h_r} \sum_{k \in I_r} \|\Phi_k(c) - \Phi(c)\| \geq \varepsilon \right\} \right| = 0,$$

for all complex c at which $\Phi(c)$ *is continuous.*

4. Main Results

In this section we establish the results of this article.

Theorem 4.1. *Let* $\theta = \{k_r\}$ *be a lacunary sequence and* $\xi = \{\xi_k\}$ *be a sequence of complex uncertain variables. Then,*

$$\xi_k \to L(N_\theta^U) \text{ implies } \xi_k \to L(S_\theta^U) \text{ i.e. } N_\theta^U \subset S_\theta^U.$$

Proof. Let $\varepsilon > 0$, be given and $\xi \in N_\theta^U$, then we can write

$$\frac{1}{h_r} \sum_{k \in I_r} \|\xi_k(\gamma) - L(\gamma)\|$$
$$\geq \sum_{k \in I_r} \|\xi_k(\gamma) - L(\gamma)\| \|\xi_k(\gamma) - L(\gamma)\|$$
$$\geq \varepsilon . \frac{1}{n} \left| \left\{ r \leq n : \frac{1}{h_r} \sum_{k \in I_r} \|\xi_k(\gamma) - L(\gamma)\| \geq \varepsilon \right\} \right|.$$

Hence $\xi \in S_\theta^U$. $\qquad\qquad\square$

Theorem 4.2. *Let* $\xi = \{\xi_k\}$ *and* $\eta = \{\eta_k\}$ *be sequences of complex uncertain variables.*

(i) If $S_\theta^U - \lim \xi_k = \xi_0$ *and* $c \in R$, *then* $S_\theta^U - \lim c\xi_k = c\xi_0$.

(ii) If $S_\theta^U - \lim \xi_k = \xi_0$ *and* $S_\theta^U - \lim \eta_k = \eta_0$, *then If* $S_\theta^U - \lim(\xi_k + \eta_k) = \xi_0 + \eta_0$.

Proof. (i) Let $\xi \in S_\theta^U$ and $c \in R$. Then

$$\sum_{k \in I_r} \|c\xi_k(\gamma) - cL(\gamma)\| = |c| \sum_{k \in I_r} \|\xi_k(\gamma) - L(\gamma)\|$$

For a given $\varepsilon > 0$ we have,

$$\lim_{n \to \infty} \frac{1}{n} \left| \left\{ r \leq n : \frac{1}{h_r} \sum_{k \in I_r} \|\xi_k(\gamma) - L(\gamma)\| \geq \varepsilon \right\} \right| \leq \lim_{n \to \infty} \frac{1}{n} \left| \left\{ r \leq n : \frac{1}{h_r} \sum_{k \in I_r} \|\xi_k(\gamma) - L(\gamma)\| \geq \frac{\varepsilon}{|c|} \right\} \right|.$$

Hence $S_\theta^U - \lim c\xi_k = c\xi_0$.

(ii) Suppose that $S_\theta^U - \lim \xi_k = \xi_0$ and $S_\theta^U - \lim \eta_k = \eta_0$. We have,

$$\sum_{k \in I_r} \|\xi_k(\gamma) + \eta_k(\gamma)\| \leq \left(\sum_{k \in I_r} \|\xi_k(\gamma)\| + \sum_{k \in I_r} \|\eta_k(\gamma)\| \right).$$

Therefore for given $\varepsilon > 0$, we have

$$\frac{1}{n} \left| \left\{ r \leq n : \frac{1}{h_r} \sum_{k \in I_r} \|\xi_k(\gamma) + \eta_k(\gamma)\| \geq \varepsilon \right\} \right|$$
$$\leq \frac{1}{n} \left| \left\{ r \leq n : \frac{1}{h_r} \left(\sum_{k \in I_r} \|\xi_k(\gamma)\| + \sum_{k \in I_r} \|\eta_k(\gamma)\| \right) \geq \varepsilon \right\} \right|$$
$$\leq \frac{1}{n} \left| \left\{ r \leq n : \frac{1}{h_r} \sum_{k \in I_r} \|\xi_k(\gamma)\| \right\} \geq \frac{\varepsilon}{2} \right\} + \frac{1}{n} \left| \left\{ r \in N : \frac{1}{h_r} \sum_{k \in I_r} \|\eta_k(\gamma)\| \geq \frac{\varepsilon}{2} \right\} \right|$$
$$= \xi_0 + \eta_0 \qquad\qquad\square$$

Theorem 4.3. *If the complex uncertain sequence $\{\xi_k\}$ lacunary statistically converges in mean to L, then $\{\xi_k\}$ lacunary strongly converges in measure to L.*

Proof. It follows from the Markov's inequality that for any given $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ r \leq n : \mathcal{M}\left[\left\{ \gamma \in \Gamma : \frac{1}{h_r} \sum_{k\in I_r} \|\xi_k(\gamma) - L(\gamma)\| \geq \varepsilon \right\} \right] \geq \delta \right\} \right|$$

$$\leq \lim_{r\to\infty} \frac{1}{n} \left| \left\{ r \leq n : \frac{E[\frac{1}{h_r}\sum_{k\in I_r} \|\xi_k(\gamma) - L(\gamma)\|]}{\varepsilon} \geq \delta \right\} \right| \to 0.$$

Thus $\{\xi_k\}$ lacunary statistically converges in measure to L and the theorem is thus proved. □

Remark 4.1 But the converse of the above theorem is not necessarily true. i.e. lacunary statistically convergence in measure does not imply lacunary statistically convergence in mean. This can be illustrated by the example given below.

Example 4.1 Consider the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, ...\}$ with power set and

$$\mathcal{M}\{\Lambda\} = \begin{cases} \sup_{\gamma_i \in \Lambda} \frac{1}{i}, & \text{if } \sup_{\gamma_i \in \Lambda} \frac{1}{i} < 0.5; \\ 1 - \sup_{\gamma_i \in \Lambda^c} \frac{1}{i}, & \text{if } \sup_{\gamma_i \in \Lambda^c} \frac{1}{i} < 0.5; \\ 0.5, & \text{otherwise,} \end{cases}$$

and the complex uncertain variables be defined by

$$\xi_i(\gamma_j) = \begin{cases} i, & \text{if } j = i; \\ 0, & \text{otherwise,} \end{cases}$$

for $i \in I_r$ and $L \equiv 0$. For $\varepsilon > 0$, we have

$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ r \leq n : \mathcal{M}\left[\left\{ \gamma \in \Gamma : \frac{1}{h_r} \sum_{k\in I_r} \|\xi_k(\gamma) - L(\gamma)\| > \varepsilon \right\} \right] \geq \delta \right\} \right|$

$= \lim_{n\to\infty} \frac{1}{n} \left| \left\{ r \leq n : \mathcal{M}\left[\left\{ \gamma \in \Gamma : \frac{1}{h_r} \sum_{k\in I_r} \|\xi_k(\gamma)\| > \varepsilon \right\} \right] \geq \delta \right\} \right|$

$= \lim_{n\to\infty} \mathcal{M}(\{\gamma_i\})$

$= \lim_{n\to\infty} \frac{1}{i} \to 0$ (since $i \in I_r$).

The sequence $\{\xi_i\}$ is lacunary strongly convergent in measure to L.
However, for each $i \in I_r$, we have the uncertainty distribution of uncertain variable $\|\xi_i - L\| = \|\xi_i\|$ is

$$\Phi_i(x) = \begin{cases} 0, & \text{if } x < 0; \\ 1 - \frac{1}{i}, & \text{if } 0 \leq x < i; \\ 1, & \text{otherwise.} \end{cases}$$

Then we have,

$E\left[\frac{1}{h_r} \sum_{k\in I_r} M\|\xi_k(\gamma) - L(\gamma)\| \right]$

$= \int_0^{+\infty} \mathcal{M}\{\xi \geq x\}dx - \int_{-\infty}^0 \mathcal{M}\{\xi \leq x\}dx$

$= \int_0^i 1 - (1 - \frac{1}{i})dx$

$= 1.$

That is, the sequence $\{\xi_i(\gamma)\}$ does not converge in mean to $L(\gamma)$.

Theorem 4.4. *Let the complex uncertain sequence $\{\xi_k\}$ has the real part $\{\zeta_k\}$ and imaginary part $\{\eta_k\}$. If uncertain sequences $\{\zeta_k\}$ and $\{\eta_k\}$ are lacunary statistically convergent in measure to L_1 and L_2 respectively, then complex uncertain sequence $\{\xi_k\}$ lacunary statistically uniformly convergent in distribution to $L = L_1 + iL_2$.*

Proof. Let $c = a + ib$ be a point at which the complex uncertainty distribution Φ is continuous. For any $\alpha > a, \beta > b$, we have,

$$\{\zeta_k \le a, \eta_k \le b\} = \{\zeta_k \le a, \eta_k \le b, \mathrm{L}_1 \le \alpha, \mathrm{L}_2 \le \beta\} \cup \{\zeta_k \le a, \eta_k \le b, \mathrm{L}_1 > \alpha, \mathrm{L}_2 > \beta\}$$

$$\cup\{\zeta_k \le a, \eta_k \le b, \mathrm{L}_1 \le \alpha, \mathrm{L}_2 > \beta\} \cup \{\zeta_k \le a, \eta_k \le b, \mathrm{L}_1 > \alpha, \mathrm{L}_2 \le \beta\}$$

$$\subset \{\mathrm{L}_1 \le \alpha, \mathrm{L}_2 \le \beta\} \cup \{\|\zeta_k(\gamma) - \mathrm{L}_1(\gamma)\| \ge \alpha - a\} \cup \{\|\eta_k(\gamma) - \mathrm{L}_2(\gamma)\| \ge \beta - b\}.$$

It follows from the subadditivity axiom that
$$\Phi_k(c) = \Phi_k(a + ib)$$
$$\le \Phi(\alpha + i\beta) + \mathcal{M}\left\{\gamma \in \Gamma : \frac{1}{h_r}\sum_{k \in I_r} \|\zeta_k(\gamma) - L_1(\gamma)\| \ge \alpha - a\right\}$$
$$+\mathcal{M}\left\{\gamma \in \Gamma : \frac{1}{h_r}\sum_{k \in I_r} \|\eta_k(\gamma) - L_2(\gamma)\| \ge \beta - b\right\}.$$

Since $\{\zeta_k\}$ and $\{\eta_k\}$ are lacunary statistically convergent in measure to L_1 and L_2 respectively. So for $\varepsilon > 0$ and $k \in I_r$ we have,

$$\lim_{n \to \infty} \frac{1}{n}\left|\left\{r \le n : \mathcal{M}\left[\left\{\gamma \in \Gamma : \frac{1}{h_r}\sum_{k \in I_r} \|\zeta_k(\gamma) - L_1(\gamma)\| \ge (\alpha - a)\right\}\right] \ge \varepsilon\right\}\right| = 0$$

$$\text{and} \quad \lim_{n \to \infty} \frac{1}{n}\left|\left\{r \le n : \mathcal{M}\left[\left\{\gamma \in \Gamma : \frac{1}{h_r}\sum_{k \in I_r} \|\eta_k(\gamma) - L_2(\gamma)\| \ge (\beta - b)\right\}\right] \ge \varepsilon\right\}\right| = 0.$$

Thus we have,
$$\limsup_{r \to \infty} \Phi_k(c) \le \Phi(\alpha + i\beta),$$

for any $\alpha > a, \beta > b$.
Taking $\alpha + i\beta \to a + ib$, we get,

$$\limsup_{r \to \infty} \Phi_k(c) \le \Phi(c). \tag{4.1}$$

On the other hand, for any $x < a, y < b$ we have,

$$\{L_1 \le x, L_2 \le y\}$$
$$= \{\zeta_k \le a, \eta_k \le b, L_1 \le x, L_2 \le y\} \cup \{\zeta_k \le a, \eta_k \le b, L_2 \le x, L_2 \le y\}$$

$$\cup\{\zeta_k > a, \eta_k \le b, L_1 \le x, L_2 \le y\} \cup \{\zeta_k > a, \eta_k > b, L_1 \le x, L_2 \le y\}$$

$$\subset \{\zeta_k \le a, \eta_k \le b\} \cup \left\{\frac{1}{h_r}\sum_{k \in I_r} \|\zeta_k - L_1\| \ge a - x\right\} \cup \left\{\frac{1}{h_r}\sum_{k \in I_r} \|\eta_k - L_2\| \ge b - y\right\}.$$

which implies,

$$\Phi(x + iy)$$
$$\le \Phi_k(a + ib) + \mathcal{M}\left\{\gamma \in \Gamma : \frac{1}{h_r}\sum_{k \in I_r} \|\zeta_k(\gamma) - L_1(\gamma)\| \ge a - x\right\}$$
$$+\mathcal{M}\left\{\gamma \in \Gamma : \frac{1}{h_r}\sum_{k \in I_r} \|\eta_k(\gamma) - L_2(\gamma)\| \ge b - y\right\}.$$

We have,

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ r \le n : \mathcal{M}\left[\left\{ \gamma \in \Gamma : \frac{1}{h_r} \sum_{k \in I_r} \|\zeta_k(\gamma) - L_1(\gamma)\| \ge (a - x) \right\} \right] \ge \varepsilon \right\} \right| = 0$$

and

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ r \le n : \mathcal{M}\left[\left\{ \gamma \in \Gamma : \frac{1}{h_r} \sum_{k \in I_r} \|\eta_k(\gamma) - L_2(\gamma)\| \ge (b - y) \right\} \right] \ge \varepsilon \right\} \right| = 0.$$

Thus we have,

$$\Phi(x + iy) \le \lim_{r\to\infty} \inf \Phi_k(a + ib)$$

for any $x < a, y < b$.

Taking $x + iy \to a + ib$, we get

$$\Phi(c) \le \liminf_{n\to\infty} \Phi_k(c). \tag{4.2}$$

It follows from (4.1) and (4.2) that $\Phi_k(c) \to \Phi(c)$ as $r \to \infty$ and $k \in I_r$. That is the complex uncertain sequence $\{\xi_k\}$ is lacunary statistically convergent in distribution to $L = L_1 + iL_2$. $\square$

Remark 4.2 The converse of the above theorem is not necessarily true in general. That is lacunary statistically convergent in distribution does not imply lacunary statistically convergence in measure. The following example illustrates this.

Example 4.2 Consider the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2\}$ with $\mathcal{M}\{\gamma_1\} = \mathcal{M}\{\gamma_2\} = \frac{1}{2}$. Define a complex uncertain variable ξ by

$$\xi(\gamma) = \begin{cases} 1, & \text{if } \gamma = \gamma_1; \\ -1, & \text{if } \gamma = \gamma_2. \end{cases}$$

Define $\{\xi_k\}$ by $\xi_k = -\xi$, for $k \in I_r$. Then $\{\xi_k\}$ and ξ have the same distribution and thus $\{\xi_k\}$ converges in distribution to ξ. However, for any given $\varepsilon, \delta > 0$, we have

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ r \le n : \mathcal{M}\left[\left\{ \gamma \in \Gamma : \frac{1}{h_r} \sum_{k \in I_r} \|\xi_k(\gamma) - \xi(\gamma)\| > \varepsilon \right\} \right] \ge \delta \right\} \right|$$

$$= \lim_{n\to\infty} \frac{1}{n} \left| \left\{ r \le n : \mathcal{M}\left[\left\{ \gamma \in \Gamma : \frac{1}{h_r} \sum_{k \in I_r} \|2\xi_k(\gamma)\| > \varepsilon \right\} \right] \ge \delta \right\} \right|$$

$$\ne 0.$$

Therefore, the sequence $\{\xi_k\}$ does not lacunary statistically converge in measure to ξ.

Theorem 4.5. *Let $\xi_1, \xi_2, \xi_3, \dots$ be complex uncertain variables. Then $\{\xi_k\}$ is lacunary statistically convergent almost surely to L if and only if for any $\varepsilon > 0$, we have,*

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ r \le n : \mathcal{M}\left[\bigcap_{r \in I_{r_k}} \bigcup_{k \in I_r} \{\gamma \in \Gamma : \frac{1}{h_r} \sum_{k \in I_r} \|\xi_k(\gamma) - L(\gamma)\| > \varepsilon \} \right] \ge \delta \right\} \right| = 0.$$

Proof. By the definition of lacunary strongly convergent almost surely we have that there exists an event Λ with $\mathcal{M}(\Lambda) = 1$, such that

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ r \leq n : \mathcal{M}\left[\left\{ \gamma \in \Gamma : \frac{1}{h_r} \sum_{k\in I_r} \|\xi_k(\gamma) - L(\gamma)\| \geq \varepsilon \right\} \right] \geq \delta \right\} \right| = 0,$$

for every $\gamma \in \Lambda$. Then for any $\varepsilon > 0$ there exists m such that $\frac{1}{h_r}\sum_{k\in I_r} \|\xi_k(\gamma) - L(\gamma)\| < \varepsilon$ where $k > m$, for any $\gamma \in \Lambda$, which is equivalent to

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ r \leq n : \mathcal{M}\left[\bigcup_{r\in I_{r_k}} \bigcap_{k\in I_r} \left\{ \gamma \in \Gamma : \frac{1}{h_r} \sum_{k\in I_r} \|\xi_k(\gamma) - L(\gamma)\| > \varepsilon \right\} \right] \geq \delta \right\} \right| = 1$$

But using the duality axiom it follows that

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ r \leq n : \mathcal{M}\left[\bigcap_{r\in I_{r_k}} \bigcup_{k\in I_r} \left\{ \gamma \in \Gamma : \frac{1}{h_r} \sum_{k\in I_r} \|\xi_k(\gamma) - L(\gamma)\| > \varepsilon \right\} \right] \geq \delta \right\} \right| = 0$$

Hence the result is proved. □

Theorem 4.6. *Let $\xi_1, \xi_2, \xi_3, \dots$ be complex uncertain variables. If $\{\xi_k\}$ is lacunary statistically convergent uniformly almost surely to L, then $\{\xi_k\}$ is lacunary statistically convergent in measure to L.*

Proof. If $\{\xi_k\}$ is lacunary statistically convergent uniformly almost surely to L then

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ \mathcal{M}\left[\bigcup_{k\in I_r} \left\{ \gamma \in \Gamma : \frac{1}{h_r} \sum_{k\in I_r} \|\xi_k(\gamma) - L(\gamma)\| > \varepsilon \right\} \right] \geq \delta \right\} \right| = 0,$$

from the above theorem.
But,

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ \mathcal{M}\left[\left\{ \gamma \in \Gamma : \frac{1}{h_r} \sum_{k\in I_r} \|\xi_k(\gamma) - L(\gamma)\| > \varepsilon \right\} \right] \geq \delta \right\} \right|$$

$$\leq \lim_{n\to\infty} \frac{1}{n} \left| \left\{ \mathcal{M}\left[\bigcup_{k\in I_r} \left\{ \gamma \in \Gamma : \frac{1}{h_r} \sum_{k\in I_r} \|\xi_k(\gamma) - L(\gamma)\| > \varepsilon \right\} \right] \geq \delta \right\} \right|.$$

Therefore, $\{\xi_k\}$ is lacunary statistically convergent in measure to L. □

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