Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 2023 (41)** : 1–7.
ISSN-0037-8712 in press
doi:10.5269/bspm.51598

Operators in Terms of $*$ and ψ

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ABSTRACT: Through this paper we consider three operators in terms of operators $*$ and ψ in an ideal topological space. Many properties of these operators have been discussed. Characterizations of Hayashi-Samuel spaces are obtained as applications of the properties.

Key Words: Ideal, local function, ψ-operator, Hayashi-Samuel space, operator $\Delta_i (i = 1, 2, 3)$.

Contents

1. Introduction and Preliminaries

If (X, τ) is a topological space and $\mathcal{I}$ is an ideal on X, then for $A \subseteq X$, the local function [7] is defined as $A^*(\mathcal{I}, \tau) = \{x \in X : U_x \cap A \notin \mathcal{I} \text{ for every } U_x \in \tau(x)\}$, where $\tau(x)$ is the collection of all open sets containing x. $A^*(\mathcal{I}, \tau)$ is simply denoted as $A^*(\mathcal{I})$ or A^*. For the simplest ideals $\{\emptyset\}$ and $\wp(X)$, we observe that $A^*(\{\emptyset\}) = cl(A)$ ($cl(A)$ denotes the closure of A) and $A^*(\wp(X)) = \emptyset$ for every $A \subseteq X$.

The complement set-operator of the set-operator $()^*$ is ψ [13] and it is defined as $\psi(A) = X \setminus (X \setminus A)^*$. It is notable that $()^*$ is not a closure operator and ψ is not an interior operator. However, the set operator $C : \wp(X) \to \wp(X)$ defined by $C(A) = A \cup A^*$ makes a closure operator [7,8,16] and it is denoted as 'cl^*', that is $cl^*(A) = A \cup A^*$. This closure operator induces a topology on X and it is called $*$-topology [6,5,4,9,14,1]. This topology is denoted as τ^* and its interior operator is denoted as 'int^*'.

In the study of ideal topological spaces, two ideals are important: one is codense ideal [3]; and another is compatible ideal [12]. An ideal $\mathcal{I}$ on a topological space (X, τ) is called a codense ideal if $\mathcal{I} \cap \tau = \{\emptyset\}$. Such type of spaces are called Hayashi-Samuel spaces [2]. Some authors called it τ-boundary [4,11].

In this paper, by using $()^*$ and ψ-operator, we introduce some new types of set-operators. These new set-operators give us new characterizations of Hayashi-Samuel spaces and various relationships between $()^*$ and ψ-operator. A topological space (X, τ) with an ideal $\mathcal{I}$ on X is called an ideal topological space and is denoted by $(X, \tau, \mathcal{I})$.

Lemma 1.1. *[7] Let $(X, \tau, \mathcal{I})$ be an ideal topological space. The following are equivalent.*

1. *$X^* = X$,*
2. *$\tau \cap \mathcal{I} = \{\emptyset\}$,*
3. *If $I \in \mathcal{I}$, then $int(I) = \emptyset$, and*
4. *For every $U \in \tau$, $U \subseteq U^*$.*

Lemma 1.2. *[10] Let $(X, \tau, \mathcal{I})$ be a Hayashi-Samuel space. Then for $A \subseteq X$, $\psi(A) \subseteq A^*$.*

2010 *Mathematics Subject Classification:* 54A05.
Submitted December 22, 2020. Published September 01, 2020

Lemma 1.3. *[4,7] Let $(X, \tau, \mathcal{I})$ be an ideal topological space, and let A and B be subsets of X. Then the following properties hold:*

1. *If $A \subseteq B$, then $A^* \subseteq B^*$,*
2. *$A^* = cl(A^*) \subseteq cl(A)$,*
3. *$(A^*)^* \subseteq A^*$,*
4. *$(A \cup B)^* = A^* \cup B^*$,*
5. *If $I \in \mathcal{I}$, then $(A \cup I)^* = A^* = (A \setminus I)^*$,*
6. *If $U \in \tau^*$, then $U \subseteq \psi(U)$.*

The present authors [15] defined the operators $\veebar$, $\bar{\wedge}$ and $\wedge$ on an ideal topological space $(X, \tau, \mathcal{I})$ as follows: for a subset A of X, $\veebar(A) = \psi(A) \cap \psi(X \setminus A)$, $\bar{\wedge}(A) = A \setminus A^*$ and $\wedge(A) = \psi(A) \setminus A$, respectively. Further they have shown that $\veebar(A) = \psi(A) \setminus A^*$.

Before starting the main sections, we consider the following definition from literature:

Definition 1.4. *A set-valued set function $p : \wp(X) \to \wp(X)$ is said to be grounded (resp. idempotent, sub-additive, additive) if $p(\emptyset) = \emptyset$ (resp. $p(p(A)) = p(A)$, $p(A \cup B) \subseteq p(A) \cup p(B)$, $p(\bigcup_{l \in \Lambda} A_l) = \bigcup_{l \in \Lambda} p(A_l)$), where $A, B, A_l \in \wp(X)$ for all $l \in \Lambda$(index set).*

2. The operator Δ_1

We define the operator Δ_1 on an ideal topological space $(X, \tau, \mathcal{I})$ in the following way: for a subset A of X, $\Delta_1(A) = \veebar(A) \cup \wedge(A) = (\psi(A) \setminus A^*) \cup (\psi(A) \setminus A)$.

For the ideal topological space $(X, \tau, \mathcal{I})$, if $\mathcal{I} = \{\emptyset\}$ (resp. $\mathcal{I} = \wp(X)$), then $\Delta_1(A) = \emptyset$ (resp. X) for every subset A of X.

Lemma 2.1. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and A a subset of X. Then $\Delta_1(A) = \psi(A) \setminus (A^* \cap A)$.*

Proof. $\Delta_1(A) = (\psi(A) \setminus A^*) \cup (\psi(A) \setminus A) = (\psi(A) \cap (X \setminus A^*)) \cup (\psi(A) \cap (X \setminus A)) = \psi(A) \cap [(X \setminus A^*) \cup (X \setminus A)] = \psi(A) \cap [X \setminus (A^* \cap A)] = \psi(A) \setminus (A^* \cap A)$. □

Theorem 2.2. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then following statements hold:*

1. *for $A,\ B \subseteq X$, $\psi(A \cup B) \supseteq \Delta_1(A) \cup \Delta_1(B)$.*
2. *for $A \subseteq X$, $\Delta_1(A) = \psi(A) \setminus (\psi(A) \cap A^* \cap A)$.*
3. *for $A \subseteq X$, $\Delta_1(A) = \psi(A) \setminus (int^*(A) \cap A^*)$.*
4. *for $U \in \tau$, $\Delta_1(U) \supseteq \wedge(U)$.*
5. *for $U \in \tau^*$, $\Delta_1(U) \supseteq \wedge(U)$.*
6. *for clopen subset A of X, $\Delta_1(A) = \veebar(A)$.*
7. *for $A \subseteq X$, $\Delta_1(X \setminus A) = \veebar(A) \cup \bar{\wedge}(A)$.*
8. *for $A \subseteq X$, $\Delta_1(X \setminus A) \supseteq cl^*(A) \setminus A^*$.*
9. *for $J \in \mathcal{I}$, $\Delta_1(J) = X \setminus X^* = \veebar(A)$.*
10. *for $J \in \mathcal{I}$, $\Delta_1(X \setminus J) = X \setminus (X^* \setminus J)$.*

Proof. 1. We know $\psi(A) \setminus A^* \subseteq \psi(A \cup B) \setminus A^*$, $\psi(A) \setminus A \subseteq \psi(A \cup B) \setminus A$, $\psi(B) \setminus B^* \subseteq \psi(A \cup B) \setminus B^*$ and $\psi(B) \setminus B \subseteq \psi(A \cup B) \setminus B$. Then $\Delta_1(A) \cup \Delta_1(B) \subseteq [[\psi(A \cup B) \setminus A^*] \cup [\psi(A \cup B) \setminus A]] \cup [[\psi(A \cup B) \setminus B^*] \cup [\psi(A \cup B) \setminus B]] \subseteq \psi(A \cup B)$.

5. $\Delta_1(U) = \psi(U) \setminus (Int^*(U) \cap U^*) = \psi(U) \setminus (U \cap U^*)$, since $U \in \tau^*$. Thus, $\Delta_1(U) \supseteq \psi(U) \setminus U = \wedge(U)$.

6. $\Delta_1(A) = \psi(A) \setminus (int^*(A) \cap A^*) = \psi(A) \setminus (A \cap A^*)$, since $A \in \tau^*$. This implies that $\Delta_1(A) = \psi(A) \setminus A^*$ (as $A^* \subseteq cl(A) = A) = \veebar(A)$.

7. $\Delta_1(X \setminus A) = [\psi(X \setminus A) \setminus (X \setminus A)^*] \cup [\psi(X \setminus A) \setminus (X \setminus A)] = [(X \setminus A^*) \setminus (X \setminus A)^*] \cup [(X \setminus A^*) \setminus (X \setminus A)] = (\psi(A) \setminus A^*) \cup (A \setminus A^*) = \veebar(A) \cup \barwedge(A)$.

8. $\Delta_1(X \setminus A) = \psi(X \setminus A) \setminus (int^*(X \setminus A) \cap (X \setminus A)^*) \supseteq \psi(X \setminus A) \setminus int^*(X \setminus A) = (X \setminus A^*) \setminus (X \setminus cl^*(A)) = cl^*(A) \setminus A^*$.

9. $\Delta_1(J) = \psi(J) \setminus (\psi(J) \cap J^* \cap J) = (X \setminus X^*) = \veebar(A)$, since $J^* = \emptyset$.

10. $\Delta_1(X \setminus J) = \psi(X \setminus J) \setminus \ [(X \setminus J)^* \cap (X \setminus J)] = (X \setminus J^*) \setminus [(X^* \cap (X \setminus J)] = X \setminus (X^* \setminus J)$. □

The following example shows that the operator Δ_1 is not grounded:

Example 2.3. *Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a, b\}, \{c\}\}$ and $\mathfrak{I} = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$. Then $\psi(\emptyset) = X \setminus X^* = X \setminus \{a, b\} = \{c\}$, $\emptyset^* = \emptyset$. Therefore, $\Delta_1(\emptyset) = \{c\}$.*

Therefore we conclude that the operator Δ is not subadditive and additive.

The following example shows that the operator Δ_1 operator is not an idempotent operator.

Example 2.4. *Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a, b\}, \{c\}\}$ and $\mathfrak{I} = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$. Let $A = \{b\}$. Then $\psi(A) = X \setminus \{a, c\}^* = X \setminus \emptyset = X$, $int^*(A) = A \cap \psi(A) = \{b\} \cap X = \{b\}$ and $A^* = \{b\}^* = \{a, b\}$. Thus, $\Delta_1(A) = \psi(A) \setminus (A \cap A^*) = X \setminus (\{b\} \cap \{a, b\}) = X \setminus \{b\} = \{a, c\}$. Now $\psi(\{a, c\}) = X \setminus \{b\}^* = X \setminus \{a, b\} = \{c\}$, $\{a, c\}^* = \emptyset$ and $int^*(\{a, c\}) = \{c\}$. Therefore, $\Delta_1(\Delta_1(A)) = \psi(\{a, c\}) \setminus (int^*(\{a, c\}) \cap \{a, c\}^*) = \{c\}$.*

Theorem 2.5. *Let $(X, \tau, \mathfrak{I})$ be a Hayashi-Samuel space. Then following statements hold:*

1. *for $A \subseteq X$, $\Delta_1(A) \subseteq A^* \setminus A$*
2. *for $A \subseteq X$, $\Delta_1(A) \subseteq A^*$.*
3. *for $A \subseteq X$, $cl(\Delta_1(A)) \subseteq A^*$.*
4. *for $A \subseteq X$, $cl^*(\Delta_1(A)) \subseteq A^*$.*
5. *for regular open set U, $\Delta_1(U) \subseteq U$.*
6. *for $U \in \tau$, $\Delta_1(U) = \wedge(U)$.*
7. *for $J \in \mathfrak{I}$, $\Delta_1(J) = \emptyset$.*
8. *for $J \in \mathfrak{I}$, $\Delta_1(X \setminus J) = J$.*

Proof. 1. By Lemma 1.2, $\Delta_1(A) = (\psi(A) \setminus A^*) \cup (\psi(A) \setminus A) \subseteq (A^* \setminus A^*) \cup (A^* \setminus A) = A^* \setminus A$.

5. It follows from Theorem 5 of [4] that $\Delta_1(U) = \psi(U) \setminus (U \cap U^*) = U \setminus (U \cap U^*) \subseteq U$.

6. By Lemma 1.1, $\Delta_1(U) = \psi(U) \setminus (U \cap U^*) = \psi(U) \setminus U = \wedge(U)$.

7. From Theorem 2.2(9), $\Delta_1(J) = X \setminus X^* = \emptyset$, since $X = X^*$.

8. The proof is obvious from Theorem 2.2(10). □

Theorem 2.6. *Let $(X, \tau, \mathfrak{I})$ be an ideal topological space and $J \in \mathfrak{I}$. If $\Delta_1(X \setminus J) = J$, then $X^{**} = X^*$.*

Proof. From Theorem 2.2(10) we have $\Delta_1(X \setminus J) = X \setminus (X^* \setminus J)$. Given that $X \setminus (X^* \setminus J) = J$. Then $X^* \setminus J = X \setminus J$. By Lemma 1.3, $X^{**} = (X^* \setminus J)^* = (X \setminus J)^* = X^*$ and hence $X^{**} = X^*$. □

By the next example, $X^{**} = X^*$ need not imply that the space is Hayashi-Samuel.

Example 2.7. *Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}\}$, $\mathcal{I} = \{\emptyset, \{a\}\}$. Then $X^* = \{b, c\}$ and $X^{**} = \{b, c\}$. But the space $(X, \tau, \mathcal{I})$ is not Hayashi-Samuel.*

Theorem 2.8. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then the space is Hayashi-Samuel if one of the following conditions hold:*

1. *for each $J \in \mathcal{I}$, $\Delta_1(J) = \emptyset$;*
2. *for each $U \in \tau$, $\Delta_1(U) = \emptyset$.*

Proof. 1. $\Delta_1(J) = X \setminus X^* = \emptyset$. Then $X = X^*$. Thus from Lemma 1.1, the space is Hayashi-Samuel.

2. $\Delta_1(U) = \psi(U) \setminus (int^*(U) \cap U^*) = \psi(U) \setminus (U \cap U^*) = \emptyset$ (given). Thus $U \subseteq \psi(U) \subseteq U \cap U^* \subseteq U^*$. This implies that $U \subseteq U^*$. Therefore from Lemma 1.1, the space is Hayashi-Samuel. □

Corollary 2.9. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then following statements are equivalent:*

1. *The space $(X, \tau, \mathcal{I})$ is Hayashi-Samuel;*
2. *For $J \in \mathcal{I}$, $\Delta_1(J) = \emptyset$;*
3. *For each $U \in \tau$, $\Delta_1(U) = \emptyset$.*

3. The operator Δ_2

We define the operator Δ_2 on an ideal topological space $(X, \tau, \mathcal{I})$ in the following way: for a subset A of X, $\Delta_2(A) = \veebar(A) \cup \barwedge(A) = (\psi(A) \setminus A^*) \cup (A \setminus A^*)$.

Lemma 3.1. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then $\Delta_2(A) = (\psi(A) \cup A) \setminus A^*$ for every subset A of X.*

Proof. $\Delta_2(A) = (\psi(A) \setminus A^*) \cup (A \setminus A^*) = (\psi(A) \cap (X \setminus A^*)) \cup (A \cap (X \setminus A^*)) = (\psi(A) \cup A) \cap (X \setminus A^*) = (\psi(A) \cup A) \setminus A^*$.

□

For the ideal topological space $(X, \tau, \{\emptyset\})$, the value of Δ_2 on any subset A of X is $\emptyset$. Further, the value of $\Delta_2(A)$ on the ideal topological space $(X, \tau, \wp(X))$ is X.

Theorem 3.2. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then followings hold:*

1. *$\Delta_2(\emptyset) = \veebar(A)$, for any $A \subseteq X$.*
2. *$\Delta_2(X) = \Delta_2(\emptyset) = \veebar(A)$, for any $A \subseteq X$.*
3. *if $U \in \tau^*$, then $\Delta_2(U) = \psi(U) \setminus U^* = \veebar(U)$.*
4. *if A is clopen, then $\wedge(A) \subseteq \Delta_2(A) \subseteq \psi(A)$.*
5. *if $J \in \mathcal{I}$, then $\Delta_2(X \setminus J) = X \setminus X^*$.*
6. *for any $A \subseteq X$, $\Delta_2(X \setminus A) = \Delta_1(A)$.*

Proof. 1. $\Delta_2(\emptyset) = (\psi(\emptyset) \cup \emptyset) \setminus \emptyset^* = X \setminus X^* = \veebar(A)$.

2. $\Delta_2(X) = (\psi(X) \cup X) \setminus X^* = X \setminus X^* = \Delta_2(\emptyset) = \veebar(A)$.

3. $\Delta_2(U) = (\psi(U) \cup U) \setminus U^* = \psi(U) \setminus U^*$ (as $U \subseteq \psi(U)$) $= \veebar(U)$.

4. First, $\wedge(A) = \psi(A) \setminus A \subseteq \psi(A) \setminus A^* \subseteq (\psi(A) \cup A) \setminus A^* = \Delta_2(A)$.

Next, we show that $\Delta_2(A) \subseteq \psi(A)$. Since A is clopen, $X \setminus A$ is clopen and by Lemma 3.1 $(X \setminus A)^* \subset cl(X \setminus A) = X \setminus A$.

Therefore, $\psi(A) = X \setminus (X \setminus A)^* \supseteq X \setminus (X \setminus A) = A \supseteq A^*$. Hence $A^* \cup \psi(A) = A \cup \psi(A) = \psi(A)$ and hence we have $\Delta_2(A) = (A \cup \psi(A)) \setminus A^* = \psi(A) \setminus A^* \subseteq \psi(A)$.

5. $\Delta_2(X \setminus J) = [\psi(X \setminus J) \cup (X \setminus J)] \setminus X^* = [(X \setminus J^*) \cup (X \setminus J)] \setminus X^* = X \setminus X^*$.

6. $\Delta_2(X \setminus A) = [\psi(X \setminus A) \setminus (X \setminus A)^*] \cup [(X \setminus A) \setminus (X \setminus A)^*] = [(X \setminus A^*) \setminus (X \setminus A)^*] \cup (\psi(A) \setminus A) = (\psi(A) \setminus A^*) \cup (\psi(A) \setminus A) = \Delta_1(A)$. □

In general, the operator Δ_2 is neither grounded, nor isotonic, nor subadditive, nor additive.

Theorem 3.3. *Let $(X, \tau, \mathcal{I})$ be a Hayashi-Samuel space. Then following hold:*

1. $\Delta_2(\emptyset) = \emptyset$.
2. $\Delta_2(X) = \emptyset$.
3. *for* $A \subseteq X$, *then* $\Delta_2(A) \subseteq A$.
4. *if* A *is clopen, then* $\wedge(A) \subseteq \Delta_2(A) \subseteq A$.
5. *if* $U \in \tau$, *then* $\Delta_2(U) = \emptyset$.
6. *if* $J \in \mathcal{I}$, *then* $\Delta_2(J) = J$.
7. *if* $J \in \mathcal{I}$, *then* $\Delta_2(X \setminus J) = \emptyset$.

Proof. 1. $\Delta_2(\emptyset) = \psi(\emptyset) \setminus \emptyset^* = X \setminus X^* = \emptyset$, since the space is Hayashi-Samuel.
2. $\Delta_2(X) = X \setminus X^* = \emptyset$, since the space is Hayashi-Samuel.
3. By Lemma 1.2, $\Delta_2(A) = (\psi(A) \cup A) \setminus A^* \subseteq (A^* \cup A) \setminus A^* \subseteq A$.
5. By Lemmas 1.1 and 1.2, $\Delta_2(U) = (\psi(U) \cup U) \setminus U^* \subseteq (U^* \cup U) \setminus U^* \subseteq U^* \setminus U^* = \emptyset$.
6. Since $J^* = \emptyset, \Delta_2(J) = (\psi(J) \cup J) \setminus J^* = [X \setminus (X \setminus J)^*] \cup J = (X \setminus X^*) \cup J = J$.
7. By Theorem 3.2(5), $\Delta_2(X \setminus J) = X \setminus X^* = \emptyset$.

□

Theorem 3.4. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then the space is Hayashi-Samuel if one of the following conditions is hold:*

1. $\Delta_2(\emptyset) = \emptyset$;
2. $\Delta_2(X) = \emptyset$;
3. *for each* $U \in \tau$, $\Delta_2(U) = \emptyset$;
4. *for each* $J \in \mathcal{I}$, $\Delta_2(X \setminus J) = \emptyset$.

Proof. 1. Given that $\Delta_2(\emptyset) = \veebar(A) = X \setminus X^* = \emptyset$. Then $X \subseteq X^*$ and hence from Lemma 1.1, the space is Hayashi-Samuel.
2. Given that $\Delta_2(X) = \Delta_2(\emptyset) = X \setminus X^* = \emptyset$. Thus from Lemma 1.1, the space is Hayashi-Samuel.
3. For each $U \in \tau$, by Theorem 3.2(3), $\Delta_2(U) = \psi(U) \setminus U^* = \veebar(U) = X \setminus X^*$. Since $\Delta_2(U) = \emptyset$, $X \setminus X^* = \emptyset$. This implies that $(X, \tau, \mathcal{I})$ is the Hayashi-Samuel space.
4. By Theorem 3.2(5), $\Delta_2(X \setminus J) = X \setminus X^* = \emptyset$. Thus from Lemma 1.1, the space is Hayashi-Samuel.

□

Corollary 3.5. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then following statements are equivalent:*

1. *The space $(X, \tau, \mathcal{I})$ is Hayashi-Samuel;*
2. $\Delta_2(\emptyset) = \emptyset$;
3. $\Delta_2(X) = \emptyset$;
4. *for each* $U \in \tau$, $\Delta_2(U) = \emptyset$;
5. *for each* $J \in \mathcal{I}$, $\Delta_2(X \setminus J) = \emptyset$.

Theorem 3.6. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $J \in \mathcal{I}$. If $\Delta_2(J) = J$, then $X^* = X^{**}$.*

Proof. Given that $\Delta_2(J) = \psi(J) \cup J = J$. Then $\psi(J) \subseteq J$ implies $X \setminus X^* \subseteq J$. Thus, $(X \setminus X^*)^* \subseteq J^* = \emptyset$. By Lemma 1.3, $X^* = ((X \setminus X^*) \cup X^*)^* = (X \setminus X^*)^* \cup X^{**} = \emptyset \cup X^{**} = X^{**}$ and hence $X^* = X^{**}$.

□

By Example 2.7, $X^* = X^{**}$ need not imply that the space is Hayashi-Samuel.

4. The operator Δ_3

We define the operator Δ_3 on an ideal topological space $(X, \tau, \mathcal{I})$ in the following way: for any subset A of X, $\Delta_3(A) = \overline{\wedge}(A) \cup \wedge(A) = (A \setminus A^*) \cup (\psi(A) \setminus A)$.

For the ideal topological space $(X, \tau, \{\emptyset\})$, the value of Δ_3 on any subset A of X is $\emptyset$. Further, the value of $\Delta_3(A)$ on the ideal topological space $(X, \tau, \wp(X))$ is X.

Theorem 4.1. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then followings hold:*

1. $\Delta_3(\emptyset) = \veebar(A)$ *for any* $A \subseteq X$.
2. $\Delta_3(X) = X \setminus X^*$.
3. *for* $U \in \tau^*$, $\Delta_3(U) \supseteq U \setminus U^*$.
4. *for* $J \in \mathcal{I}$, $\Delta_3(J) = (X \setminus X^*) \cup J$.
5. *for any* $A \subseteq X$, $\Delta_3(X \setminus A) = \Delta_3(A)$.

Proof. 1. $\Delta_3(\emptyset) = X \setminus X^* = \veebar(A)$ for any $A \subseteq X$.

3. $\Delta_3(U) = (U \setminus U^*) \cup (\psi(U) \setminus U) \supseteq (U \setminus U^*) \cup (U \setminus U) = U \setminus U^*)$.

4. $\Delta_3(J) = (J \setminus J^*) \cup (\psi(J) \setminus J) = J \cup \psi(J) = (X \setminus X^*) \cup J$.

5. $\Delta_3(X \setminus A) = [(X \setminus A) \setminus (X \setminus A)^*] \cup [(\psi(X \setminus A) \setminus (X \setminus A)] = (\psi(A) \setminus A) \cup [(X \setminus A^*) \setminus (X \setminus A)] = (\psi(A) \setminus A) \cup (A \setminus A^*) = \Delta_3(A)$.

□

Theorem 4.2. *Let $(X, \tau, \mathcal{I})$ be a Hayashi-Samuel space. Then following hold:*

1. $\Delta_3(\emptyset) = \emptyset$.
2. $\Delta_3(X) = \emptyset$.
3. *for any subset* A *of* X, $\Delta_3(A) \subseteq A \Delta A^*$ *(Δ stands for symmetric difference).*
4. *for any open set* $U \in \tau$, $\Delta_3(U) = \psi(U) \setminus U$.
5. *for regular open set* U, $\Delta_3(U) = \emptyset$.
6. *for* $J \in \mathcal{I}$, $\Delta_3(J) = \Delta_3(X \setminus J) = J$.

Proof. 1.2. They are obvious from Lemma 1.1.

3. $\Delta_3(A) = (A \setminus A^*) \cup (\psi(A) \setminus A) \subseteq (A \setminus A^*) \cup (A^* \setminus A)$ (by Lemma 1.2). Thus $\Delta_3(A) \subseteq A \Delta A^*$.

5. $\Delta_3(U) = (U \setminus U^*) \cup (\psi(U) \setminus U) = \emptyset \cup \emptyset$, since $U \subseteq U^*$ and $\psi(U) = U$ [4]. Thus, $\Delta_3(U) = \emptyset$.

6. By using Theorem 4.1(5), $\Delta_3(J) = \Delta_3(X \setminus J) = [(X \setminus J) \setminus (X \setminus J)^*] \cup [\psi(X \setminus J) \setminus (X \setminus J)] = [(X \setminus J) \setminus X^*] \cup [(X \setminus J^*) \setminus (X \setminus J)] = [\emptyset] \cup [(X \setminus \emptyset) \cap J] = J$.

□

Corollary 4.3. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then following statements are equivalent:*

1. *The space $(X, \tau, \mathcal{I})$ is Hayashi-Samuel;*
2. $\Delta_3(\emptyset) = \emptyset$;
3. $\Delta_3(X) = \emptyset$.

Theorem 4.4. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $J \in \mathcal{I}$. If $\Delta_3(X \setminus J) = \Delta_3(J) = J$, then $X^* = X^{**}$.*

Proof. Given that $\Delta_3(J) = \psi(J) \cup J = J$. Then $\psi(J) \subseteq J$ implies $X \setminus X^* \subseteq J$. Thus, $(X \setminus X^*)^* \subset J^* = \emptyset$ and hence $X^* = ((X \setminus X^*) \cup X^*)^* = (X \setminus X^*)^* \cup X^{**} = X^{**}$. Therefore, $X^* = X^{**}$.

□

By Example 2.7, $X^* = X^{**}$ need not imply that the space is Hayashi-Samuel.

5. Conclusion

This paper deals three new operators in an ideal topological space which have been induced from $\veebar$, $\bar{\wedge}$ and $\wedge$ operators. The result of this paper is an application of the above mentioned operators. We also characterize some more results of an ideal topological space through these new operators Δ_1, Δ_2 and Δ_3. We also consider complement operators of the operator Δ_1, Δ_2 and Δ_3.

Acknowledgments

The authors are thankful to the referee for his/her value suggestions.

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