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Stochastic Differential Equations for Orthogonal Eigenvectors of $(G,\varepsilon)-$Wishart Process Related to Multivariate $G-$fractional Brownian Motion

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Abstract: In the present paper, we introduce a new process called multivariate $G-$fractional Brownian motion $\left(B_t^H\right)$ where the Hurst parameter H is a diagonal matrix. Moreover, we give an approximation $\left(R_t^\varepsilon\right)$ of Riemann-Liouville process of $\left(B_t^H\right)$ by $G-$Itô's processes. Then we give stochastic differential equations for orthogonal eigenvectors of $(G,\varepsilon)-$Wishart fractional process defined by $R_t^\varepsilon\left(R_t^\varepsilon\right)^*$, which has 0 and $\left|R_t^\varepsilon\right|^2$ as eigenvalues. An intermediate asymptotic comparison result concerning the eigenvalue $\left|R_t^\varepsilon\right|^2$ is also obtained.

Key Words: $G-$Brownian motion, $G-$multivariate fractional Brownian motion, random matrices, $G-$Wishart process, eigenvalues, eigenvectors.

Contents

1. Introduction

In the last decade the fractional Brownian motion model (fBm for short) has found a wide range of applications in several fields since the seminal paper of Mandelbrot and Van Ness, 1968 [14]. Examples are mathematical finance, telecommunication engineering, internet traffic analysis, physical sciences, geosciences and neurosurgery. This model was introduced as the unique Gaussian process having stationary increments and self-similarity (see also [1], [11], [14] for a review of the basic properties). Since then, based on the study of fBm,various extensions were introduced by many authors, for example, the multivariate fractional Brownian motion (mfBm for short) which is an extension of the well-known fractional Brownian motion, has also seen considerable and fruitful research in both applications and theory (see [2], [3], [9]).

On the other hand, aspects of model ambiguity such as volatility uncertainty have been studied by Peng (for more details see [15], [16], [17]) who introduced as a typical and important case, a new theory of nonlinear expectation space; the so-called $G-$expectation in an intrinsic way which does not rely on any particular probability space. It reveals the probability distribution uncertainty in a fundamental way which is crucial in many situations such as modeling risk uncertainty in mathematical finance. It can be regarded as a counterpart of the Wiener probability space in the linear case. Within this framework, a new kind of Brownian motion called $G-$Brownian motion (GBm for short) was constructed and the corresponding stochastic calculus was established. Moreover, a stochastic integral of Itô's type under $G-$expectation was developed. A very interesting new phenomenon of the $G-$Brownian motion B is that its quadratic variation process $\langle B\rangle$ is a stochastic process (not deterministic in general) and has independent and stationary increments which are identically distributed.

Inspired by the $G-$Brownian motion framework as presented by Peng [17], the $G-$fractional Brownian motion (GfBm for short) with Hurst parameter $h \in (0,1)$ has been defined, firstly, by Wein Chen (see [5])

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in one dimensional case, as a centered $G-$Gaussian process with stationary increments. Our first objective is to present a new process called the multivariate $G-$fractional Brownian motion process (mGfBm for short) in terms of the $G-$Brownian motion. This model is characterized by one parameter called the Hurst exponent H which is, in our case, a diagonal matrix.

Recently, the theory of random matrices which has received an increasing interest in different application fields such as physics, economics, psychology and so on, has in turn led to the discovery of a new interesting theory of "random matrices processes" in the nonlinear framework, for a recent account we refer the reader to [12], [18]. Furthermore, historically the earliest studied ensemble of random matrices is the Wishart ensemble, introduced by Wishart [13] in 1928.

In the present paper, we introduce two new stochastic processes that is, the $G-$multivariate fractional Brownian motion process $\left(B_t^H\right)$ and the $(G,\varepsilon)-$Wishart process related to its Riemann-Liouville part. Our ultimate goal is to investigate the processes of orthogonal eigenvectors of $G-$Wishart process. Since the eigenvalues collide, then we could not use Bru's approach [6], which was used in [12], [18]. Our approach is mainly based on algebraic technics.

In summary, the structure of the paper is organized in the following way. In Section 2, we provide the necessary background for the paper and some definitions. In Section 3, we introduce the multivariate $G-$fractional Brownian motion as an integral representation with respect to $d-$dimensional $G-$Brownian motion. Moreover, we give an approximation (R_t^ε) of Riemann-Liouville process related to $\left(B_t^H\right)$ in $L_G^2(\Omega)$ by $G-$Itô processes. Then, in Section 4 we define the corresponding $G-$Wishart processes $R_t^\varepsilon\left(R_t^\varepsilon\right)^*$. Finally, we give stochastic differential equations (SDEs for short) for its orthogonal eigenvectors and an asymptotic comparison result which concerns the eigenvalue $|R_t^\varepsilon|^2$.

2. Preliminaries

We begin with a brief survey of the theory of sublinear expectation space and some main results from the $G-$framework of $G-$stochastic calculus, essentially based on the references [4], [15], [16] and [17], which are needed for what follows.

$G-$expectation and $G-$Brownian motion.

Briefly speaking, a $G-$Brownian motion is a continuous process with independent and stationary increments under a given sublinear expectation. Throughout, we let $\Omega := \left\{\omega \in C\left(\mathbb{R}_+,\mathbb{R}^d\right) : \omega(0)=0\right\},\ d\geq 1$, be the space of all $\mathbb{R}^d-$valued continuous path functions $(\omega_t)_{t\in\mathbb{R}_+}$ such that $\omega_0=0$, endowed with the distance

$$\rho\left(\omega^1,\omega^2\right) := \sum_{i=1}^{\infty} 2^{-i}\max_{0\leq t\leq i}\left|\omega_t^1-\omega_t^2\right|\wedge 1.$$

Let $\mathcal{B}(\Omega)$ be the associated Borel $\sigma-$algebra, $\Omega_t := \{\omega_{.\wedge t} : \omega\in\Omega\}$ and B be the canonical process. Consider the following space of random variables:

$$Lip\left(\Omega_t\right) := \left\{\varphi\left(B_{t_1\wedge t},...,B_{t_n\wedge t}\right) : t_1,...,t_n\in[0,\infty),\varphi\in C_{b,Lip}\left(\mathbb{R}^d\right)^n\right\},$$

where $C_{b,lip}\left(\mathbb{R}^d\right)^n$ denotes the space of all Lipschitzian and bounded functions on $\left(\mathbb{R}^d\right)^n$. We further define

$$Lip\left(\Omega\right) = \bigcup_{n=1}^{\infty} Lip\left(\Omega_n\right).$$

Peng [17] constructed the $G-$expectation $\mathbb{E} : \mathcal{H} := Lip(\Omega) \to \mathbb{R}$ as a sublinear expectation on the lattice $\mathcal{H}$ of real functions that satisfies: for all $X, Y \in \mathcal{H}$,

(a) Monotonicity: $\mathbb{E}[X] \geq \mathbb{E}[Y]$ if $X \geq Y$.

(b) Preservation of constants: $\mathbb{E}[c] = c$ for all $c\in\mathbb{R}$.

(c) Sub-additivity: $\mathbb{E}[X] - \mathbb{E}[Y] \leq \mathbb{E}[X-Y]$.

(d) Positive homogeneity : $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X]$, for all $\lambda \geq 0$.

The triple $(\Omega, \mathcal{H}, \mathbb{E})$ is called a sublinear expectation space. Let $X = (X_1, ..., X_d)$ be a $d-$dimensional random variable on $(\Omega, \mathcal{H}, \mathbb{E})$ and let

$$G(A) := \frac{1}{2}\mathbb{E}[(AX, X)], A \in \mathbb{S}_d,$$

where $\mathbb{S}_d$ is the space of $d \times d$ symmetric matrices and $(.,.)$ the Euclidian inner product of $\mathbb{R}^d$. Then there exists a non empty, bounded and closed subset Γ of $\mathbb{R}^{d\times d}$ such that

$$G(A) := \frac{1}{2}\sup_{\gamma\in\Gamma}\{tr[\gamma\gamma^* A] : \gamma^* \text{ is the transpose of } \gamma\}.$$

X is called $G-$normal distributed if for each $\varphi \in C_{b,Lip}(\mathbb{R}^d)$, the function

$$u(t,x) := \mathbb{E}\left[\varphi\left(x + \sqrt{t}X\right)\right], (t,x) \in [0,\infty) \times \mathbb{R}^d,$$

is the unique viscosity solution of the following nonlinear partial differential equation, called the $G-$heat equation:

$$\frac{\partial u}{\partial t} - G\left(D^2 u\right) = 0 \text{ on } (t,x) \in [0,\infty) \times \mathbb{R}^d \text{ and } u(0,x) = \varphi(x),$$

where $D^2 u = \left(\partial^2_{x_i x_j} u\right)_{i,j=1}^d$ is the Hessian matrix of u.

The set Γ is a collection of parameters that represents the variance uncertainty of the $G-$distributed random vector X. This $G-$normal distribution is denoted by $\mathcal{N}(0, \Sigma)$, where $\Sigma := \{\gamma\gamma^* : \gamma \in \Gamma\}$. For more details, the reader may refer to [16].

Remark 2.1. *When $d = 1$, Σ is an interval that is $\Sigma = \left[\underline{\sigma}^2, \overline{\sigma}^2\right]$ with $0 \leq \underline{\sigma} \leq \overline{\sigma}$. Here $G = G_{\underline{\sigma},\overline{\sigma}}$, is the following sublinear function parameterized by $\underline{\sigma}$ and $\overline{\sigma}$:*

$$G(\alpha) = \frac{1}{2}\left(\overline{\sigma}^2\alpha^+ - \underline{\sigma}^2\alpha^-\right), \ \alpha \in \mathbb{R}.$$

Recall that $\alpha^+ = max\{0, \alpha\}$ and $\alpha^- = -min\{0, \alpha\}$. In fact $\overline{\sigma}^2 = \mathbb{E}\left[X^2\right]$ and $\underline{\sigma}^2 = -\mathbb{E}\left[-X^2\right]$.

Definition 2.2. *A random vector $Y = (Y_1, Y_2, ..., Y_n)$, $Y_i \in \mathcal{H}$, is said to be independent under $\mathbb{E}$ from another random vector $X = (X_1, X_2, ..., X_m)$, $X_i \in \mathcal{H}$, if for each test function $\varphi \in C_{b,Lip}(\mathbb{R}^{m+n})$ we have*

$$\mathbb{E}[\varphi(X,Y)] = \mathbb{E}[\mathbb{E}[\varphi(x,Y)]|_{x=X}].$$

Definition 2.3. *Let X_1 and X_2 be two $n-$dimensional random vectors defined on the sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$. They are called identically distributed, denoted by $X_1 \overset{d}{=} X_2$, if*

$$\mathbb{E}[\varphi(X_1)] = \mathbb{E}[\varphi(X_2)] \quad \textit{for all } \varphi \in C_{b,Lip}(\mathbb{R}^n).$$

In [16], Peng showed that under the $G-$expectation $\mathbb{E}$, the $d-$dimensional canonical process $\{B_t(\omega) = \omega_t, \ t \geq 0\}$ is a $G-$Brownian motion, that is,

(i) $B_0 = 0$,

(ii) For any $s, t \geq 0$, B_t and $B_{t+s} - B_s$ are $\mathcal{N}(0, t\Sigma)-$distributed and

(iii) For any $s, t \geq 0$, the increment $B_{t+s} - B_t$ is independent of $(B_{t_1}, ..., B_{t_m})$ for each $t_1, ..., t_m \in [0, t]$.

Note that (a, B_t) is a real $G_{\underline{\sigma}_a}, \overline{\sigma}_a-$Brownian motion for each $a \in \mathbb{R}^d$, where $\overline{\sigma}_a^2 = \mathbb{E}\left((a, B_1)^2\right)$ and $\underline{\sigma}_a^2 = -\mathbb{E}\left(-(a, B_1)^2\right)$ (for more details, see [15]). In particular each component of (B_t) is a real $G-$Brownian motion. In what follows, we write $\overline{\sigma}_i$ (resp. $\underline{\sigma}_i$) instead of $\overline{\sigma}_{e_i}$ (resp. $\underline{\sigma}_{e_i}$), where $(e_i)_{i=1}^n$ is the canonical basis of $\mathbb{R}^d$.

Remark 2.4. *In [16], Peng showed that $X \sim \mathcal{N}(0, \Sigma)$ if and only if $aX + b\overline{X} \overset{d}{=} \sqrt{a^2+b^2}X$, for each $a, b \geq 0$ and for each random variable $\overline{X}$ independent of X such that $\overline{X} \overset{d}{=} X$. Consequently, since the random variable*

$$\frac{B_{t_1}^i}{\sqrt{t_1}}, \frac{B_{t_2}^i - B_{t_1}^i}{\sqrt{t_2 - t_1}}, ..., \frac{B_{t_n}^i - B_{t_{n-1}}^i}{\sqrt{t_n - t_{n-1}}}, \text{ for } 0 < t_1 < t_2 < ... < t_n,$$

are $\mathcal{N}(0, \Sigma)-$distributed then it is not hard to prove by recurrence that, for each $b_1, b_2, ..., b_n \in \mathbb{R}$ the random variable

$$b_1 B_{t_1}^i + b_2\left(B_{t_2}^i - B_{t_1}^i\right) + ... + b_n\left(B_{t_n}^i - B_{t_{n-1}}^i\right) \sim \mathcal{N}(0, [\underline{\sigma}_i^2 A, \overline{\sigma}_i^2 A]),$$

where $A = b_1^2 t_1 + ... + b_n^2 (t_n - t_{n-1})$.

For $p \geq 1$, we denote by $L_G^p(\Omega)$ the closure of $Lip(\Omega)$ under the Banach norm $\|X\|_{p,G} = (\mathbb{E}[|X|^p])^{\frac{1}{p}}$. Derived in [8], the $G-$expectation $\mathbb{E}$ can be viewed as an upper expectation of ordinary expectations, *i.e.* there exists a weakly compact set of probability measure $\mathcal{P}$ on $(\Omega, \mathcal{B}(\Omega))$, such that for each $X \in L_G^1(\Omega)$

$$\mathbb{E}[X] := \sup_{P \in \mathcal{P}} \mathbb{E}^P[X],$$

where $\mathbb{E}^P$ stands for the linear expectation under the probability measure P.

Definition 2.5. *A property holds 'quasi-surely' (q.s., for short) if it holds p.s. for each $P \in \mathcal{P}$.*

Note that the convergence in L_G^2 implies the convergence $q.s.$ Now, we introduce the definition of $d-$dimensional two-sided $G-$Brownian motion.

Definition 2.6. *Let $(\widetilde{B}_t)_{t\geq 0}$ be another $d-$dimensional $G-$Brownian motion on $(\Omega, \mathcal{H}, \mathbb{E})$ independent of $(B_t)_{t\geq 0}$. The process $(W_t)_{t\in\mathbb{R}}$ defined by*

$$W_t = \begin{cases} B_t \text{ if } t \geq 0, \\ \widetilde{B}_{-t} \text{ if } t < 0, \end{cases}$$

is called a two-sided $G-$Brownian motion.

$G-$Itô processes.

In this subsection we recall some notions on $G-$stochastic calculus with respect to $\left(B_t^i\right)_{t\geq 0}$. Let $T > 0$ and let $\pi_T = \{t_0, ..., t_N\}$ be a partition of $[0, T]$. Let $M_G^{p,0}(0, T)$ be the collection of processes in the following form

$$\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) \mathbf{1}_{[t_j, t_{j+1})}(t),$$

where $\xi_j \in L_G^p\left(\Omega_{t_j}\right)$; $j = 0, ..., N-1$. For each $\eta \in M_G^{2,0}(0, T)$, the $G-$stochastic integral is defined pointwisely by

$$I(\eta) = \int_0^T \eta_s dB_s^i := \sum_{j=0}^{N-1} \xi_j \left(B_{t_{j+1}}^i - B_{t_j}^i\right).$$

A remarkable result of Peng [15] is that the quadratic variation process $\langle B^i \rangle$ of B^i is not a deterministic process, unlike the classical case and it is defined by

$$\langle B^i \rangle_t := \left(B^i_t\right)^2 - 2\int_0^t B^i_s dB^i_s.$$

It was proved in [8] that $\underline{\sigma}_i^2 t \leq \langle B^i \rangle_t \leq \overline{\sigma}_i^2 t$. We define the mutual variation process by

$$\langle B^i, B^j \rangle = \frac{1}{4}\left[\langle B^i + B^j \rangle - \langle B^i - B^j \rangle\right].$$

Note that $\langle B^i, B^i \rangle = \langle B^i \rangle$. Following Peng, a $G-$Itô process is defined as follows:

Definition 2.7. *A $d-$dimensional $G-$Itô process X is defined by*

$$X_t = X_0 + \sum_{i=1}^d \int_0^t \eta^i_s dB^i_s + \sum_{i,j=1}^d \int_0^t \theta^{ij}_s d\langle B^i, B^j \rangle_s + \int_0^t \alpha_s ds,$$

where the initial condition $X_0 \in \mathcal{H}$, α, $\theta^{ij} \in M^1_G(0,T)$ and $\eta^i \in M^2_G(0,T)$.

We have the following Burkholder-Davis-Gundy-type estimates (BDG, in short) formulated, for each $p \geq 2$ and $0 \leq s \leq t$, by

$$\mathbb{E}\left[\sup_{0\leq s\leq t}\left|\int_0^s \eta^i_u dB^i_s\right|^p\right] \leq K t^{\frac{p}{2}-1}\int_0^t \mathbb{E}\left[\left|\eta^i_u\right|^p\right] du,$$

where K is a positive constant independent of η. (For further details see [10]).

Remark 2.8. *It follows from Remark 2.4, for each deterministic function $f \in L^2(\mathbb{R}_+, dt)$, that*

$$\int_0^t f(s)\, dB^i_s \sim \mathcal{N}\left(0, \left[\underline{\sigma}_i^2(t), \overline{\sigma}_i^2(t)\right]\right),$$

where $\underline{\sigma}_i^2(t) = \underline{\sigma}_i^2 \int_0^t f^2(s)ds$ and $\overline{\sigma}_i^2(t) = \overline{\sigma}_i^2 \int_0^t f^2(s)ds$.

Furthermore, for each convex function $\varphi \in C_{b,Lip}\left(\mathbb{R}^d\right)^n$, we have

$$\mathbb{E}\left[\varphi\left(\int_0^t f(s)dB_s\right)\right] = \frac{1}{\overline{\sigma}_i(t)\sqrt{2\pi}}\int_{-\infty}^{+\infty} \varphi(y)\exp\left(-\frac{y^2}{2\overline{\sigma}_i^2(t)}\right) dy.$$

In particular, we have

$$\mathbb{E}\left(\int_0^t f(s)dB_s\right)^4 = 3\overline{\sigma}_i^4(t).$$

We will need the vectorial $G-$Itô formula [15].

Theorem 2.9. *Let $f \in C^2(\mathbb{R}^n)$ be a real function with bounded derivatives such that $\left\{\partial^2_{x_k x_l} f\right\}_{k,l=1}^n$ are uniformly Lipschitz and let $X_t = \left(X^i_t\right)_{i=1}^n$ be an $n-$dimensionnal vector of $G-$Itô processes. Then, we have*

$$df(X_t) = \sum_{k=1}^n \frac{\partial f}{\partial x_k}(X_t)\, dX^k_t + \frac{1}{2}\left[\sum_{k,l=1}^n \frac{\partial^2 f}{\partial x_k \partial x_l}(X_t)\, d\langle X^k, X^l \rangle_t\right].$$

Note that $d\left\langle X^{k},X^{l}\right\rangle_{t}=dX_{t}^{k}dX_{t}^{l}$. In particular, as a consequence of $G-$Itô formula, if $f\left(x,y\right)=xy$, then we obtain the $G-$integration formula by parts in terms of Itô differential:

$$d\left(X_{t}Y_{t}\right)=X_{t}dY_{t}+Y_{t}dX_{t}+dX_{t}dY_{t},$$

for each $\left(X_{t}\right),\ \left(Y_{t}\right)$ two unidimensional $G-$Itô processes.

3. Multivariate $G-$fractional Brownian motion

We are interested here in the multivariate $G-$fractional Brownian motion, called also operator $G-$fractional Brownian motion. For further readings on the classical case see [2], [3] and [9].

In the rest of this paper, we consider the $d-$dimensional two side $G-$Brownian motion $\left(W_{t}\right)_{t\in\mathbb{R}}$ defined on $\left(\Omega,\mathcal{H},\mathbb{E}\right)$ where $d>1$. Let $H=diag\left[h_{1},h_{2},...,h_{d}\right]$ be a diagonal matrix such that $h_{i}\in\left(0,1\right)$ for $i=1,2,...,d$ and let $K_{H}\left(t,u\right)$ be the matrix of kernels defined by

$$K_{H}\left(t,s\right):=\left(t-s\right)_{+}^{D}-\left(-s\right)_{+}^{D}=diag\left[k_{h_{1}}\left(t,s\right),k_{h_{2}}\left(t,s\right),...,k_{h_{d}}\left(t,s\right)\right],$$

where $D=H-\frac{1}{2}I$ and $k_{h_{i}}\left(t,s\right)=\left(t-s\right)_{+}^{h_{i}-\frac{1}{2}}-\left(-s\right)_{+}^{h_{i}-\frac{1}{2}}$. In this notation, a^{D} is understood as the exponential of the matrix $D\log a$ for each $a>0$ and $M_{H}=\Gamma\left(I+D\right)^{-1}$ depending only on H, where $\Gamma\left(x\right)$ is the usual Gamma function and I is the identity matrix. Note that the matrix $\Gamma\left(I+D\right)$ is invertible.

Definition 3.1. *The process* $\left(B_{t}^{H}\right)_{t\in\mathbb{R}}$ *defined by*

$$B_{t}^{H}=\int\limits_{\mathbb{R}}K_{H}\left(t,s\right)M_{H}dW_{s},$$

is called multivariate $G-$*fractional Brownian motion.*

Remark 3.2. *Let* $C_{h_{i}}:=\Gamma\left(h_{i}-\frac{1}{2}\right)^{-1}$ *for each* $i=1,...,d$. *Since,*

$$B_{t}^{H,i}=C_{h_{i}}\int\limits_{\mathbb{R}}k_{h_{i}}\left(t,s\right)dW_{s}^{i},$$

then all the components of $\left(B_{t}^{H}\right)_{t\in\mathbb{R}}$ *are real fractional* $G-$*Brownian motions.*

Remark 3.3. *Clearly, we can easily check that* $B_{t}^{\frac{1}{2}I}=B_{t},\ t\geq0$.

Proposition 3.1. *The mGfBm* $(B_{t}^{H})_{t\in\mathbb{R}}$ *satisfies the following properties:*

(i) $H-$*self-similarity, that is* $B_{at}^{H}\overset{d}{=}a^{H}B_{t}^{H}$.

(ii) *Stationarity of increments.*

Proof. **(i)** Since B is $\frac{1}{2}I-$self-similar then W is also $\frac{1}{2}I-$self-similar, so that

$$\begin{aligned}K_{H}\left(at,s\right)M_{H}dW_{s} &= a^{D}K_{H}\left(t,\frac{s}{a}\right)M_{H}dW_{s}\\ &\overset{d}{=}a^{D}K_{H}\left(t,\frac{s}{a}\right)a^{\frac{1}{2}I}M_{H}dW_{\frac{s}{a}}\\ &\overset{d}{=}a^{H}K_{H}\left(t,\frac{s}{a}\right)M_{H}dW_{\frac{s}{a}},\end{aligned}$$

then for each $a>0$, we have

$$B_{at}^{H}\overset{d}{=}a^{H}B_{t}^{H}.$$

(ii) Observe that, following Peng [15], [16], [17], the process $\left(\widetilde{W}_v\right)_{v\in\mathbb{R}}$ defined by $\widetilde{W}_v = W_{u-v} - W_u$ is a $G-$Brownian motion and then $\widetilde{W}_v \overset{d}{=} W_v$. It follows that

$$B_t^H - B_s^H = \int\limits_{\mathbb{R}} \left((t-u)_+^D - (s-u)_+^D\right) M_H dW_s = \int\limits_{\mathbb{R}} \left((t-s-v)_+^D - (-v)_+^D\right) M_H d\widetilde{W}_v \overset{d}{=} B_{t-s}^H.$$

The proof is complete.

$\square$

Recall that, for each $t \geq 0$,

$$B_t^H := \int\limits_{-\infty}^{0} \left[(t-s)^D - (-s)^D\right] M_H dW_s + \int\limits_0^t (t-s)^D M_H dW_s.$$

The first one of these two Wiener integrals is called the low-frequency part of mGfBm and the other one its high-frequency part. Roughness of paths of mGfBm is mainly due to its high-frequency part, which is also called the Riemann-Liouville process given by

$$R_t^H := \int\limits_0^t (t-s)^D M_H dW_s = \int\limits_0^t (t-s)^D M_H dB_s.$$

In the rest of this section, we assume that:

$$\text{either all } h_i \in \left(0, \frac{1}{2}\right), \text{either all } h_i \in \left(\frac{1}{2}, 1\right), \text{for } i = 1, 2, ..., d.$$

We introduce at first an approximation of R_t^H by $G-$Itô processes as follows:

$$R_t^{H,\varepsilon} = \int\limits_0^t (t-s+\varepsilon)^D M_H dB_s.$$

(see [7], [19] for the classical case).

Next, we prove the following lemma which plays an important role in our results.

Lemma 3.4. *For every $\varepsilon > 0$, the process $\left(R_t^{H,\varepsilon}\right)_{t\geq 0}$ is a $d-$dimensional $G-$Itô process satisfying the following $G-$SDE*

$$dR_t^{H,\varepsilon} = \varepsilon^D M_H dB_t + \left(\int\limits_0^t D(t-s+\varepsilon)^{D-I} M_H dB_s\right) dt \tag{3.1}$$

Proof. Let

$$F(s, B_s) = \varphi(s) B_s,$$

where $\varphi(s) = (t-s+\varepsilon)^D M_H$ is the deterministic diagonal $d \times d$ matrix, which is differentiable in s. By using the $G-$integration formula by parts, we obtain

$$d\left(\varphi_s^{ij} B_s^j\right) = \varphi_s^{ij} dB_s^j + d\varphi_s^{ij} B_s^j$$

and then

$$\int\limits_0^t \varphi(s) dB_s = \varphi(t) B_t - \int\limits_0^t d\varphi(s) B_s.$$

Taking into account the facts that

$$\varphi\left(t\right)=\varepsilon^{D}M_{H}\text{ and }d\varphi\left(s\right)=-D\left(t-s+\varepsilon\right)^{D-I}M_{H}ds,$$

we have

$$R_{t}^{H,\varepsilon}=\varepsilon^{D}M_{H}B_{t}+I\left(t\right) \tag{3.2}$$

where

$$I\left(t\right):=\int_{0}^{t}M_{H}\left(t-s+\varepsilon\right)^{D-I}DB_{s}ds.$$

We set $f(t,s)=M_{H}\left(t-s+\varepsilon\right)^{D-I}DB_{s}$. Note that the deterministic function $I\left(t\right)$ is derivable and

$$\begin{aligned}I'\left(t\right) &= \int_{0}^{t}\frac{\partial}{\partial t}f(t,s)ds+f\left(t,t\right)\\ &= \int_{0}^{t}\frac{\partial}{\partial t}\left(M_{H}\left(t-s+\varepsilon\right)^{D-I}D\right)B_{s}ds+M_{H}\varepsilon^{D-I}DB_{t}.\end{aligned}$$

Since

$$\frac{\partial}{\partial t}\left(M_{H}\left(t-u+\varepsilon\right)^{D-I}D\right)=-\frac{\partial}{\partial u}\left(M_{H}\left(t-u+\varepsilon\right)^{D-I}D\right),$$

then

$$I'\left(t\right)=-\int_{0}^{t}\frac{\partial}{\partial u}\left(M_{H}\left(t-u+\varepsilon\right)^{D-I}D\right)B_{u}du+M_{H}\varepsilon^{D-I}DB_{t}.$$

By using the $G-$integration formula by parts, we have

$$\int_{0}^{t}\frac{\partial}{\partial u}\left(M_{H}\left(t-s+\varepsilon\right)^{D-I}D\right)B_{s}ds=M_{H}\varepsilon^{D-I}DB_{t}-\int_{0}^{t}M_{H}\left(t-s+\varepsilon\right)^{D-I}DdB_{s},$$

then

$$I'\left(t\right)=\int_{0}^{t}M_{H}\left(t-s+\varepsilon\right)^{D-I}DdB_{s}.$$

Consequently, the $G-$Itô differential of (3.2) is given by

$$dR_{t}^{H,\varepsilon}=d\left(\varepsilon^{D}M_{H}B_{t}\right)+I'\left(t\right)dt=\varepsilon^{D}M_{H}dB_{t}+\left(\int_{0}^{t}D\left(t-s+\varepsilon\right)^{D-I}M_{H}dB_{s}\right)dt.$$

Then, we obtain the desired result. $\square$

We will need the following lemma.

Lemma 3.5. *Assume that $\frac{1}{2}<h_{i}<1$. Then, for each $a,b>0$, we have*

$$\left\|\left(a+b\right)^{D}-a^{D}\right\|\leq\left\|b^{D}\right\|,$$

where $\|A\|:=\left(\sum_{i,j}A_{ij}^{2}\right)^{\frac{1}{2}}$ denotes the classical norm of the matrix A.

Proof. We have

$$\left\|(a+b)^D - a^D\right\|^2 = \sum_{i=1}^{d}\left((a+b)^{h_i-\frac{1}{2}} - a^{h_i-\frac{1}{2}}\right)^2$$

Note that $0 < h_i - \frac{1}{2} < \frac{1}{2}$, then by using the inequality $(a+b)^p \le a^p + b^p$ if $0 < p < 1$, we have

$$(a+b)^{h_i-\frac{1}{2}} - a^{h_i-\frac{1}{2}} \le b^{h_i-\frac{1}{2}}$$

and so

$$\left\|(a+b)^D - a^D\right\|^2 \le \sum_{i=1}^{d} b^{2\left(h_i-\frac{1}{2}\right)} = \left\|b^D\right\|^2.$$

The proof is complete. □

Proposition 3.2. *For each deterministic matrix process $\nu \in L^2\left([0,\infty[\,, dt\right)$, there exists a positive constant μ independent of ν such that*

$$\mathbb{E}\left[\left|\int_0^t v_s dB_s\right|^2\right] \le \mu\int_0^t \|v_s\|^2 ds \tag{3.3}$$

Proof. We have

$$\begin{aligned}\mathbb{E}\left[\left|\int_0^t v_s dB_s\right|^2\right] &= \mathbb{E}\left[\sum_{i=1}^{d}\left(\sum_{j=1}^{d}\int_0^t v_s^{i,j} dB_s^j\right)^2\right]\\ &\le \sum_{i=1}^{d}\mathbb{E}\left[\sum_{j=1}^{d}\int_0^t v_s^{i,j} dB_s^j\right]^2.\end{aligned}$$

On the other hand

$$\begin{aligned}\mathbb{E}\left[\sum_{j=1}^{d}\int_0^t v_s^{i,j} dB_s^j\right]^2 &= \mathbb{E}\left[\sum_{j=1}^{d}\left(\int_0^t v_s^{i,j} dB_s^j\right)^2 + \sum_{m\neq k}\int_0^t v_s^{i,m} dB_s^m \int_0^t v_s^{i,k} dB_s^k\right]\\ &\le \sum_{j=1}^{d}\mathbb{E}\left[\int_0^t v_s^{i,j} dB_s^j\right]^2 + \sum_{m\neq k}\mathbb{E}\left(\int_0^t v_s^{i,m} dB_s^m \int_0^t v_s^{i,k} dB_s^k\right)\\ &\le \sum_{j=1}^{d}\overline{\sigma}_j^2\left[\int_0^t \left(v_s^{i,j}\right)^2 ds\right] + 2\sum_{m<k}\mathbb{E}\left[\int_0^t v_s^{i,m} dB_s^m \int_0^t v_s^{i,k} dB_s^k\right],\end{aligned}$$

where $\overline{\sigma}_j^2 = \mathbb{E}\left[\left(B_1^j\right)^2\right]$. Therefore

$$\mathbb{E}\left[\left|\int_0^t v_s dB_s\right|^2\right] \le \sum_{i=1}^{d}\left\{\sum_{j=1}^{d}\overline{\sigma}_j^2\left[\int_0^t \left(v_s^{i,j}\right)^2 ds\right] + 2\sum_{m<k}\mathbb{E}\left[\int_0^t v_s^{i,m} dB_s^m \int_0^t v_s^{i,k} dB_s^k\right]\right\} \tag{3.4}$$

Thanks to the $G-$integration formula by parts, we have

$$\begin{aligned}d\left[\int_0^t v_s^{i,m} dB_s^m \int_0^t v_s^{i,k} dB_s^k\right] &= v_t^{i,k}\left(\int_0^t v_s^{i,m} dB_s^m\right) dB_t^k + v_t^{i,m}\left(\int_0^t v_s^{i,k} dB_s^k\right) dB_t^m\\ &\quad + v_t^{i,k} v_t^{i,m} d\left\langle B^k, B^m\right\rangle_s.\end{aligned}$$

Thus

$$\int_0^t v_s^{i,m} dB_s^m \int_0^t v_s^{i,k} dB_s^k = \int_0^t v_s^{i,k}\left(\int_0^s v_u^{i,m} dB_u^m\right) dB_s^k + \int_0^t v_s^{i,m}\left(\int_0^s v_u^{i,k} dB_u^k\right) dB_s^m$$
$$+\int_0^t v_s^{i,k} v_s^{i,m} d\left\langle B^k, B^m\right\rangle_s .$$

Taking into account the fact that

$$\mathbb{E}\left[\int_0^t v_s^{i,k}\left(\int_0^s v_u^{i,m} dB_u^m\right) dB_s^k\right] = \mathbb{E}\left[\int_0^t v_s^{i,m}\left(\int_0^s v_u^{i,k} dB_u^k\right) dB_s^m\right] = 0,$$

we have

$$\mathbb{E}\left[\int_0^t v_s^{i,m} dB_s^m \int_0^t v_s^{i,k} dB_s^k\right] \leq \mathbb{E}\left[\int_0^t v_s^{i,k} v_s^{i,m} d\left\langle B^k, B^m\right\rangle_s\right].$$

Recall that for each $t \geq 0$, $B_t^k \pm B_t^m = (e_k \pm e_m, B_t)$ is a $G_\pm-$Brownian motion, where $G_\pm(\alpha) = \frac{1}{2}\left(\overline{\sigma}_\pm^2(k,m)\,\alpha^+ - \underline{\sigma}_\pm^2(k,m)\,\alpha^-\right)$ with

$$\overline{\sigma}_\pm^2(k,m) = \mathbb{E}\left((e_k \pm e_m, B_1)^2\right) \text{ and } \underline{\sigma}_\pm^2(k,m) = -\mathbb{E}\left(-(e_k \pm e_m, B_1)^2\right).$$

Hence

$$\left\langle B^k, B^m\right\rangle_t = \frac{1}{4}\left[\left\langle B^k + B^m\right\rangle_t - \left\langle B^k - B^m\right\rangle_t\right]$$
$$\leq \frac{1}{4}\left(\overline{\mu}^2 - \underline{\mu}^2\right) t,$$

where $\overline{\mu} = \max\limits_{k<m} \overline{\sigma}_+(k,m)$ and $\underline{\mu} = \min\limits_{k<m} \underline{\sigma}_-(k,m)$. Finally, we obtain by the inequality (3.4)

$$\mathbb{E}\left[\left|\int_0^t v_s dW_s\right|^2\right] \leq \sum_{i=1}^d \int_0^t \left(\sum_{j=1}^d \overline{\sigma}_j^2\left(v_s^{i,j}\right)^2 + \frac{1}{2}\left(\overline{\mu}^2 - \underline{\mu}^2\right)\sum_{k<m} v_s^{i,k} v_s^{i,m}\right) ds.$$

Therefore, by setting $\mu = \max\left(\overline{\sigma}_j^2, \frac{1}{4}\left(\overline{\mu}^2 - \underline{\mu}^2\right);\ j = 1,2,...,d\right)$, we have

$$\mathbb{E}\left[\left|\int_0^t v_s dB_s\right|^2\right] \leq \mu \int_0^t \|v_s\|^2 ds.$$

The proof is complete. □

Lemma 3.6. *The random variables $R_t^{H,\varepsilon}$ converge to R_t^H in $L_G^2(\Omega)$ when ε tends to 0 for each $t \geq 0$.*

Proof. **The case $\frac{1}{2} < h_i < 1$:**

From the inequality (3.3) we have

$$\mathbb{E}\left[\left|R_t^{H,\varepsilon} - R_t^H\right|^2\right] \leq \mu \int_0^t \left\|\left((t-s+\varepsilon)^D - (t-s)^D\right) M_H\right\|^2 ds$$
$$\leq \mu \int_0^t \left\|(t-s+\varepsilon)^D - (t-s)^D\right\|^2 \|M_H\|^2 ds.$$

Thus, it follows from the above lemma, that

$$\mathbb{E}\left[\left|R_t^{H,\varepsilon}-R_t^{H}\right|^2\right]\leq \mu\left\|M_H\right\|^2\int\limits_0^t\left\|\varepsilon^{D}\right\|^2 ds\leq \mu\left\|M_H\right\|^2 t\sum_{i=1}^{d}\varepsilon^{2h_i-1},$$

which implies that $R_t^{H,\varepsilon}$converges to R_t^H in $L_G^2(\Omega)$ when ε tends to 0.

The case $\mathbf{0}<\mathbf{h}_i<\mathbf{\frac{1}{2}}$:

For each $i=1,2,...,d$, there exists $\theta\in(0,1)$ such that

$$(t-s+\varepsilon)^{h_i-\frac{1}{2}}-(t-s)^{h_i-\frac{1}{2}}=\varepsilon\left(h_i-\frac{1}{2}\right)(t-s+\theta\varepsilon)^{h_i-\frac{3}{2}},$$

so that

$$\begin{aligned}\left|(t-s+\varepsilon)^{h_i-\frac{1}{2}}-(t-s)^{h_i-\frac{1}{2}}\right|^2 &\leq \varepsilon^2\left(h_i-\frac{1}{2}\right)^2\sup_{0<\theta<1}|t-s+\theta\varepsilon|^{2h_i-3}\\ &= \varepsilon^2\left(h_i-\frac{1}{2}\right)^2(t-s)^{2h_i-3},\text{ if } 0\leq s\leq t.\end{aligned}$$

It follows, by proposition 3.2, that

$$\begin{aligned}\mathbb{E}\left(\left|R_t^{H,\varepsilon}-R_t^{H}\right|^2\right) &\leq \mu\int\limits_0^t\left\|(t-s+\varepsilon)^D-(t-s)^D\right\|^2\left\|M_H\right\|^2 ds\\ &\leq \mu\left\|M_H\right\|^2\varepsilon^2\sum_{i=1}^{d}\left(h_i-\frac{1}{2}\right)^2\int\limits_0^t(t-s)^{2h_i-3}ds\\ &= \mu\left\|M_H\right\|^2\varepsilon^2\sum_{i=1}^{d}\left(h_i-\frac{1}{2}\right)^2\frac{t^{2h_i-2}}{2h_i-2},\end{aligned}$$

which insure the desired convergence. □

4. $(G,\varepsilon)-$Wishart fractional process

The objective of this section is to find stochastic differential equations of eigenvalues and eigenvectors for the $(G,\varepsilon)-$Wishart fractional process $(\Sigma_t^\varepsilon)_{t\geq0}$ defined by $\Sigma_t^\varepsilon=R_t^\varepsilon\left(R_t^\varepsilon\right)^*$ where $R_t^\varepsilon:=R_t^{H,\varepsilon}$. Let $\Sigma^{\varepsilon,ij}$ be the entries of the matrix Σ^ε and $R^{\varepsilon,i}$ be the components of the vector R^ε.

In fact, the only eigenvalues of Σ^ε are 0 with multiplicity $(d-1)$ and $\lambda^\varepsilon:=|R^\varepsilon|^2$ with multiplicity 1. Indeed, the characteristic polynomial of Σ^ε is given by

$$\mathcal{P}^\varepsilon(\lambda)=\det\left(\Sigma^\varepsilon-\lambda I\right)=(-\lambda)^d\det\left(I-\frac{\Sigma^\varepsilon}{\lambda}\right),\text{ for }\lambda\neq0.$$

We conclude, by Weinstein–Aronszajn identity, that

$$\begin{aligned}\mathcal{P}^\varepsilon(\lambda)&=(-\lambda)^d\left(1-\frac{R^{\varepsilon,*}R^\varepsilon}{\lambda}\right)\\ &=(-\lambda)^d\left(1-\frac{|R^\varepsilon|^2}{\lambda}\right)\\ &=(-\lambda)^{d-1}\left(\lambda-|R^\varepsilon|^2\right).\end{aligned}$$

Proposition 4.1. *The eigenvalue $\lambda^\varepsilon = |R^\varepsilon|^2$ satisfies the following $G-SDE$:*

$$d\lambda^\varepsilon(t) = \sum_{i=1}^{d} \left(2\varepsilon^{h_i-\frac{1}{2}} C_{h_i} R_t^{\varepsilon,i} dB_t^i + \left(h_i - \frac{1}{2}\right) C_{h_i} R_t^{\varepsilon,i} \left(\int_0^t (t-s+\varepsilon)^{h_i-\frac{3}{2}} dB_s^i \right) dt + \varepsilon^{2h_i-1} C_{h_i}^2 d\left\langle B^i \right\rangle_t \right) \tag{4.1}$$

Proof. In view of the embedding (3.1), we may write

$$dR_t^{\varepsilon,i} = \varepsilon^{h_i-\frac{1}{2}} C_{h_i} dB_t^i + \left(h_i - \frac{1}{2}\right) C_{h_i} \left(\int_0^t (t-s+\varepsilon)^{h_i-\frac{3}{2}} dB_s^i \right) dt.$$

The formula (4.1) follows from $G-$Itô formula:

$$d\lambda_t^\varepsilon = \sum_{i=1}^{d} \left(2R_t^{\varepsilon,i} dR_t^{\varepsilon,i} + dR_t^{\varepsilon,i} dR_t^{\varepsilon,i} \right).$$

□

Remark 4.1. *Similarly, the process $\Sigma := RR^*$ admits only two eigenvalues: 0 with multiplicity $(d-1)$ and $\lambda := |R|^2$ with multiplicity 1 and then by Lemma 3.6, λ_t^ε converges to λ_t in $L_G^2(\Omega)$ when ε tends to 0 for each $t \geq 0$.*

In what follows, we set, for each $i, j = 1, ..., d$, $\eta_t^{i,j} = R_t^i \xi_t^j$ where,

$$\xi_t^j = \int_0^t (t-s+\varepsilon)^{h_j-\frac{3}{2}} dB_s^j.$$

We will need the following lemma.

Lemma 4.2. *We have*

$$\mathbb{E}\left(\left(\eta_t^{i,j}\right)^2 \right) \leq \frac{3}{8} \overline{\sigma}^4 \left[\left(\frac{(t+\varepsilon)^{2h_i} - \varepsilon^{2h_i}}{h_i} \right)^2 + \left(\frac{(t+\varepsilon)^{2h_j-2} - \varepsilon^{2h_j-2}}{h_j - 1} \right)^2 \right],$$

with $\overline{\sigma} = \max_{1 \leq i \leq d} \overline{\sigma}_i$.

Proof. We have, by using inequalities $ab \leq \frac{a^2+b^2}{2}$ and $(c+d)^2 \leq 2\left(c^2+d^2\right)$ for $a, b, c, d \in \mathbb{R}$,

$$\left|\eta_t^{i,j}\right|^2 \leq \left(\frac{\left|R_t^i\right|^2 + \left|\xi_t^j\right|^2}{2} \right)^2 \leq \frac{\left|R_t^i\right|^4 + \left|\xi_t^j\right|^4}{2},$$

and so,

$$\mathbb{E}\left(\left|\eta_t^{i,j}\right|^2 \right) \leq \frac{\mathbb{E}\left(\left|R_t^i\right|^4 \right) + \mathbb{E}\left(\left|\xi_t^j\right|^4 \right)}{2}.$$

By Remark 2.8 with $f(s) = (t-s+\varepsilon)^{h_i-\frac{1}{2}}$ (resp. $(t-s+\varepsilon)^{h_j-\frac{3}{2}}$), we then obtain

$$\mathbb{E}\left(\left|R_t^i\right|^4 \right) = 3\overline{\sigma}^4 \left(\int_0^t (t-s+\varepsilon)^{2h_i-1} ds \right)^2 \leq \frac{3}{4} \overline{\sigma}^4 \left(\frac{(t+\varepsilon)^{2h_i} - \varepsilon^{2h_i}}{h_i} \right)^2$$

(resp. $\mathbb{E}\left(\left|\xi_t^j\right|^4\right) \leq \frac{3}{4}\overline{\sigma}_i^4\left(\frac{(t+\varepsilon)^{2h_j-2}-\varepsilon^{2h_j-2}}{h_j-1}\right)^2$), which implies that

$$\mathbb{E}\left(\left(\eta_t^{i,j}\right)^2\right) \leq \frac{3}{8}\overline{\sigma}^4\left[\left(\frac{(t+\varepsilon)^{2h_i}-\varepsilon^{2h_i}}{h_i}\right)^2+\left(\frac{(t+\varepsilon)^{2h_j-2}-\varepsilon^{2h_j-2}}{h_j-1}\right)^2\right].$$

Then, we obtain the desired result. □

The next theorem expresses an asymptotic comparison of λ^ε.

Theorem 4.3. *We have, for each $t \geq 0$*

1. $\lambda_t^\varepsilon \in L_G^2(\Omega)$,
2. $\varepsilon^\alpha\ \lambda_t^\varepsilon = o(1)$ *as $\varepsilon \to 0$, q.s. for each $\alpha > 0$.*

Proof. 1. Let O_t^ε be an eigenvector associated to λ_t^ε. Since $\Sigma_t O_t = \lambda_t^\varepsilon O_t$ and $\Sigma_0 = 0$, then $\lambda_0^\varepsilon = 0$. It follows from the SDE (4.1) and inequality $\left(\sum_{i=1}^n a_i\right)^2 \leq n\sum_{i=1}^n a_i^2$ for $a_i \geq 0$, that

$$|\lambda_t^\varepsilon|^2 \leq 3d\sum_{i=1}^d\left(4\varepsilon^{2h_i-1}C_{h_i}^2\left|\int_0^t R_s^{\varepsilon,i}dB_s^i\right|^2+\left(h_i-\frac{1}{2}\right)^2C_{h_i}^2\left|\int_0^t \eta_u^{i,i}du\right|^2+\left|\varepsilon^{2h_i-1}C_{h_i}^2\left\langle B^i\right\rangle_t\right|^2\right) \tag{4.2}$$

Then, by using Hölder's inequality and the fact that $\left\langle B^i\right\rangle_t \leq \overline{\sigma}^2 t$, we get

$$\begin{aligned}\mathbb{E}\left(|\lambda_t^\varepsilon|^2\right) &\leq 3d\sum_{i=1}^d\left(4\varepsilon^{2h_i-1}C_{h_i}^2\overline{\sigma}^2\int_0^t\mathbb{E}\left(R_s^{\varepsilon,i}\right)^2ds+\left(h_i-\frac{1}{2}\right)^2C_{h_i}^2 t\int_0^t\mathbb{E}\left(\left|\eta_s^{i,i}\right|^2\right)ds+\overline{\sigma}^4\varepsilon^{4h_i-2}C_{h_i}^4t^2\right)\\ &\leq 3d\sum_{i=1}^d\left(4\varepsilon^{2h_i-1}C_{h_i}^2\overline{\sigma}^2\int_0^t\mathbb{E}\left(\lambda_s^\varepsilon\right)ds+\left(h_i-\frac{1}{2}\right)^2C_{h_i}^2 t\int_0^t\mathbb{E}\left(\left|\eta_s^{i,i}\right|^2\right)ds+\overline{\sigma}^4\varepsilon^{4h_i-2}C_{h_i}^4t^2\right)\end{aligned}$$

Thanks to the inequality $|\lambda_t^\varepsilon| \leq \frac{1+|\lambda_t^\varepsilon|^2}{2}$, we have

$$\begin{aligned}\mathbb{E}\left(|\lambda_t^\varepsilon|^2\right) \leq\ & 3d\sum_{i=1}^d\left(2\varepsilon^{2h_i-1}C_{h_i}^2\overline{\sigma}^2\int_0^t\mathbb{E}\left(\lambda_s^\varepsilon\right)^2ds+\left(h_i-\frac{1}{2}\right)^2C_{h_i}^2 t\int_0^t\mathbb{E}\left(\left|\eta_s^{i,i}\right|^2\right)ds\right.\\ &\left.+2\varepsilon^{2h_i-1}C_{h_i}^2\overline{\sigma}^2t+2\overline{\sigma}^4\varepsilon^{4h_i-2}C_{h_i}^4t^2\right).\end{aligned}$$

On the other hand, we may apply Lemma 4.2 to obtain,

$$\mathbb{E}\left(|\lambda_t^\varepsilon|^2\right) \leq L\left(\varepsilon,t,h_i,d\right)+A(\varepsilon,t,h_i,d,\overline{\sigma})+6\overline{\sigma}^2d\sum_{i=1}^d\varepsilon^{2h_i-1}C_{h_i}^2\int_0^t\mathbb{E}\left(|\lambda_s^\varepsilon|\right)^2ds, \tag{4.3}$$

where $L\left(\varepsilon,t,h,d\right) = 6d\sum_{i=1}^d\left(\varepsilon^{2h_i-1}C_{h_i}^2\overline{\sigma}^2t+\overline{\sigma}^4\varepsilon^{4h_i-2}C_{h_i}^4t^2\right)$ and

$$A(\varepsilon,t,h,d,\overline{\sigma}) = \frac{9}{8}\overline{\sigma}^4dt\sum_{i=1}^d\left(h_i-\frac{1}{2}\right)^2C_{h_i}^2\int_0^t\left[\left(\frac{(s+\varepsilon)^{2h_i}-\varepsilon^{2h_i}}{h_i}\right)^2+\left(\frac{(s+\varepsilon)^{2h_i-2}-\varepsilon^{2h_i-2}}{h_i-1}\right)^2\right]ds.$$

Thanks to Gronwall's lemma, the inequality (4.3) implies

$$\mathbb{E}\left(|\lambda_t^\varepsilon|^2\right) \leq \left(L\left(\varepsilon,t,h,d\right)+A(\varepsilon,t,h,d,\overline{\sigma})\right)\exp\left(6\overline{\sigma}^2d\sum_{i=1}^d\varepsilon^{2h_i-1}C_{h_i}^2t\right) < \infty.$$

2. Consequently, $\mathbb{E}\left[\left(\varepsilon^{\alpha}(\lambda_t^{\varepsilon}\right)^2\right]$ converges to 0 when ε goes to 0 and then $\varepsilon^{\alpha}\lambda_t^{\varepsilon}$ tends to 0 $q.s.$ when ε goes to 0. The proof is complete.

□

5. Stochastic differential equations for orthogonal eigenvectors of Σ^{ε}

The purpose of this section is to study stochastic differential equations satisfied by orthogonal eigenvectors of Σ^{ε}. Let $\Lambda_t^{\varepsilon} = O_t^{\varepsilon,*}\Sigma_t^{\varepsilon}O_t^{\varepsilon}$ be the factorization with O_t^{ε} an orthogonal matrix, where $\Lambda_t^{\varepsilon} := diag\left(\lambda_t^{\varepsilon}, 0, ..., 0\right)$ and let $\tau^{\varepsilon} = \inf\left\{t > 0 : R_t^{\varepsilon,i} = 0 \text{ for some } i \in \overline{1,d}\right\}$. In the sequel, we write for notational convenience O_t (resp. $R_t, \Sigma_t, \Lambda_t, O_t^*, \tau$) instead of O_t^{ε} (resp. $\overset{\varepsilon}{R_t}, \Sigma_t^{\varepsilon}, \Lambda_t^{\varepsilon}, O_t^{\varepsilon,*}, \tau^{\varepsilon}$).

Our approach is mainly based on algebraic technics. Indeed, the eigenvalues collide, so that we can not use Bru's approach [6] as in [12], [18]. To simplify the proofs and also to avoid the difficulties that the components of B are not necessarily independent, we make the following assumptions:

(**H1**) There exists an increasing real process u such that $\left\langle B^i, B^j\right\rangle_t = \delta_{ij}u_t$ $q.s.$ for each $i,j \in \overline{1,d}$ and $t \geq 0$, where δ_{uv} is the Kronecker symbol.

(**H2**) $h_i = h \in (0,1) \setminus \left\{\frac{1}{2}\right\}$, for each $i \in \overline{1,d}$.

We have then $\underline{\sigma}^2 t \leq u_t \leq \overline{\sigma}^2 t$ where $\underline{\sigma} := \min\limits_i \underline{\sigma}_i$. Note that in the classical case, the assumption (**H1**) is satisfied with $u_t = t$ for each $t \geq 0$.

Lemma 5.1. *Let the processes* $\alpha^{ij} := \frac{R^iR^j}{(R^1)^2+(R^i)^2}$ *and* $\beta^{ij} := R^1\alpha^{ij}$ *for* $1 < i < j$. *Then, it holds that*

(i)

$$d\alpha^{ij} = \frac{1}{\left(R^1\right)^2 + \left(R^i\right)^2}\left[-2\beta^{ij}dR^1 + R^idR^j + \frac{R^j\left(\left(R^1\right)^2 - \left(R^i\right)^2\right)}{\left(R^1\right)^2 + \left(R^i\right)^2}dR^i\right] \tag{5.1}$$

(ii)

$$\begin{aligned} d\beta^{ij} &= \frac{1}{\left(R^1\right)^2 + \left(R^i\right)^2}\left[\alpha^{ij}\left(1 - 2\left(R^1\right)^2\right)dR^1 + R^1R^idR^j \right. \\ &\left. + \frac{R^1R^j\left(\left(R^1\right)^2 - \left(R^i\right)^2\right)}{\left(R^1\right)^2 + \left(R^i\right)^2}dR^i - 2\varepsilon^{2h-1}C_h^2\beta^{ij}du\right] \end{aligned} \tag{5.2}$$

Note that $\xi_t^i = \int\limits_0^t (t-s+\varepsilon)^{h-\frac{3}{2}}\, dB_s^i$ and the formula (3.1) becomes

$$dR_t^i = \varepsilon^{h-\frac{1}{2}}C_h dB_t^i + \left(h - \frac{1}{2}\right)C_h\xi_t^i dt.$$

Proof. **(i)** The formula (5.1) follows from the $G-$Itô formula with

$$f\left(x_1, x_2, x_3\right) = \frac{x_2x_3}{x_1^2 + x_2^2}, X_t = \left(R^1, R^i, R^j\right)$$

and the following facts:

- $$d\left\langle R^i, R^j\right\rangle_t = \varepsilon^{2h-1}C_h^2\delta_{ij}du_t,$$

  which follows from the assumption (**H1**).

- $$\frac{\partial f}{\partial x_1}(x_1,x_2,x_3)=\frac{-2x_1x_2x_3}{\left(x_1^2+x_2^2\right)^2},\ \frac{\partial f}{\partial x_2}(x_1,x_2,x_3)=\frac{x_3\left(x_1^2-x_2^2\right)}{\left(x_1^2+x_2^2\right)^2}$$
  and
  $$\frac{\partial f}{\partial x_3}(x_1,x_2,x_3)=\frac{x_2}{x_1^2+x_2^2}.$$
- $$\frac{\partial^2 f}{\partial x_1^2}(x_1,x_2,x_3)=-\frac{\partial^2 f}{\partial x_2^2}(x_1,x_2,x_3)=\frac{2x_2x_3\left(3x_1^2-x_2^2\right)}{\left(x_1^2+x_2^2\right)^3}.$$

(ii) The formula (5.2) follows from formula (5.1) and $G-$integration by parts formula. The proof is complete.

□

Now, we are able to state the main result of the paper.

Theorem 5.2. *The orthogonal eigenvectors satisfy the following $G-$SDEs on $\{t<\tau\}$,*

$$\begin{aligned} dO_t^{i1} &= dR_t^i, \qquad dO_t^{i2}=-\delta_{i1}dR_t^2+\delta_{i2}dR_t^1,\\ dO_t^{ij} &= \delta_{i1}\left(-dR_t^j+\sum_{k=2}^{j-1}\left[R_t^k d\alpha^{kj}+\alpha^{kj}dR_t^k+\varepsilon^{2h-1}C_h^2\frac{R^j\left(\left(R^1\right)^2-\left(R^k\right)^2\right)}{\left[\left(R^1\right)^2+\left(R^k\right)^2\right]^2}du_t\right]\right)\\ &\quad -\sum_{k=2}^{j-1}\delta_{ik}d\beta^{kj}+\delta_{ij}dR_t^1 \end{aligned} \tag{5.3}$$

for each $i\in\overline{1,d}$ and for each $j\in\overline{3,d}$, where α^{kj} and β^{kj} are defined in the above lemma.

Proof. We give the proof in two steps.
Step 1: Firstly, we construct an orthogonal basis of $\mathbb{R}^d$. Obviously, since $\Sigma R=\lambda R$ then $V^1:=R$ is an eigenvector associated to $\lambda=|R|^2$. It is easy to check that

$$V^2:=\left(-R^2,R^1,0,...,0\right)^*,V^3:=\left(-R^3,0,R^1,0,...,0\right)^*,...,V^d:=\left(-R^d,0,...,0,R^1\right)^*,$$

are eigenvectors associated to the eigenvalue 0. Then $\left\{V^1,V^2,...,V^d\right\}$ is a basis of $\mathbb{R}^d$ but not orthogonal. Let O be the orthogonal matrix which columns are $O^1,O^2,...,O^d$, obtained by Gram-shmidt's orthogonalization process. Then we have

$$O^1=V^1=R, \tag{5.4}$$
$$O^2=V^2-\frac{\left(V^2,V^1\right)}{\left|V^1\right|^2}V^1=V^2,$$

and

$$O^j=V^j-\sum_{k=1}^{j-1}\frac{\left(V^j,V^k\right)}{\left|V^k\right|^2}V^k,\ \text{for } j\in\overline{3,d}.$$

Since $\left|V^k\right|^2=\left(R^1\right)^2+\left(R^k\right)^2$, $\left(V^k,V^1\right)=0$ and $\left(V^k,V^l\right)=R^kR^l$ for each $k,l\in\overline{2,d}$, then

$$\begin{aligned} O^{i2} &= -\delta_{i1}R^2+\delta_{i2}R^1,\\ O^{ij} &= \delta_{i1}\left(-R^j+\sum_{k=2}^{j-1}\frac{R^j\left(R^k\right)^2}{\left(R^1\right)^2+\left(R^k\right)^2}\right)-\sum_{k=2}^{j-1}\delta_{ik}\frac{R^1R^kR^j}{\left(R^1\right)^2+\left(R^k\right)^2}+\delta_{ij}R^1\\ &= \delta_{i1}\left(-R^j+\sum_{k=2}^{j-1}R^k\alpha^{kj}\right)-\sum_{k=2}^{j-1}\delta_{ik}\beta^{kj}+\delta_{ij}R^1, j\in\overline{3,d} \end{aligned} \tag{5.5}$$

Step 2: We have by $G-$integration by parts formula:

$$\begin{aligned} d\left(R^k\alpha^{kj}\right) &= R^k d\alpha^{kj} + \alpha^{kj} dR^k + d\alpha^{kj} dR^k \\ &= R^k d\alpha^{kj} + \alpha^{kj} dR^k + \varepsilon^{2h-1} C_h^2 \frac{R^j\left(\left(R^1\right)^2 - \left(R^k\right)^2\right)}{\left[\left(R^1\right)^2 + \left(R^k\right)^2\right]^2} du_t \end{aligned} \tag{5.6}$$

Hence, the formula (5.3) follows from (5.4) ,(5.5) and (5.6) . □

Remark 5.3. *To avoid the explosion of the solutions of system (5.3), it is necessary to have $\tau = +\infty$ q.s..*

To see this, it suffices to repeat the same proof used in corollary 2 [12] with $U := -\sum_{i=1}^{d} \log\left(R^i\right)^2$.

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