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Blow up Result for a Viscoelastic Plate Equation with Nonlinear Source*

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ABSTRACT: We consider a viscoelastic plate equation with nonlinear source and partially hinged boundary conditions. Our goal is to show analytically that the solution blows up in finite time. The background of the problem comes from the modeling of the downward displacement of a suspension bridge using a thin rectangular plate. This result shows that in the present of a nonlinear source such as the earthquake shocks, the bridge will collapse in a finite time.

Key Words: Blow up, suspension bridge, plate equation, viscoelastic, fourth-order.

Contents

1. Introduction

In this paper, we consider the following problem

$$\begin{cases} u_{tt}(x,y,t) + \Delta^2 u(x,y,t) - \displaystyle\int_0^t g(t-s)\Delta^2 u(x,y,s)ds \\ \qquad\qquad + h(u_t(x,y,t)) = u(x,y,t)|u(x,y,t)|^{p-2}, \text{in } \Omega\times(0,T), \\ u(0,y,t) = u_{xx}(0,y,t) = 0, & \text{for } (y,t)\in(-\ell,\ell)\times(0,T), \\ u(L,y,t) = u_{xx}(L,y,t) = 0, & \text{for } (y,t)\in(-\ell,\ell)\times(0,T), \\ u_{yy}(x,\pm\ell,t) + \nu u_{xx}(x,\pm\ell,t) = 0, & \text{for } (x,t)\in(0,L)\times(0,T), \\ u_{yyy}(x,\pm\ell,t) + (2-\nu)u_{xxy}(x,\pm\ell,t) = 0, & \text{for } (x,t)\in(0,L)\times(0,T), \\ u(x,y,0) = u_0(x,y), \quad u_t(x,y,0) = u_1(x,y), & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $u = u(x,y,t)$ is the downward displacement of a suspension bridge, $\Omega = (0,L)\times(-\ell,\ell)\subset\mathbb{R}^2$, $0<\nu<\frac{1}{2}$, g and h are given functions to be specified later, $p>2$ and u_0, u_1 are given data. Our aim is to show that that the solution of problem (1.1) blows up in a finite time. This model is formulated from the ground work of Ferrero and Gazzola [3], where a suspension bridge is modelled through a rectangular thin plate Ω assumed to be hinged on the vertical edges

$$u(0,y) = u_{xx}(0,y) = u(L,y) = u_{xx}(L,y),\ \forall y\in(-\ell,\ell)$$

and free on the horizontal edges

$$u_{yy}(x,\pm\ell) + \nu u_{xx}(x,\pm\ell) = u_{yyy}(x,\pm\ell) + (2-\nu)u_{xxy}(x,\pm\ell) = 0,\ \forall x\in(0,L).$$

The present model (1.1), takes into consideration the viscoelatic damping of the material and also the appearance of nonlinear external source. In the presence of a non-linear external force such as extreme earthquake shocks, the suspension bridge is set into unstable oscillations leading to it collapsing. See

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videos available on the web [22] of the Tacoma Narrows bridge collapsing. This, unfortunately, is not the only case of bridges collapsing, many other bridges have collapsed in history, see for example [1,8]. For $g = 0$, $h(u_t) = \delta u_t, \delta > 0$ and a source term $f \in L^2(\Omega \times (0,T))$ in (1.1), Ferrero and Gazzola [3] established the existence and uniqueness of a global solution and discussed many stationary problems. Wang [21] studied the following plate equation

$$u_{tt} + \delta u_t + \Delta^2 u + au = |u|^{m-2}u, \tag{1.2}$$

where $a = a(x,y,t)$ is a bounded and measurable sign-changing function and $2 < m < +\infty$. He supplemented (1.2) with the boundary conditions of (1.1) and initial data and proved the existence and uniqueness of a local solution and a finite-time blow-up result. Gazzola and Wang [5] modelled suspension bridges through the Von Karman quasilinear plate equations. Messaoudi in [20] considered the Petrovsky system

$$\begin{cases} u_{tt} + \Delta^2 u + a|u_t|^{m-2}u_t = b|u|^{p-2}u, & \text{in} \quad \Omega \times (0,T), \\ u = \frac{\partial u}{\partial \eta} = 0, & \text{on} \quad \partial\Omega \times [0,T), \\ u(x,0) = v_0(x), \quad u_t(x,0) = v_1(x), & \text{in} \quad \Omega, \end{cases} \tag{1.3}$$

where $a, b > 0$ are constants and $\Omega \subset \mathbb{R}^{\mathrm{N}}$, $\mathrm{N} \geq 1$, is a bounded domain with a smooth boundary $\partial\Omega$. For $p > m$, he established the existence and uniqueness of a weak local solution. In addition, he proved that for negative initial energy ($E(0) < 0$), the local solution blows up in finite time. He also established the existence of global solution when $m \geq p$. The result in [20] has been improved by Chen and Zhou in [2]. We refer the reader to [14,15,12,13,16,17,21] and references therein for results related to problem (1.1). The paper is organized as follows: In Section 2, we introduce some fundamental materials and useful assumptions on the relaxation function g and the function h. In Section 3, we state and prove some technical lemmas. Finally, in Section 4, we establish a blow-up result for problem (1.1).

2. Preliminaries

Throughout the paper, C or c are generic positive constants that may change within lines. We recall some useful materials and state our assumptions. For this, we assume the functions g and h admit the following conditions:
$(A1)$ $g : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a decreasing C^1- function such that

$$1 - \int_0^\infty g(s)ds = l_0 > 0. \tag{2.1}$$

$(A2)$ $h : \mathbb{R} \to \mathbb{R}$ is an increasing C^1- function with $h(0) = 0$ such that

$$\begin{cases} |\xi(s)| \leq |h(s)| \leq |\xi^{-1}(s)|, \quad |s| \leq 1, \\ \\ c_1|s|^{m-1} \leq |h(s)| \leq c_2|s|^{m-1}, \quad |s| > 1, \end{cases} \tag{2.2}$$

where $\xi : [-1,1] \to \mathbb{R}$ is an increasing and odd function, $c_1, c_2 > 0$ are constants,
$2 \leq m < p < +\infty$ and ξ^{-1} denotes the inverse of ξ.
We consider the Hilbert space (see [3])

$$H_*^2(\Omega) = \left\{ w \in H^2(\Omega) : w = 0 \text{ on } \{0, L\} \times (-\ell, \ell) \right\},$$

together with the inner product

$$(u,v)_{H_*^2(\Omega)} = \int_\Omega [(\Delta u \Delta v + (1-\nu)(2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx})]dxdy,$$

and denote by $\mathcal{H}(\Omega)$ the dual of $H_*^2(\Omega)$.

Lemma 2.1. *[21] Suppose that $1 \leq p < +\infty$. Then, there exists an embedding constant $S_p = S_p(\Omega, p) > 0$ such that*

$$\|u\|_{L^p(\Omega)} \leq S_p \|u\|_{H^2_*(\Omega)}, \quad \forall\ u \in H^2_*(\Omega). \tag{2.3}$$

For completeness, we state without proof a local existence result. The proof can be established using similar methods as in [6,7,10,18].

Theorem 2.2. *Let $(u_0, u_1) \in H^2_*(\Omega) \times L^2(\Omega)$ be given. Assume g and h satisfy (A1) and (A2). Then, there exists a unique local solution to problem (1.1) in the class*

$$u \in L^\infty\left([0, T_{max}), H^2_*(\Omega)\right), u_t \in L^\infty\left([0, T_{max}), L^2(\Omega)\right) \cap L^m(\Omega \times (0, T_{max})),$$
$$u_{tt} \in L^\infty([0, T_{max}), \mathcal{H}(\Omega)), \quad \text{for some} \quad T_{max} > 0.$$

The energy functional associated to problem (1.1) is given by

$$E(t) = \frac{1}{2}\|u_t(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|u(t)\|^2_{H^2_*(\Omega)} + \frac{1}{2}(g \diamond u)(t) - \frac{1}{p}\|u(t)\|^p_{L^p(\Omega)}, \tag{2.4}$$

where

$$(g \diamond u)(t) = \int_0^t g(t-s)\|u(t) - u(s)\|^2_{H^2_*(\Omega)} ds.$$

We also define the following functionals:

$$\Phi(t) = \Phi(u(t)) = \left(1 - \int_0^t g(s)ds\right)\|u(t)\|^2_{H^2_*(\Omega)} + (g \diamond u)(t) - \|u(t)\|^p_{L^p(\Omega)}, \tag{2.5}$$

$$\chi(t) = \chi(u(t)) = \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|u(t)\|^2_{H^2_*(\Omega)} + \frac{1}{2}(g \diamond u)(t) - \frac{1}{p}\|u(t)\|^p_{L^p(\Omega)}, \tag{2.6}$$

and for $t \geq 0$, we define

$$\beta(t) = \inf \sup_{\lambda \geq 0} \chi(\lambda u(t)). \tag{2.7}$$

Next, we state and prove some useful Lemmas needed in the next section.

3. Technical Lemmas

Lemma 3.1. *Let u be the solution to problem (1.1) and assume (A1) and (A2) hold. Then, the energy functional (2.4) satisfies*

$$E'(t) \quad = \quad \frac{1}{2}(g' \diamond u)(t) - \frac{1}{2}g(t)\|u(t)\|^2_{H^2_*(\Omega)} - \int_\Omega h(u_t(x,y,t))u_t(x,y,t)dxdy \leq 0. \tag{3.1}$$

for almost all $t \in [0, T_{max})$.

Proof. Multiplying $(1.1)_1$ by u_t and integrating over Ω, using integration by parts and (A2), we obtain (3.1) for any regular solution. This result remains valid for weak solutions by simple density argument. The reader is refered to [12,16] for detailed computations. This implies the energy functional E is non-increasing. □

Lemma 3.2. *For $t \geq 0$, the following inequality holds:*

$$0 < \beta_1 \leq \beta(t) \leq \sup_{\lambda \geq 0} \chi(\lambda u), \tag{3.2}$$

where

$$\beta_1 = \frac{(p-2)}{2p}\left(\frac{l_0}{S_q^2}\right)^{\frac{p}{p-2}} \tag{3.3}$$

and

$$\sup_{\lambda \geq 0} \chi(\lambda u(t)) = \frac{(p-2)}{2p}\left(\frac{\left(1 - \int_0^t g(s)ds\right)\|u(t)\|^2_{H^2_*(\Omega)} + (g \diamond u)(t)}{\|u(t)\|^2_{L^p(\Omega)}}\right)^{\frac{p}{p-2}}. \tag{3.4}$$

Proof. For $\lambda \geq 0$, let

$$\chi(\lambda u(t)) = \frac{\lambda^2}{2}\left(1-\int_0^t g(s)ds\right)\|u(t)\|^2_{H^2_*(\Omega)} + \frac{\lambda^2}{2}(g\diamond u)(t) - \frac{\lambda^p}{p}\|u(t)\|^p_{L^p(\Omega)}. \tag{3.5}$$

Thus, we have

$$\frac{d\left[\chi(\lambda u(t))\right]}{d\lambda} = \lambda\left(1-\int_0^t g(s)ds\right)\|u(t)\|^2_{H^2_*(\Omega)} + \lambda(g\diamond u)(t) - \lambda^{p-1}\|u(t)\|^p_{L^p(\Omega)}. \tag{3.6}$$

Solving $\frac{d[\chi(\lambda u(t))]}{d\lambda} = 0$, we obtain two critical points

$$\lambda_1 = 0, \quad \lambda_2 = \left(\frac{\left(1-\int_0^t g(s)ds\right)\|u(t)\|^2_{H^2_*(\Omega)} + (g\diamond u)(t)}{\|u(t)\|^p_{L^p(\Omega)}}\right)^{\frac{1}{p-2}}.$$

The second derivative of $\chi(\lambda u(t))$ with respect to λ is given by

$$\frac{d^2\left[\chi(\lambda u(t))\right]}{d\lambda^2} = \left(1-\int_0^t g(s)ds\right)\|u(t)\|^2_{H^2_*(\Omega)} + (g\diamond u)(t) - (p-1)\lambda^{p-2}\|u(t)\|^p_{L^p(\Omega)}. \tag{3.7}$$

Simple computations gives

$$\frac{d^2\left[\chi(\lambda_1 u(t))\right]}{d\lambda^2} > 0 \quad and \quad \frac{d^2\left[\chi(\lambda_2 u(t))\right]}{d\lambda^2} < 0.$$

Thus, we get

$$\begin{aligned}\sup_{\lambda\geq 0}\chi(\lambda u(t)) = \chi(\lambda_2 u(t)) &= \frac{p-2}{2p}\left(\frac{\left(1-\int_0^t g(s)ds\right)\|u(t)\|^2_{H^2_*(\Omega)} + (g\diamond u)(t)}{\|u(t)\|^2_{L^p(\Omega)}}\right)^{\frac{p}{p-2}} \\ &\geq \frac{p-2}{2p}\left(\frac{l_0}{S_q^2}\right)^{\frac{p}{p-2}} > 0.\end{aligned} \tag{3.8}$$

This completes the proof. □

Lemma 3.3. *Assume (A1) and (A2) hold. For any $\epsilon < 1$ fixed, let $(u_0, u_1) \in H^2_*(\Omega)\times L^2(\Omega)$ and satisfy*

$$\Phi(0) < 0 \ and \ \ E(0) < \epsilon\beta_1. \tag{3.9}$$

Assume further that

$$\int_0^{+\infty} g(s)ds < \frac{p-2}{p-2+\frac{1}{(1-\epsilon_0)^2(p-2)+2(1-\epsilon_0)}}, \tag{3.10}$$

where $\epsilon_0 = \max(0,\epsilon)$. Then, there exist $T > 0$ such that

$$\Phi(t) < 0, \quad \forall t \in [0,T) \tag{3.11}$$

and

$$\beta_1 < \frac{p-2}{2p}\left(\left(1-\int_0^t g(s)ds\right)\|u(t)\|^2_{H^2_*(\Omega)} + (g\diamond u)(t)\right) < \frac{p-2}{2p}\|u(t)\|^p_{L^p(\Omega)}, \quad t\in[0,T). \tag{3.12}$$

Proof. Using (3.1) and (3.9), it follows that $E(t) < \epsilon\beta_1$, for all $t \in [0,T)$. Similarly, we can obtain $\Phi(t) < 0$, for all $t \in [0,T)$. Suppose by contradiction that there exist $t^* > 0$ such that

$$\Phi(t^*) = 0 \quad and \quad \Phi(t) < 0, \quad 0 \le t < t^*.$$

Using the definition of Φ in (2.5), we obtain

$$\left(1 - \int_0^t g(s)ds\right) \|u(t)\|^2_{H^2_*(\Omega)} + (g \diamond u)(t) < \|u(t)\|^p_{L^p(\Omega)}, \quad 0 \le t < t^*. \tag{3.13}$$

Applying lemma 3.2, we get

$$\begin{aligned} \beta_1 &< \frac{p-2}{2p} \left[\frac{\left(1 - \int_0^t g(s)ds\right) \|u(t)\|^2_{H^2_*(\Omega)} + (g \diamond u)(t)}{\left[\left(1 - \int_0^t g(s)ds\right) \|u(t)\|^2_{H^2_*(\Omega)} + (g \diamond u)(t)\right]^{\frac{2}{p}}} \right]^{\frac{p}{p-2}} \\ &= \frac{p-2}{2p} \left[\left(1 - \int_0^t g(s)ds\right) \|u(t)\|^2_{H^2_*(\Omega)} + (g \diamond u)(t)\right], \ 0 \le t < t^*. \end{aligned} \tag{3.14}$$

Combining (3.13) and (3.14), we arrive at

$$0 < \beta_1 < \frac{p-2}{2p} \|u(t)\|^p_{L^p(\Omega)}, \quad 0 \le t < t^*.$$

By the continuity of $t \longmapsto \|u(t)\|^p_{L^p(\Omega)}$, we have that $u(t^*) \neq 0$. It follows from lemma 3.2 and (2.6) that

$$\beta_1 \le \frac{p-2}{2p} \|u(t^*)\|^p_{L^p(\Omega)} = \chi(u(t^*)).$$

But this is impossible because

$$\chi(u(t^*)) \le E(u(t^*)) < \beta_1.$$

By using Lemma 3.2 again, we obtain the estimate (3.12). This completes the proof. □

Lemma 3.4. *Under the assumptions of Lemma 3.3, the functional F defined by*

$$F(t) = \epsilon_0 \beta_1 - E(t) \tag{3.15}$$

is increasing and satisfies

$$0 < F(0) \le F(t) \le \epsilon_0\beta_1 + \frac{1}{p}\|u(t)\|^p_{L^p(\Omega)} \le p_0 \|u(t)\|^p_{L^p(\Omega)}, \ t \in [0,T), \tag{3.16}$$

where $p_0 = \frac{(p-2)\epsilon_0}{2p} + \frac{1}{p}$.

Proof. Using (3.1), (3.9) and (3.12), we obtain the result easily. □

Lemma 3.5. *Under the assumptions of Lemma 3.3, the solution of problem (1.1) satisfies*

$$\|u(t)\|^s_{L^p(\Omega)} \le c\left(-F(t) - \|u_t(t)\|^2_{L^2(\Omega)} - (g \diamond u)(t) + \|u(t)\|^p_{L^p(\Omega)}\right), \ \forall \ t \in [0,T), \ 2 \le s \le p. \tag{3.17}$$

Proof. We follow the ideas of Messaoudi [10,11]. From (2.1) and (2.4), we have

$$\begin{aligned} \frac{(1-l_0)}{2}\|u(t)\|^2_{H^2_*(\Omega)} &\le \left(1 - \int_0^t g(s)ds\right)\|u(t)\|^2_{H^2_*(\Omega)} \\ &\le E(t) - \frac{1}{2}\|u_t(t)\|^2_{L^2(\Omega)} - \frac{1}{2}(g \diamond u)(t) + \frac{1}{p}\|u(t)\|^p_{L^p(\Omega)} \\ &\le \epsilon_0\beta_1 - F(t) - \frac{1}{2}\|u_t(t)\|^2_{L^2(\Omega)} - \frac{1}{2}(g \diamond u)(t) + \frac{1}{p}\|u(t)\|^p_{L^p(\Omega)}. \end{aligned} \tag{3.18}$$

A combination of (3.12) and (3.18) yields

$$\|u(t)\|_{H_*^2(\Omega)}^2 \le c\left(-F(t) - \|u_t(t)\|_{L^2(\Omega)}^2 - (g \diamond u)(t) + \|u(t)\|_{L^p(\Omega)}^p\right), \ \forall t \in [0, T). \tag{3.19}$$

Now, if $\|u(t)\|_{L^p(\Omega)} \le 1$, it follows from Lemma 2.1 that

$$\|u(t)\|_{L^p(\Omega)}^s \le \|u(t)\|_{L^p(\Omega)}^2 \le S_q^2 \|u(t)\|_{H_*^2(\Omega)}^2, \quad 2 \le s \le p. \tag{3.20}$$

Also, if $\|u(t)\|_{L^p(\Omega)} > 1$, then we get

$$\|u(t)\|_{L^p(\Omega)}^s \le \|u(t)\|_{L^p(\Omega)}^p, \quad 2 \le s \le p. \tag{3.21}$$

It follows from (3.20) and (3.21) that

$$\|u(t)\|_{L^p(\Omega)}^s \le C\left(\|u(t)\|_{H_*^2(\Omega)}^2 + \|u(t)\|_{L^p(\Omega)}^p\right), \forall u \in H_*^2(\Omega), \ 2 \le s \le p. \tag{3.22}$$

Thus, we obtain (3.17) from (3.19) and (3.22). This completes the proof. □

4. Blow-up result

In this section, we state and prove the blow-up result for problem (1.1). We adopt the ideas and method used in [9,10,11,19] with necessary modifications to establish our result. Our result read as follows:

Theorem 4.1. *Assume (A1) and (A2) hold, $m < p$ and the conditions of Lemma 3.3 remain valid. Furthermore, assume that*

$$\xi^{-1}(1) < \left(\frac{\theta\epsilon_0\beta_1 p\mu}{(p-1)|\Omega|}\right)^{\frac{p-1}{p}}, \tag{4.1}$$

where

$$0 < \mu^{p-1} < \theta < \min\{k_1, k_2\},$$

and for some $\eta > 0$

$$k_1 = (1-\epsilon_0)\left(\frac{p-2}{2}\right) + (1-\eta) > 0, \tag{4.2}$$

$$k_2 = (1-\epsilon_0)\left(\frac{p-2}{2}\right) - \left((1-\epsilon_0)\left(\frac{p-2}{2}\right) + \frac{1}{4\eta}\right)\int_0^\infty g(s)ds > 0. \tag{4.3}$$

Then, there exists a finite time at which the solution of problem (1.1) blows up.

Proof. Define

$$H(t) = F^{(1-\sigma)}(t) + \varrho \int_\Omega u(x,y,t)u_t(x,y,t)dxdy, \tag{4.4}$$

for $\varrho > 0$ small to be chosen later and for

$$0 < \sigma < \min\left\{\frac{p-2}{2p}, \frac{p-m}{p(m-1)}\right\}. \tag{4.5}$$

Differentiating (4.4) and using Eq. $(1.1)_1$, we obtain

$$\begin{aligned} H'(t) &= (1-\sigma)F^{-\sigma}(t)F'(t) + \varrho\|u_t(t)\|_{L^2(\Omega)}^2 - \varrho\|u(t)\|_{H_*^2(\Omega)}^2 + \varrho\|u(t)\|_{L^p(\Omega)}^p \\ &\quad + \varrho\int_0^t g(t-s)\left(u(s), u(t)\right)_{H_*^2(\Omega)} ds - \varrho\int_\Omega h(u_t(x,y,t)u(x,y,t)dxdy. \end{aligned} \tag{4.6}$$

Using Schwarz inequality, we have for any $\eta > 0$,

$$\begin{aligned}
&\int_0^t g(t-s)\left(u(s),u(t)\right)_{H_*^2(\Omega)} ds \\
&= \int_0^t g(t-s)\left(u(s)-u(t),u(t)\right)_{H_*^2(\Omega)} ds + \left(\int_0^t g(s)ds\right)\|u(t)\|^2_{H_*^2(\Omega)} \\
&\geq -\eta(g\diamond u)(t) + \left(1-\frac{1}{4\eta}\right)\left(\int_0^t g(s)ds\right)\|u(t)\|^2_{H_*^2(\Omega)}.
\end{aligned} \tag{4.7}$$

Thus, the estimate (4.6) becomes

$$\begin{aligned}
H'(t) \geq\, &(1-\sigma)F^{-\sigma}(t)F'(t) + \varrho\|u_t(t)\|^2_{L^2(\Omega)} - \varrho\|u(t)\|^2_{H_*^2(\Omega)} + \varrho\|u(t)\|^p_{L^p(\Omega)} \\
&- \varrho\eta(g\diamond u)(t) + \varrho\left(1-\frac{1}{4\eta}\right)\left(\int_0^t g(s)ds\right)\|u(t)\|^2_{H_*^2(\Omega)} \\
&- \varrho\int_\Omega h(u_t(x,y,t)u(x,y,t)dxdy.
\end{aligned} \tag{4.8}$$

Substituting $\|u(t)\|^p_{L^p(\Omega)}$ from (2.4), we get

$$\begin{aligned}
H'(t) \geq\, &(1-\sigma)F^{-\sigma}(t)F'(t) + \varrho\|u_t(t)\|^2_{L^2(\Omega)} - \varrho\|u(t)\|^2_{H_*^2(\Omega)} - \varrho\eta(g\diamond u)(t) \\
&+ \varrho\left(-pE(t) + \frac{p}{2}\|u_t(t)\|^2_{L^2(\Omega)} + \frac{p}{2}\left(1-\int_0^t g(s)ds\right)\|u(t)\|^2_{H_*^2(\Omega)} + \frac{p}{2}(g\diamond u)(t)\right) \\
&+ \varrho\left(1-\frac{1}{4\eta}\right)\left(\int_0^t g(s)ds\right)\|u(t)\|^2_{H_*^2(\Omega)} - \varrho\int_\Omega h(u_t(x,y,t)u(x,y,t)dxdy.
\end{aligned} \tag{4.9}$$

Using $E(t) = \epsilon_0\beta_1 - F(t)$, we arrive at

$$\begin{aligned}
H'(t) \geq\, &(1-\sigma)F^{-\sigma}(t)F'(t) + \varrho\left(\frac{p}{2}+1\right)\|u_t(t)\|^2_{L^2(\Omega)} \\
&+ \varrho\left[\frac{p}{2} - \eta - \epsilon_0\left(\frac{p}{2}-1\right)\right](g\diamond u)(t) + \varrho pF(t) \\
&+ \varrho\left[\left(\frac{p}{2}-1\right) - \left(\left(\frac{p}{2}-1\right) + \frac{1}{4\eta}\right)\int_0^t g(s)ds\right]\|u(t)\|^2_{H_*^2(\Omega)} \\
&- \varrho p\epsilon_0\beta_1 - \varrho\int_\Omega h(u_t(x,y,t)u(x,y,t)dxdy.
\end{aligned} \tag{4.10}$$

Using (3.12), then estimate (4.10) takes the form

$$\begin{aligned}
H'(t) \geq\, &(1-\sigma)F^{-\sigma}(t)F'(t) + \varrho\left(\frac{p}{2}+1\right)\|u_t(t)\|^2_{L^2(\Omega)} \\
&+ \varrho\left[\frac{p}{2} - \eta - \epsilon_0\left(\frac{p}{2}-1\right)\right](g\diamond u)(t) + \varrho pF(t) \\
&+ \varrho\left[\left(\frac{p}{2}-1\right) - \left(\left(\frac{p}{2}-1\right) + \frac{1}{4\eta}\right)\int_0^t g(s)ds\right]\|u(t)\|^2_{H_*^2(\Omega)} \\
&- \frac{\varrho\epsilon_0(p-2)}{2}\left(1-\int_0^t g(s)ds\right)\|u(t)\|^2_{H_*^2(\Omega)} - \frac{\varrho\epsilon_0(p-2)}{2}(g\diamond u)(t) \\
&- \varrho\int_\Omega h(u_t(x,y,t)u(x,y,t)dxdy,
\end{aligned}$$

which yields

$$
\begin{aligned}
H'(t) \geq & (1-\sigma)F^{-\sigma}(t)F'(t) + \varrho\left(\frac{p}{2}+1\right)\|u_t(t)\|^2_{L^2(\Omega)} \\
& + \varrho\left[(1-\epsilon_0)\left(\frac{p}{2}-1\right) + (1-\eta)\right](g\diamond u)(t) + \varrho p F(t) \\
& + \varrho\left[(1-\epsilon_0)\left(\frac{p}{2}-1\right) - \left((1-\epsilon_0)\left(\frac{p}{2}-1\right)+\frac{1}{4\eta}\right)\int_0^t g(s)ds\right]\|u(t)\|^2_{H^2_*(\Omega)} \\
& - \varrho\int_\Omega h(u_t(x,y,t)u(x,y,t)dxdy.
\end{aligned}
\tag{4.11}
$$

Now, choosing $\eta > 0$ small enough such that

$$
0 < \eta < (1-\epsilon_0)\left(\frac{p}{2}-1\right) + 1
$$

and taking note of (4.2) and (4.3), the estimate in (4.11) becomes

$$
\begin{aligned}
H'(t) \geq & (1-\sigma)F^{-\sigma}(t)F'(t) + \varrho\left(\frac{p}{2}+1\right)\|u_t(t)\|^2_{L^2(\Omega)} + \varrho k_1(g\diamond u)(t) \\
& + \varrho p F(t) + \varrho k_2\|u(t)\|^2_{H^2_*(\Omega)} - \varrho\int_\Omega h(u_t(x,y,t)u(x,y,t)dxdy.
\end{aligned}
\tag{4.12}
$$

To estimate the last term on right-hand side of (4.12), we partition Ω as follows:

$$
I_1 = \{(x,y)\in\Omega : |u_t(x,y,t)| \leq 1\}, \quad I_2 = \{(x,y)\in\Omega : |u_t(x,y,t)| > 1\}.
$$

Using Young's inequality and (2.2), we have the following estimates (see [9] for details on similar computations)

$$
\int_{I_1} h(u_t(x,y,t)u(x,y,t)dxdy \leq \frac{\mu^{p-1}}{p}\|u(t)\|^p_{L^p(\Omega)} + \frac{(p-1)|\Omega|}{p\mu}\left(\xi^{-1}(1)\right)^{\frac{p}{p-1}}, \quad \mu > 0
\tag{4.13}
$$

and

$$
\int_{I_2} h(u_t(x,y,t)u(x,y,t)dxdy \leq \frac{m-1}{m}\gamma^{-\frac{m}{m-1}}F'(t) + \frac{\gamma^m}{m}\|u(t)\|^m_{L^p(\Omega)}, \quad \gamma > 0.
\tag{4.14}
$$

Substituting (4.13) and (4.14) into (4.12), we obtain

$$
\begin{aligned}
H'(t) \geq & (1-\sigma)F^{-\sigma}(t)F'(t) + \varrho\left(\frac{p}{2}+1\right)\|u_t(t)\|^2_{L^2(\Omega)} + \varrho k_1(g\diamond u)(t) + \varrho p F(t) \\
& + \varrho k_2\|u(t)\|^2_{H^2_*(\Omega)} - \frac{\varrho\mu^{p-1}}{p}\|u(t)\|^p_{L^p(\Omega)} - \frac{\varrho(p-1)|\Omega|}{p\mu}\left(\xi^{-1}(1)\right)^{\frac{p}{p-1}} \\
& - \frac{\varrho(m-1)}{m}\gamma^{-\frac{m}{m-1}}F'(t) - \frac{\varrho\gamma^m}{m}\|u(t)\|^m_{L^p(\Omega)}.
\end{aligned}
\tag{4.15}
$$

We observe that (4.15) remains valid even if γ is time-dependent since the integral is done over (x,y) variables. Making use of (2.4) and (3.15), and adding $\varrho\theta F(t) - \varrho\theta F(t)$ to the right-hand side of (4.15), for some θ to be specified later, we get

$$
\begin{aligned}
H'(t) \geq & \left[(1-\sigma)F^{-\sigma}(t) - \frac{\varrho(m-1)}{m}\gamma^{-\frac{m}{m-1}}\right]F'(t) + \varrho\left(\frac{p}{2}+1-\frac{\theta}{2}\right)\|u_t(t)\|^2_{L^2(\Omega)} \\
& + \varrho\left(k_1 - \frac{\theta}{2}\right)(g\diamond u)(t) + \varrho\left(k_2 - \frac{\theta}{2}\left(1-\int_0^t g(s)ds\right)\right)\|u(t)\|^2_{H^2_*(\Omega)} \\
& + \varrho(p-\theta)F(t) + \varrho\left(\frac{\theta}{p} - \frac{\mu^{p-1}}{p}\right)\|u_t(t)\|^p_{L^p(\Omega)} - \frac{\varrho(p-1)|\Omega|}{p\mu}\left(\xi^{-1}(1)\right)^{\frac{p}{p-1}} \\
& - \frac{\varrho\gamma^m}{m}\|u(t)\|^m_{L^p(\Omega)} + \varrho\theta\epsilon_0\beta_1.
\end{aligned}
\tag{4.16}
$$

Now, we set $\gamma^{-\frac{m}{m-1}} = bF^{-\sigma}(t)$, for some b to be chosen properly and substituting in (4.16), we obtain

$$\begin{aligned} H'(t) \geq & \left[(1-\sigma) - \frac{b\varrho(m-1)}{m}\right] F^{-\sigma}(t)F'(t) + \varrho\left(\frac{p}{2}+1-\frac{\theta}{2}\right)\|u_t(t)\|^2_{L^2(\Omega)} \\ & + \varrho\left(k_1 - \frac{\theta}{2}\right)(g\diamond u)(t) + \varrho\left(k_2 - \frac{\theta}{2}\left(1-\int_0^t g(s)ds\right)\right)\|u(t)\|^2_{H^2_*(\Omega)} \\ & + \varrho(p-\theta)F(t) + \varrho\left(\frac{\theta}{p} - \frac{\mu^{p-1}}{p}\right)\|u_t(t)\|^p_{L^p(\Omega)} - \frac{\varrho(p-1)|\Omega|}{p\mu}\left(\xi^{-1}(1)\right)^{\frac{p}{p-1}} \\ & - \frac{b^{1-m}\varrho}{m}F^{\sigma(m-1)}(t)\|u(t)\|^m_{L^p(\Omega)} + \varrho\theta\epsilon_0\beta_1. \end{aligned} \tag{4.17}$$

Using Lemma 3.4, we have

$$-\frac{b^{1-m}\varrho}{m}F^{\sigma(m-1)}(t)\|u(t)\|^m_{L^p(\Omega)} \geq -\frac{b^{1-m}\varrho}{m}p_0^{\sigma(m-1)}\|u(t)\|^{\sigma p(m-1)+m}_{L^p(\Omega)}. \tag{4.18}$$

Making use of (4.5), Lemma 3.5 with $s = \sigma p(m-1)+m \leq p$ and (4.18), we get from (4.17) that

$$\begin{aligned} H'(t) \geq & \left[(1-\sigma) - \frac{b\varrho(m-1)}{m}\right] F^{-\sigma}(t)F'(t) \\ & + \varrho\left[\frac{p}{2}+1-\frac{\theta}{2}+\frac{cb^{1-m}\varrho}{m}p_0^{\sigma(m-1)}\right]\|u_t(t)\|^2_{L^2(\Omega)} \\ & + \varrho\left[k_1 - \frac{\theta}{2}+\frac{cb^{1-m}\varrho}{m}p_0^{\sigma(m-1)}\right](g\diamond u)(t) \\ & + \varrho\left[k_2 - \frac{\theta}{2}\left(1-\int_0^t g(s)ds\right)\right]\|u(t)\|^2_{H^2_*(\Omega)} \\ & + \varrho\left[p-\theta+\frac{cb^{1-m}\varrho}{m}p_0^{\sigma(m-1)}\right]F(t) \\ & + \varrho\left[\frac{\theta}{p} - \frac{\mu^{p-1}}{p} - \frac{cb^{1-m}\varrho}{m}p_0^{\sigma(m-1)}\right]\|u(t)\|^p_{L^p(\Omega)} \\ & - \frac{\varrho(p-1)|\Omega|}{p\mu}\left(\xi^{-1}(1)\right)^{\frac{p}{p-1}} + \varrho\theta\epsilon_0\beta_1. \end{aligned} \tag{4.19}$$

Now, we choose our parameters carefully. First, we select θ small enough such that

$$0 < \theta < \min\{k_1, k_2\}.$$

Secondly, we choose μ small enough so that

$$\theta - \mu^{p-1} > 0.$$

Then we select b large enough so that

$$\frac{\theta}{p} - \frac{\mu^{p-1}}{p} - \frac{cb^{1-m}\varrho}{m}p_0^{\sigma(m-1)} > 0$$

and select ϱ so small such that

$$(1-\sigma) - \frac{b\varrho(m-1)}{m} > 0$$

and, hence, because of (4.1), we deduce that

$$\varrho\theta\epsilon_0\beta_1 - \frac{\varrho(p-1)|\Omega|}{p\mu}\left(\xi^{-1}(1)\right)^{\frac{p}{p-1}} > 0.$$

With the above choices, we obtain

$$H'(t) \geq \lambda \left(F(t) + \|u_t(t)\|_{L^2(\Omega)} + \|u(t)\|^p_{L^p(\Omega)} + (g \diamond u)(t) \right), \tag{4.20}$$

where $\lambda > 0$ is a constant. Now, using Schwarz and Young's inequalities, we have

$$\begin{aligned} \left| \int_\Omega u(x,y,t) u_t(x,y,t) dxdy \right|^{\frac{1}{1-\sigma}} &\leq \|u(t)\|^{\frac{1}{1-\sigma}}_{L^2(\Omega)} \|u_t(t)\|^{\frac{1}{1-\sigma}}_{L^2(\Omega)} \\ &\leq C \|u(t)\|^{\frac{1}{1-\sigma}}_{L^p(\Omega)} \|u_t(t)\|^{\frac{1}{1-\sigma}}_{L^2(\Omega)} \\ &\leq C \left(\|u(t)\|^{\frac{r_1}{1-\sigma}}_{L^p(\Omega)} + \|u_t(t)\|^{\frac{r_2}{1-\sigma}}_{L^2(\Omega)} \right), \end{aligned} \tag{4.21}$$

such that $\frac{1}{r_1} + \frac{1}{r_2} = 1$. From (4.5), we need $s = \frac{r_1}{1-\sigma} = \frac{2}{1-2\sigma} \leq p$. Therefore we select $r_2 = 2(1-\sigma)$ and arrive at

$$\left| \int_\Omega u(x,y,t) u_t(x,y,t) dxdy \right|^{\frac{1}{1-\sigma}} \leq C \left(\|u(t)\|^s_{L^p(\Omega)} + \|u_t(t)\|^2_{L^2(\Omega)} \right). \tag{4.22}$$

Applying Lemma 3.5, we get, for all $t \geq 0$,

$$\left| \int_\Omega u(x,y,t) u_t(x,y,t) dxdy \right|^{\frac{1}{1-\sigma}} \leq C \left(F(t) + \|u_t(t)\|_{L^2(\Omega)} + \|u(t)\|^p_{L^p(\Omega)} + (g \diamond u)(t) \right). \tag{4.23}$$

Thus, using the definition of $H(t)$, we get

$$\begin{aligned} H(t)^{\frac{1}{1-\sigma}} &= \left(F^{1-\sigma}(t) + \varrho \int_\Omega u(x,y,t) u_t(x,y,t) dxdy \right)^{\frac{1}{1-\sigma}} \\ &\leq 2^{\frac{1}{1-\sigma}} \left(F(t) + \left| \int_\Omega u(x,y,t) u_t(x,y,t) dxdy \right|^{\frac{1}{1-\sigma}} \right) \\ &\leq C \left(F(t) + \|u_t(t)\|_{L^2(\Omega)} + \|u(t)\|^p_{L^p(\Omega)} + (g \diamond u)(t) \right), \forall t \geq 0. \end{aligned} \tag{4.24}$$

A combination of (4.20) and (4.24) leads to

$$H'(t) \geq C \left(H(t) \right)^{\frac{1}{1-\sigma}}, \ \forall t \geq 0. \tag{4.25}$$

Integrating (4.25) over $(0,t)$ gives

$$H(t) \geq \frac{1}{H(0)^{-\frac{\sigma}{1-\sigma}} - t\frac{C\sigma}{1-\sigma}} \tag{4.26}$$

and, consequently, we obtain that $H(t)$ blows up in a finite time

$$T^* \leq \frac{1-\sigma}{C\sigma \left(H(0) \right)^{\frac{\sigma}{1-\sigma}}}. \tag{4.27}$$

This completes the proof. □

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References

1. B. Akesson, *Understanding bridges collapses*, CRC Press, Taylor and Francis Group, London (2008).
2. W. Chen and Y. Zhou, *Global nonexistence for a semilinear Petrovsky equations*, Nonlinear Anal. TMA 70(9), 3203-3208, (2009).
3. A. Ferrero and F. Gazzola, *A partially hinged rectangular plate as a model for suspension bridges*, Discrete Contin. Dyn. Syst. 35, 5879-5908, (2015).
4. F. Gazzola, *Mathematical models for Suspension Bridges: Nonlinear Structural Instability*, Modeling, Simulation and Applications 15, Springer-Verlag (2015).
5. F. Gazzola and Y. Wang, *Modeling suspension bridges through the Von Karman quasilinear plate equations, Progress in Nonlinear Differential Equations and Their Applications*, In Contributions to Nonlinear Differential Equations and Systems, a tribute to Djairo Guedes de Figueiredo on occasion of his 80th birthday, 269-297, (2015).
6. V. Georgiev, G. Todorova, *Existence of solutions of the wave equation with nonlinear damping and source terms* , J. Differential Equations 109, 295-308, (1994).
7. T.G. Ha, *Asymptotic stability of the semilinear wave equation with boundary damping and source term*, C. R. Math.Acad. Sci. Paris 352, 213-218, (2014).
8. T. Kawada, *History of the modern suspension bridge: solving the dilemma between economy and stiffness*, ASCE Press (2010).
9. W.J. Liu, J. Yu, *On decay and blow-up of the solution for a viscoelastic wave equation with boundary damping and source terms*, Nonlinear Anal. 74, 2175-2190, (2011).
10. S.A. Messaoudi, *Blow-up of positive-initial-energy solutions of a nonlinear viscoelastic hyperbolic equation*, J. Math. Anal. Appl. 320, 902-915, (2006).
11. S.A. Messaoudi, *Blow up and global existence in a nonlinear viscoelastic wave equation*, Math. Nachr. 260, 58-66, (2003).
12. S.A. Messaoudi and S.E. Mukiawa, *Existence and decay of solutions to a viscoelastic plate equation*, Electronic J. Differ. Equ. 22, 1-14, (2016).
13. S.A. Messaoudi and S.E. Mukiawa, *A Suspension Bridge Problem: Existence and Stability*, Mathematics Across Contemporary Sciences. Springer, Geneva (2016).
14. S.E. Mukiawa, *Existence and general decay estimate for a nonlinear plate problem*, Boundary Value Problems (2018),https://doi.org/10.1186/s13661-018-0931-0, 2018:11.
15. S.E. Mukiawa, *Asymptotic behaviour of a suspension bridge problem*, Arab Journal of Mathematical Sciences 24, 31-42, (2018).
16. S.E. Mukiawa, *Decay Result for a Delay Viscoelastic Plate Equation*, Bull. Braz. Math. Soc., New Series 51 (2), 333-356, (2020).
17. S.E. Mukiawa, *Blow-up result for a plate equation with fractional damping and nonlinear source terms*,Open Journal of Mathematical Analysis 4,32-41, (2020).
18. J.Y. Park, T.G. Ha, *Existence and asymptotic stability for the semilinear wave equation with boundary damping and source term*, J. Math. Phys. 49, (2008).
19. S. H. Park, M. J. Lee, J.R. Kang, *Blow-up results for viscoelastic wave equations with weak damping*, Apl. Math. Letters 80, 20-26, (2018).
20. S.A. Messaoudi, *Global existence and nonexistence in a system of Petrovsky*, Journal of Mathematical Analysis and Applications 265, 296-308, (2002).
21. Y. Wang, *Finite time blow-up and global solutions for fourth order damped wave equations*, Journal of Mathematical Analysis and Applications, 418, 713-733, (2014).
22. Tacoma Narrows Bridge collapse, http://www.youtube.com/watch?v=3mclp9QmCGs, (1940).

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