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# The Fekete-Szegö Estimates for a New Class of Analytic Functions Associated With the Convolution

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ABSTRACT: In the present investigation, we discuss the sharpness of the bound of the Fekete-Szegö functional $|a_3 - \mu a_2^2|$ for the functions belonging to certain subclass $\mathcal{R}^{\epsilon}_{\nu,\mathcal{L}_g}(\psi)$ of analytic functions by means of convolution. The significant and useful consequences with the relevance of this class with some known classes are also pointed out.

Key Words: Analytic functions, starlike functions, convex functions, subordination, Fekete-Szego inequality.

## Contents



## 1. Introduction

Let $\mathcal{H}$ be the class of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, normalized by $f(0) = 0$ and $f'(0) = 1$. Further the subclass of $\mathcal{H}$ consisting of univalent functions is denoted by $S$. Flash back that a function $f \in \mathcal{H}$ is starlike if $f(\mathbb{U})$ is a starlike domain and convex if $f(\mathbb{U})$ is a convex domain. The classes comprise of starlike and convex functions are usually denoted by $S^*$ and $C$, respectively. The function $f(z)$ is subordinate to the function $g(z)$, written $f(z) \prec g(z)$, provided that there is an analytic function $w(z)$ defined on $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ such that $f(z) = g(w(z))$. A function is starlike if and only if $Re(zf'(z)/f(z)) > 0$, or in other words if $zf'(z)/f(z) \prec (1 + z)/(1 - z)$. The superordinate function $\phi(z) = (1 + z)/(1 - z)$ is a convex function. Ma and Minda [16] have given a unification of various subclasses consisting of starlike and convex functions for which either one of the expressions $zf'(z)/f(z)$ or $1 + zf''(z)/f'(z)$ is subordinate to a more general superordinate function. Literally, they considered the analytic function $\psi$ with positive real part in the unit disk $\mathbb{U}$, $\psi(0) = 1$, $\psi'(0) > 0$, where $\psi$ maps $\mathbb{U}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. The unified class $S^*(\psi)$ initiated by Ma and Minda [16] consists of functions $f \in \mathcal{H}$ satisfying

$$\frac{zf'(z)}{f(z)} \prec \psi(z) \quad (z \in \mathbb{U})$$

They also investigated the corresponding class $C(\psi)$ of convex functions $f \in \mathcal{H}$ satisfying

$$1 + \frac{zf''(z)}{f'(z)} \prec \psi(z) \quad (z \in \mathbb{U}).$$

A function $f \in S^*(\psi)$ is said to be starlike function with respect to $\psi$, and a function $f \in C(\psi)$ is a convex function with respect to $\psi$. In geometric function theory, bounds for the coefficient $|a_i|$ play the

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significant role, as it reveals the geometric properties for the corresponding function. For instance, the bound for the second coefficient $|a_2|$ of given functions in the class $S$ describes the growth and distortion bounds together with covering theorems. The Fekete-Szegö coefficient functional also comes up obviously in the exploration of univalency of analytic functions. A number of authors have scrutinized the Fekete-Szegö functional for functions in various subclasses of univalent, multivalent and close-to-convex functions [17], [18], [11], [26], [14], [27], [19], [2]] and recently by R. Agrawal et al. [1].

In 1933; Fekete and Szegö [10] obtained the sharp bound for $|a_3 - \mu a_2^2|$ as a function of the real parameter $\mu$ and proved that

$$|a_3 - \mu a_2^2| \leq 1 + 2\exp\left(-\frac{2\mu}{1-\mu}\right) \quad (0 \leq \mu \leq 1),$$

for functions in the class $S$. Later the problem of finding sharp bound for the non-linear functional $|a_3 - \mu a_2^2|$ of any compact family of functions $f \in S$ is identified as Fekete-Szegö problem.

If $f \in \mathcal{H}$ is given by (1.1) and $g \in \mathcal{H}$ is given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \tag{1.2}$$

then the convolution (or Hadamard product) $f * g$ of $f$ and $g$ is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z). \tag{1.3}$$

In this article $\psi$ is assumed to be an analytic function with positive real part in the unit disk $\mathbb{U}$ and has the Taylor's series expansion of the form

$$\psi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + ...;$$

with $B_1 > 0$ and $B_2$ is any real number.
Stimulated by the class $R_\gamma^\tau(\beta)$ in paper [25], we introduce the following class.

**Definition 1.1.** *Let $0 \leq \nu \leq 1$, $\epsilon \in \mathbb{C}\backslash\{0\}$. A function $f \in \mathcal{H}$ is in the class $\mathcal{R}^\epsilon_{\nu,\mathcal{L}_g}(\psi)$ if*

$$1 + \frac{1}{\epsilon}\left(\mathcal{L}_g' f(z) + \nu z \mathcal{L}_g'' f(z) - 1\right) \prec \psi(z), \tag{1.4}$$

where $\mathcal{L}_g(f(z)) = f(z) * g(z)$ and $\psi(z)$ defined same as the above.
If we set

$$\psi(z) = \frac{1 + Az}{1 + Bz} \quad (-1 \leq B < A \leq 1;\ z \in \mathbb{U}),$$

in (1.4), we get

$$\mathcal{R}^\epsilon_{\nu,\mathcal{L}_g}\left(\frac{1+Az}{1+Bz}\right) = \mathcal{R}^\epsilon_{\nu,\mathcal{L}_g}(A,B) = \left\{f \in \mathcal{H} : \left|\frac{\mathcal{L}_g' f(z) + \nu z \mathcal{L}_g'' f(z) - 1}{\epsilon(A-B) - B(\mathcal{L}_g' f(z) + \nu z \mathcal{L}_g'' f(z) - 1)}\right| < 1\right\}$$

which is again a new class. Here we discuss some applications of particular case when $g(z) = \frac{z}{(1-z)}$ of our class discussed in the literature.

**(1)** $R_\gamma^\tau(1-2\beta, -1) = R_\gamma^\tau(\beta)$ for $0 \leq \beta < 1$, $\tau = \mathbb{C}\backslash\{0\}$ was discussed recently by Swaminathan [25].
**(2)** The class $R_\gamma^\tau(1-2\beta, -1)$ for $\tau = e^{i\eta}\cos\eta$, where $-\pi/2 < \eta < \pi/2$ is considered in [20] and the properties of certain integral transforms of the type

$$V_\lambda(f) = \int_0^1 \lambda(t)\frac{f(tz)}{t}dt, \quad f \in R_\gamma^{(e^{i\eta}\cos\eta)}(\beta)$$

under a suitable restriction on $\lambda(t)$ was discussed using duality techniques for various values of in [20].

**(3)** The class $R_{\gamma}^{\tau}(0,-1)$ with $\tau = e^{i\eta}\cos\eta$ was considered in [12] with reference to the univalencey of partial sums.

**(4)** if $f \in R_{\gamma}^{e^{i\eta}\cos\eta}(1-2\beta,-1)$ whenever $zf'(z) \in P_{\gamma}^{\tau}(\beta)$, the class considered in [24].

Here we list some more particular cases of this class discussed in the literature:

**(1)** Let $g(z) = z + \sum_{k=2}^{\infty}\left(\frac{(l+1)+\theta(k-1)}{l+1}\right)^m z^k$, where $\theta > 0,\ l \geq 0$ and $m \in \mathbb{N}_0$ in (1.4), then the class $\mathcal{R}^{\epsilon}_{\nu,\mathcal{L}_g}(\psi)$ reduces to the class $\mathcal{R}^{\epsilon}_{\nu,I_g}(\psi)$, which is defined by:

$$1 + \frac{1}{\epsilon}\left(I_g'(\theta,l)f(z) + \nu z I_g''(\theta,l)f(z) - 1\right) \prec \psi(z), \tag{1.5}$$

where $I_g(\theta,l)$ is the generalized multiplier transformation which was introduced and studied by Catas et *al.* [6].

**(2)** Let $g(z) = z + \sum_{k=2}^{\infty}\left(\frac{l+k}{l+1}\right)^m z^k$, where $l \geq 0$ and $m \in \mathbb{N}_0$ in (1.4), then the class $\mathcal{R}^{\epsilon}_{\nu,\mathcal{L}_g}(\psi)$ reduces to the class $\mathcal{R}^{\epsilon}_{\nu,\mathcal{S}_g}(\psi)$ which is defined by

$$1 + \frac{1}{\epsilon}\left(\mathcal{S}_g'(l)f(z) + \nu z \mathcal{S}_g''(l)f(z) - 1\right) \prec \psi(z), \tag{1.6}$$

where $\mathcal{S}_g(l)$ is the generalized multiplier transformation see [7,8].

**(3)** Let $g(z) = z + \sum_{k=2}^{\infty}(1+\theta(k-1))^m z^k$, where $\theta > 0$ and $m \in \mathbb{N}_0$ in(1.4), then the class $\mathcal{R}^{\epsilon}_{\nu,\mathcal{L}_g}(\psi)$ reduces to the class $\mathcal{R}^{\epsilon}_{\nu,\mathcal{D}_g}(\psi)$ which is defined by

$$1 + \frac{1}{\epsilon}\left(\mathcal{D}_g'f(z) + \nu z \mathcal{D}_g''(l)f(z) - 1\right) \prec \psi(z), \tag{1.7}$$

where the operator $\mathcal{D}_g(\theta)$ is the Salagean operator, see ( [21]).

**(4)** Let $g(z) = z + \sum_{k=2}^{\infty}\left(\frac{1+b}{k+b}\right)^s z^k$ where $b \in \mathbb{C}\backslash\mathbb{Z}_0^-,\ s \in \mathbb{C}$ in (1.4), then the class $\mathcal{R}^{\epsilon}_{\nu,\mathcal{L}_g}(\psi)$ reduces to the class $\mathcal{R}^{\epsilon}_{\nu,J_{s,b}}(\psi)$ which is defined by

$$1 + \frac{1}{\epsilon}\left(J_{s,b}'f(z) + \nu z J_{s,b}''(l)f(z) - 1\right) \prec \psi(z), \tag{1.8}$$

where the operator $J_{s,b}$ wae introduced and studied by Srivastava and Attiya [23].

**(5)** Let $g(z) = z + \sum_{k=2}^{\infty}\left(\frac{2}{k+1}\right)^{\alpha} z^k$ where $\alpha \geq 0$ in (1.4), then the class $\mathcal{R}^{\epsilon}_{\nu,\mathcal{L}_g}(\psi)$ reduces to the class $\mathcal{R}^{\epsilon}_{\nu,I^{\alpha}}(\psi)$ which is defined by

$$1 + \frac{1}{\epsilon}\left(I^{\alpha\prime}f(z) + \nu z I^{\alpha\prime\prime}f(z) - 1\right) \prec \psi(z), \tag{1.9}$$

where the operator $I^{\alpha}$ was introduced and studied by Jung et *al.* [13].

**(6)** Let $g(z) = z + \frac{\Gamma(1+\alpha+\beta)}{\Gamma(1+\beta)}\sum_{k=2}^{\infty}\frac{\Gamma(k+\beta)}{\Gamma(k+\alpha+\beta)}z^k$ where $\alpha \geq 0, \beta > -1$ in (1.4), then the class $\mathcal{R}^{\epsilon}_{\nu,\mathcal{L}_g}(\psi)$ reduces to the class $\mathcal{R}^{\epsilon}_{\nu,Q^{\alpha}_{\beta}}(\psi)$ which is defined by

$$1 + \frac{1}{\epsilon}\left(Q^{\alpha\prime}_{\beta}f(z) + \nu z Q^{\alpha\prime\prime}_{\beta}f(z) - 1\right) \prec \psi(z), \tag{1.10}$$

where the operator $Q^{\alpha}_{\beta}$ was introduced and studied by Jung et *al.* [13].

**(7)** Let $g(z) = z + \sum_{k=2}^{\infty} \frac{(1+\mu)^{\nu}}{(k+\mu)^{\nu}} \Gamma_k(a_1) z^k$, where $\Gamma_k(a_1) = \frac{(a_1)_{k-1} \dots (a_l)_{k-1}}{(b_1)_{k-1} \dots (b_m)_{k-1} (1)_{k-1}}$ $\big($with $a_i \in \mathbb{C}, i = 1, 2 \dots l;\ b_j \in \mathbb{C} \backslash \mathbb{Z}_0^- = \{0, -1, -2, \dots\}, j = 1, 2 \dots m,\ l \leq m+1,\ \ l, m, \in \mathbb{N}_0, z \in \mathbb{U}\big)$. and $(\nu)_k$ is the Pochhammer symbol defined by

$$(\nu)_k := \begin{cases} 1, & (k = 0,\ \ \nu \in \mathbb{C}^* = \mathbb{C} \backslash \{0\}), \\ \nu(\nu+1) \dots (\nu+k-1), & (k \in \mathbb{N}, \nu \in \mathbb{C}). \end{cases}$$

Then the class $\mathcal{R}^{\epsilon}_{\nu, \mathcal{L}_g}(\psi)$ reduces to the class $\mathcal{R}^{\epsilon}_{\nu, \mathcal{K}(a_1, b_1, l, m, \mu, \nu)}(\psi)$ which is defined by

$$1 + \frac{1}{\epsilon} \left( \mathcal{K}'(a_1, b_1, l, m, \mu, \nu) f(z) + \nu z \mathcal{K}''(a_1, b_1, l, m, \mu, \nu) f(z) - 1 \right) \prec \psi(z), \tag{1.11}$$

where the operator $\mathcal{K}(a_1, b_1, l, m, \mu, \nu(\psi)$ wae introduced and studied by Selvaraj and Karthikeyan [22].

**(8)** Let $g(z) = z + \sum_{k=2}^{\infty} \left( \frac{1+b}{k+b} \right)^s \frac{(k+\mu-2)! \rho!}{(\mu-1)!(k+\rho-1)!} z^k$ $(b \in \mathbb{C} \backslash \mathbb{Z}_0^-, s \in \mathbb{C}, \mu > 0,\ \rho > -1)$ then the class $\mathcal{R}^{\epsilon}_{\nu, \mathcal{L}_g}(\psi)$ reduces to the class $\mathcal{R}^{\epsilon}_{\nu, J^{\rho,\mu}_{s,b}}(\psi)$ which is defined by

$$1 + \frac{1}{\epsilon} \left( J^{\rho,\mu \prime}_{s,b} f(z) + \nu z J^{\rho,\mu \prime\prime}_{s,b} f(z) - 1 \right) \prec \psi(z), \tag{1.12}$$

where the operator $J^{\rho,\mu}_{s,b}$ was introduced and studied by Al-Shaqsi et*al.* [3] and Darus et*al.* [9].

In the present investigation, we derive the Fekete-Szegö inequality for the class $\mathcal{R}^{\epsilon}_{\nu, \mathcal{L}_g}(\psi)$ and deduce the such type of results for some special classes also. Here are lemmas that are required in order to prove our main results. Lemma 1.2 of Ali et al. [4], is a reformulation of the corresponding result for functions with positive real part due to Ma and Minda [16].

Let $\Omega$ be the class of analytic functions $\omega$, normalized by the condition $\omega(0) = 0$ and satisfying $|\omega(z)| < 1$.

**Lemma 1.2.** *[4] If $\omega(z) = \omega_1 z + \omega_2 z^2 + \dots \in \Omega \ \ (z \in \mathbb{U})$, then*

$$|\omega_2 - t\omega_1^2| \leq \begin{cases} -t & (t \leq -1) \\ 1 & (-1 \leq t \leq 1) \\ t & (t \geq 1). \end{cases} \tag{1.13}$$

*For $t < -1$ or $t > 1$, equality holds if and only if $\omega(z) = z$ or one of its rotations. For $-1 < t < 1$, equality holds if and only if $\omega(z) = z^2$ or one of its rotations. Equality holds for $t = -1$ if and only if $\omega(z) = z(\lambda + z)/(1 + \lambda z) \ \ (0 \leq \lambda \leq 1)$ or one of its rotations, while for $t = 1$, equality holds if and only if $\omega(z) = -z(\lambda + z)/(1 + \lambda z) \ \ (0 \leq \lambda \leq 1)$ or one of its rotations. Also the sharp upper bound in the inequality (1.13) can be improved as follows when $-1 < t < 1$;*

$$|\omega_2 - t\omega_1^2| + (1+t)|\omega_1|^2 \leq 1, \quad (-1 < t \leq 0) \tag{1.14}$$

*and*

$$|\omega_2 - t\omega_1^2| + (1-t)|\omega_1|^2 \leq 1, \quad (0 < t \leq 1). \tag{1.15}$$

**Lemma 1.3.** *(see [15]) If $p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \dots (z \in \mathbb{U})$ is a function with positive real part, then for any complex number $\mu$*

$$|c_2 - \mu c_1^2| \leq 2 \max\{1, |2\mu - 1|\},$$

*and the result is sharp for the function given by*

$$p(z) = \frac{1+z^2}{1-z^2}, \quad p(z) = \frac{1+z}{1-z} \quad (z \in \mathbb{U}).$$

## 2. The Fekete-Szego Inequality

We begin with the following results for the class in $\mathcal{R}^{\epsilon}_{\nu,\mathcal{L}_g}(\psi)$.

**Theorem 2.1.** *let $g(z)$ be given by (1.2) with $b_2$, $b_3$ non zero real numbers. Also assume that $\epsilon > 0$, $0 \leq \nu \leq 1$ and $\psi(z) = 1 + B_1 z + B_2 z^2 + \cdots$. If $f \in \mathcal{R}^{\epsilon}_{\nu,\mathcal{L}_g}(\psi)$ and $\mu$ is any real number then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{\epsilon}{3|b_3|(1+2\nu)}\left\{B_2 - \frac{3b_3\mu\epsilon B_1^2(1+2\nu)}{4b_2^2(1+\nu)^2}\right\} & \text{if } \mu \leq \sigma_1 \\ \frac{\epsilon B_1}{3|b_3|(1+2\nu)} & \text{if } \sigma_1 \leq \mu \leq \sigma_2 \\ \frac{\epsilon}{3|b_3|(1+2\nu)}\left\{-B_2 + \frac{3b_3\mu\epsilon B_1^2(1+2\nu)}{4b_2^2(1+\nu)^2}\right\} & \text{if } \mu \geq \sigma_2. \end{cases} \tag{2.1}$$

*where*

$$\sigma_1 = \frac{4b_2^2(1+\nu)^2}{3b_3\epsilon B_1(1+2\nu)}\left\{-1 + \frac{B_2}{B_1}\right\}$$

$$\text{and} \qquad \sigma_2 = \frac{4b_2^2(1+\nu)^2}{3b_3\epsilon B_1(1+2\nu)}\left\{1 + \frac{B_2}{B_1}\right\}.$$

*The inequality (2.1) is sharp.*

*Further, when $\sigma_1 \leq \mu \leq \sigma_2$ the above result can be improved as follows, for this let*

$$\sigma_3 = \frac{4b_2^2(1+\nu)^2 B_2}{3b_3\epsilon B_1^2(1+2\nu)}.$$

*If $\sigma_1 < \mu \leq \sigma_3$, then*

$$|a_3 - \mu a_2^2| + \frac{4b_2^2(1+\nu)^2}{3|b_3|\epsilon B_1(1+2\nu)}\left(1 + \frac{3\mu\epsilon(1+2\nu)B_1 b_3}{4(1+\nu)^2 b_2^2} - \frac{B_2}{B_1}\right)|a_2|^2 \leq \frac{\epsilon B_1}{3|b_3|(1+2\nu)} \tag{2.2}$$

*and if $\sigma_3 \leq \mu < \sigma_2$, then*

$$|a_3 - \mu a_2^2| + \frac{4b_2^2(1+\nu)^2}{3|b_3|\epsilon B_1(1+2\nu)}\left(1 - \frac{3\mu\epsilon(1+2\nu)B_1 b_3}{4(1+\nu)^2 b_2^2} + \frac{B_2}{B_1}\right)|a_2|^2 \leq \frac{\epsilon B_1}{3|b_3|(1+2\nu)}. \tag{2.3}$$

*Proof.* Since $f \in \mathcal{R}^{\epsilon}_{\nu,\mathcal{L}_g}(\psi)$, then there exist an analytic function $\omega(z) = \omega_1 z + \omega_2 z^2 + ... \in \Omega$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ such that

$$1 + \frac{1}{\epsilon}\left(\mathcal{L}'_g f(z) + \nu z \mathcal{L}''_g f(z) - 1\right) = \psi(\omega(z)). \tag{2.4}$$

A calculation shows that

$$1 + \frac{1}{\epsilon}\left[\mathcal{L}'_g f(z) + \nu z \mathcal{L}''_g f(z) - 1\right] = 1 + \frac{2a_2 b_2}{\epsilon}(1+\nu)z + \frac{3a_3 b_3}{\epsilon}(1+2\nu)z^2 + ... \tag{2.5}$$

substituting these values in (2.4) and comparing both of the sides we get

$$\frac{2a_2 b_2}{\epsilon}(1+\nu) = B_1\omega_1 \tag{2.6}$$

$$\frac{3a_3 b_3}{\epsilon}(1+2\nu) = B_1\omega_2 + B_2\omega^2. \tag{2.7}$$

Buy using (2.6) and (2.7), we have

$$a_3 - \mu a_2^2 = \frac{\epsilon B_1}{3b_3(1+2\nu)}\left[\omega_2 - t\omega_1^2\right] \tag{2.8}$$

where

$$t = \frac{3\mu\epsilon(1+2\nu)B_1 b_3}{4(1+\nu)^2 b_2^2} - \frac{B_2}{B_1}. \tag{2.9}$$

If $t \leq -1$, then

$$\frac{3\mu\epsilon(1+2\nu)B_1 b_3}{4(1+\nu)^2 b_2^2} - \frac{B_2}{B_1} \leq -1,$$

which implies

$$\mu \leq \frac{4b_2^2(1+\nu)^2}{3b_3\epsilon B_1(1+2\nu)}\left\{-1+\frac{B_2}{B_1}\right\} =: \sigma_1.$$

Now an application of Lemma 1.2 gives

$$|a_3 - \mu a_2^2| \leq \frac{\epsilon}{3|b_3|(1+2\nu)}\left\{B_2 - \frac{3b_3\mu\epsilon B_1^2(1+2\nu)}{4b_2^2(1+\nu)^2}\right\} \quad (\mu \leq \sigma_1),$$

which is the first part of assertion (2.1).

Next, if $t \geq 1$, then

$$\frac{3\mu\epsilon(1+2\nu)B_1 b_3}{4(1+\nu)^2 b_2^2} - \frac{B_2}{B_1} \geq 1.$$

Which implies

$$\mu \geq \frac{4b_2^2(1+\nu)^2}{3b_3\epsilon B_1(1+2\nu)}\left\{1+\frac{B_2}{B_1}\right\} =: \sigma_2,$$

applying Lemma 1.2, we have

$$|a_3 - \mu a_2^2| \leq \frac{\epsilon}{3|b_3|(1+2\nu)}\left\{-B_2 + \frac{3b_3\mu\epsilon B_1^2(1+2\nu)}{4b_2^2(1+\nu)^2}\right\} \quad (\mu > \sigma_2),$$

which is essentially the third part of assertion (2.1).
Finally if $-1 \leq t \leq 1$, then

$$-1 \leq \frac{3\mu\epsilon(1+2\nu)B_1 b_3}{4(1+\nu)^2 b_2^2} - \frac{B_2}{B_1} \leq 1.$$

Which shows that $\sigma_1 \leq \mu \leq \sigma_2$. Thus by an application of Lemma 1.2, we obtain

$$|a_3 - \mu a_2^2| \leq \frac{\epsilon B_1}{3|b_3|(1+2\nu)} \quad (\sigma_1 \leq \mu \leq \sigma_2),$$

which is the second part of assertion (2.1).

Further when $\sigma_1 < \mu < \sigma_2$ the above result can be improved as follows:
If $-1 < t \leq 0$, then

$$-1 < \frac{3\mu\epsilon(1+2\nu)B_1 b_3}{4(1+\nu)^2 b_2^2} - \frac{B_2}{B_1} \leq 0,$$

which implies that $\sigma_1 < \mu \leq \sigma_3$, where

$$\sigma_3 := \frac{4b_2^2(1+\nu)^2 B_2}{3b_3\epsilon B_1^2(1+2\nu)}.$$

Now using (1.14), (2.8) and (2.9), we have

$$\frac{3|b_3|(1+2\nu)}{\epsilon B_1}|a_3 - \mu a_2^2| + \left(1 + \frac{3\mu\epsilon(1+2\nu)B_1 b_3}{4(1+\nu)^2 b_2^2} - \frac{B_2}{B_1}\right)|\omega_1|^2 \leq 1. \tag{2.10}$$

Substituting the value of $\omega_1^2$ from (2.6) to (2.10) and simplifying, we have

$$|a_3 - \mu a_2^2| + \frac{4b_2^2(1+\nu)^2}{3|b_3|\epsilon B_1(1+2\nu)}\left(1 + \frac{3\mu\epsilon(1+2\nu)B_1 b_3}{4(1+\nu)^2 b_2^2} - \frac{B_2}{B_1}\right)|a_2|^2 \leq \frac{\epsilon B_1}{3|b_3|(1+2\nu)}, \quad (\sigma_1 < \mu \leq \sigma_3).$$

Further if $0 \leq t < 1$, then $\sigma_3 \leq \mu < \sigma_2$. Now a similar computation using (1.15), (2.6), (2.8) and (2.9) gives us

$$|a_3 - \mu a_2^2| + \frac{4b_2^2(1+\nu)^2}{3|b_3|\epsilon B_1(1+2\nu)}\left(1 - \frac{3\mu\epsilon(1+2\nu)B_1 b_3}{4(1+\nu)^2 b_2^2} + \frac{B_2}{B_1}\right)|a_2|^2 \leq \frac{\epsilon B_1}{3|b_3|(1+2\nu)}.$$

This completes the proof. □

From Theorem 2.1, we deduce the following results:

**Corollary 2.2.** *let $g(z)$ be given by (1.2) with $b_2$, $b_3$ non zero real numbers. Also assume that $\epsilon > 0$ and $-1 \leq D < C \leq 1$. If $f \in \mathcal{R}^{\epsilon}_{\nu,\mathcal{L}_g}(\frac{1+Cz}{1+Dz})$ and $\mu$ is any real number then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{\epsilon}{3|b_3|(1+2\nu)} \left\{ D(D-C) - \frac{3b_3\mu\epsilon(C-D)^2(1+2\nu)}{4b_2^2(1+\nu)^2} \right\} & \text{if } \mu \leq \sigma_1 \\ \frac{\epsilon(C-D)}{3|b_3|(1+2\nu)} & \text{if } \sigma_1 \leq \mu \leq \sigma_2 \\ \frac{\epsilon}{3|b_3|(1+2\nu)} \left\{ -D(D-C) + \frac{3b_3\mu\epsilon(C-D)^2(1+2\nu)}{4b_2^2(1+\nu)^2} \right\} & \text{if } \mu \geq \sigma_2. \end{cases} \tag{2.11}$$

*where*

$$\sigma_1 = \frac{4b_2^2(1+\nu)^2}{3b_3\epsilon(1+2\nu)} \left\{ \frac{1+D}{D-C} \right\}$$

$$\text{and} \quad \sigma_2 = \frac{4b_2^2(1+\nu)^2}{3b_3\epsilon(1+2\nu)} \left\{ \frac{1-D}{C-D} \right\}.$$

*The inequality (2.11) is sharp.*

*The above result can be improved when $\sigma_1 \leq \mu \leq \sigma_2$ as follows.*
*Let*

$$\sigma_3 = \frac{4b_2^2(1+\nu)^2 D}{3b_3\epsilon(1+2\nu)(D-C)}.$$

*If $\sigma_1 < \mu < \sigma_3$, then*

$$|a_3 - \mu a_2^2| + \frac{4b_2^2(1+\nu)^2}{3|b_3|\epsilon(1+2\nu)(C-D)} \left( 1 + D + \frac{3\mu\epsilon(1+2\nu)(C-D)b_3}{4(1+\nu)^2 b_2^2} \right) |a_2|^2 \leq \frac{\epsilon(C-D)}{3|b_3|(1+2\nu)} \tag{2.12}$$

*and if $\sigma_3 < \mu < \sigma_2$, then*

$$|a_3 - \mu a_2^2| + \frac{4b_2^2(1+\nu)^2}{3|b_3|\epsilon(1+2\nu)(C-D)} \left( 1 - D - \frac{3\mu\epsilon(1+2\nu)(C-D)b_3}{4(1+\nu)^2 b_2^2} \right) |a_2|^2 \leq \frac{\epsilon(C-D)}{3|b_3|(1+2\nu)}. \tag{2.13}$$

By taking $D = -1$ and $C = 1$ in the above Corollary 2.2, we obtain the following:
**Example** let $g(z)$ be given by (1.2) with $b_2$, $b_3$ non zero real numbers. If $f \in \mathcal{R}^{\epsilon}_{\nu,\mathcal{L}_g}(\frac{1+z}{1-z})$ and $\mu$ is any real number then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{2\epsilon}{3|b_3|(1+2\nu)} \left\{ 1 - \frac{3b_3\mu\epsilon(1+2\nu)}{2b_2^2(1+\nu)^2} \right\} & \text{if } \mu \leq \sigma_1 \\ \frac{2\epsilon}{3|b_3|(1+2\nu)} & \text{if } \sigma_1 \leq \mu \leq \sigma_2 \\ \frac{2\epsilon}{3|b_3|(1+2\nu)} \left\{ -1 + \frac{3b_3\mu\epsilon(1+2\nu)}{2b_2^2(1+\nu)^2} \right\} & \text{if } \mu \geq \sigma_2. \end{cases} \tag{2.14}$$

where

$$\sigma_1 = 0 \ \text{ and } \quad \sigma_2 = \frac{4b_2^2(1+\nu)^2}{3b_3\epsilon(1+2\nu)}.$$

The inequality (2.14) is sharp.

Using Lemma 1.3 and equation (2.8), we deduce the following:

**Theorem 2.3.** *Let $\psi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + ...$, where $\psi(0) = 0$ and $\psi'(0) > 0$. If $f(z)$ is given by (1.1) belongs to $\mathcal{R}^{\epsilon}_{\nu,\mathcal{L}_g}(\psi)$, $(0 \leq \nu \leq 1,\ \epsilon \in \mathbb{C}\backslash\{0\}\ \ z \in \mathbb{U})$, then for any complex number $\mu$*

$$|a_3 - \mu a_2^2| \leq \frac{B_1|\epsilon|}{3|b_3|(1+2\nu)} \max \left\{ 1, \left| \frac{B_2}{B_1} - \frac{3}{4}\frac{b_3}{b_2^2}\frac{\mu\epsilon B_1(1+2\nu)}{(1+\nu)^2} \right| \right\}. \tag{2.15}$$

*The result is sharp.*

**Remark 2.4.** *If we put $g(z) = \frac{z}{1-z}$ in (1.4), we have the result([5], Theorem 2.1.) of D. Bansal.*

Putting $B_1 = (A-B)$ and $B_2 = -B(A-B)$ in Theorem 2.3, we get the following corollary.

**Corollary 2.5.** *Let* $\psi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots$ *. If* $f(z)$ *is given by* (1.1) *belongs to* $\mathcal{R}^{\epsilon}_{\nu,\mathcal{L}_g}(A,B)$, $(0 \leq \nu \leq 1,\ \epsilon \in \mathbb{C}\backslash\{0\},\ z \in \mathbb{U})$, *then for any complex number* $\mu$

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)|\epsilon|}{3|b_3|(1+2\nu)} \max\left\{1, \left|B + \frac{3}{4}\frac{b_3}{b_2^2}\frac{\mu\epsilon(A-B)(1+2\nu)}{(1+\nu)^2}\right|\right\}. \tag{2.16}$$

*The result is sharp.*

Putting $g(z) = z + \sum_{k=2}^{\infty}\left(\frac{(l+1)+\theta(k-1)}{l+1}\right)^m z^k$, where $\theta > 0,\ l \geq 0$ and $m \in \mathbb{N}_0$ in (1.4), we obtain the following corollary.

**Corollary 2.6.** *Let* $\psi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots$ *. If* $f(z)$ *is given by* (1.1) *belongs to* $\mathcal{R}^{\epsilon}_{\nu,I_g}(\psi)$, $(0 \leq \nu \leq 1, \epsilon \in \mathbb{C}\backslash\{0\}\ z \in \mathbb{U})$, *then*

$$|a_3 - \mu a_2^2| \leq \frac{B_1|\epsilon||l+1|^m}{3(1+2\nu)|l+1+2\theta|^m} \max\left\{1, \left|\frac{B_2}{B_1} - \frac{3}{4}\frac{(l+1)^m(l+1+2\theta)^m}{(l+1+\theta)^m}\frac{\mu\epsilon B_1(1+2\nu)}{(1+\nu)^2}\right|\right\}. \tag{2.17}$$

Putting $g(z) = z + \sum_{k=2}^{\infty}\left(\frac{l+k}{l+1}\right)^m z^k$, where $l \geq 0$ and $m \in \mathbb{N}_0$ in (1.4), then the class $\mathcal{R}^{\epsilon}_{\nu,\mathcal{L}_g}(\psi)$ we get the following corollary.

**Corollary 2.7.** *Let* $\psi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots$ *. If* $f(z)$ *is given by* (1.1) *belongs to* $\mathcal{R}^{\epsilon}_{\nu,\mathcal{S}_g}(\psi)$, $(0 \leq \nu \leq 1, \epsilon \in \mathbb{C}\backslash\{0\},\ m \in \mathbb{N}_0,\ z \in \mathbb{U})$, *then*

$$|a_3 - \mu a_2^2| \leq \frac{B_1|\epsilon||l+1|^m}{3(1+2\nu)|l+3|^m} \max\left\{1, \left|\frac{B_2}{B_1} - \frac{3}{4}\frac{(l+3)^m(l+1)^m}{(l+2)^{2m}}\frac{\mu\epsilon B_1(1+2\nu)}{(1+\nu)^2}\right|\right\}. \tag{2.18}$$

Putting $g(z) = z + \sum_{k=2}^{\infty}(1+\theta(k-1))^m z^k$, where $\theta > 0$ and $m \in \mathbb{N}_0$, we get the following corollary.

**Corollary 2.8.** *Let* $\psi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots$ *. If* $f(z)$ *is given by* (1.1) *belongs to* $\mathcal{R}^{\epsilon}_{\nu,\mathcal{D}_g}(\psi)$, $(0 \leq \nu \leq 1, \epsilon \in \mathbb{C}\backslash\{0\},\ m \in \mathbb{N}_0,\ z \in \mathbb{U})$, *then*

$$|a_3 - \mu a_2^2| \leq \frac{B_1|\epsilon|}{3(1+2\nu)|(1+2\theta)|^m} \max\left\{1, \left|\frac{B_2}{B_1} - \frac{3}{4}\frac{(1+2\theta)^m}{(1+\theta)^m}\frac{\mu\epsilon B_1(1+2\nu)}{(1+\nu)^2}\right|\right\}. \tag{2.19}$$

## 3. Concluding Remarks and Observations

In this article we define a new subclass $\mathcal{R}^{\epsilon}_{\nu,\mathcal{L}_g}(\psi)$ of analytic function by means of convolution. We discuss the relevance of this class with some known subclasses of analytic functions.

The results depicted (see Theorem 2.1 and Theorem 2.3) which lead to various interesting results. In Theorem 2.1, we found more improved bound using the relations (1.14) and (1.15).

We deem it proper to point out some of the known special cases which arise from the results proved above. Similarly for the Theorem 2.3, we deduce the set of corollaries from 2.5 to 2.8.

Regarding the scope of future work, the sharp Fekete-Szego functional for $Q$-version of the class $\mathcal{R}^{\epsilon}_{\nu,\mathcal{L}_g}(\psi)$ using $Q$-calculus approach may be obtained.

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