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Existence of Solutions to a Discrete Problems for Fourth Order Nonlinear p-Laplacian Via Variational Method

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ABSTRACT: We consider the boundary value problem for a fourth order nonlinear p-Laplacian difference equation and to prove the existence of at least two nontrivial solutions. Our approach is mainly based on the variational method and critical point theory. One example is included to illustrate the result.

Key Words: Discrete boundary value problems, fourth order, critical point theory, variational methods.

Contents

1. Introduction

Let $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ denote the sets of all natural numbers, integers and real numbers, respectively. For $a, b \in \mathbb{Z}$, define $[a, b]_{\mathbb{Z}} = \{a, a+1, ..., b\}$ when $a \leq b$.

In this paper, we consider the following fourth order nonlinear boundary value problems:

$$(P_\alpha) \quad \begin{cases} \Delta^2(\varphi_p(\Delta^2 u(t-2))) - \Delta(\varphi_p(\Delta u(t-1))) = \alpha f(u(t)), \ t \in [1, N]_{\mathbb{Z}}, \\ u(0) = u(N+1) = \Delta u(-1) = \Delta u(N+1) = 0, \end{cases}$$

where $N \geq 1$ is an integer, $1 < p < \infty$ is a constant, φ_p is the p-Laplacian operator, that is, $\varphi_p(s) = |s|^{p-2}s$, α is a positive real parameter, $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function, Δ is the forward difference operator defined by $\Delta u(t) = u(t+1) - u(t)$, $\Delta^0 u(t) = u(t)$, $\Delta^i u(t) = \Delta^{i-1}(\Delta u(t))$ for $i = 1, 2, 3, 4$.
As usual, a solution of (P_α) be a function $u : [-1, N+2]_{\mathbb{Z}} \longrightarrow \mathbb{R}$ satisfies both equations of (P_α).

We may think of (P_α) as a discrete analogue of the following fourth-order p-Laplacian functional differential equation

$$\begin{cases} \dfrac{d^2}{dt^2}\left(\varphi_p\left(\dfrac{d^2 u(t)}{dt^2}\right)\right) - \dfrac{d}{dt}\left(\varphi_p\left(\dfrac{du(t)}{dt}\right)\right) = \alpha f(u(t)), \quad t \in \,]0, 1[\,, \\ u(0) = u(1) = u'(0) = u'(1) = 0. \end{cases}$$

For $i \in [1, N]_{\mathbb{Z}}$, let λ_i be the eigenvalues of the nonlinear boundary value problem (1.1) corresponding to the problem (P_α)

$$\begin{cases} \Delta^2(\varphi_p(\Delta^2 u(t-2))) - \Delta(\varphi_p(\Delta u(t-1))) = \lambda \varphi_p(u(t)), \quad t \in [1, N]_{\mathbb{Z}}, \\ u(0) = u(N+1) = \Delta u(-1) = \Delta u(N+1) = 0. \end{cases} \tag{1.1}$$

Nonlinear discrete problems are important mathematical models in various research fields such as computer science, astrophysics, control systems, economics, fluid mechanics, image processing and many others.

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Boundary value problems for difference equations have been extensively studied; see the monographs [1,3,4,8,9,10]. Difference equations represent the discrete counterpart of ordinary equations, and the classical theory of difference equations employs numerical analysis and features from the linear and nonlinear operator theory, such as fixed point methods; for an exhaustive description of the subject, we refer the reader to the monographs of Agarwal [1], Kelley and Peterson [7], and Lakshmikantham and Trigiante [8]. We remark that, usually, most methods yield existence results for solutions of a difference equation. As is well known, critical point theory and variational methods are powerful tools to investigate the existence of solutions of various problems.

The main goal of the present paper is to establish the existence of nontrivial solutions of (P_α), via variational methods and critical point theory.
We make the following assumptions:

(H1) $\lim\limits_{x\to 0}\dfrac{F(x)}{|x|^p}=\infty$, where $F(x)=\int_0^x f(s)ds$;

(H2) $L_\infty=\lim\inf\limits_{|x|\to\infty}\dfrac{pF(x)}{|x|^p}>0$;

(H3) there exists $d^\star>0$ such that $F(d^\star)<0$;

(H4) there exists $c>0$ such that
$$\max_{|x|\le c}F(x)<\frac{c^pL_\infty}{Np\lambda_N(N+1)^{p-1}},$$
where
$$\lambda_N=\max_{u\in E_N\setminus\{0\}}\frac{\sum\limits_{t=1}^{N+2}|\Delta^2u(t-2)|^p+|\Delta u(t-1)|^p}{\sum\limits_{t=1}^{N}|u(t)|^p},\tag{1.2}$$
with E_N defined in (2.1).

The main results in this paper are the following theorems.

Theorem 1.1. *λ_N (defined in (1.2)) is the last eigenvalue of the nonlinear eigenvalue problem (1.1).*

Theorem 1.2. *Assume that $(H1)$ hold. Let c be a positive constant. Then, there exists a non-empty open set $\Lambda_c\subset\,]0,\infty[$ such that, for each $\alpha\in\Lambda_c$, the problem (P_α) has at least one nontrivial solution $u_{\alpha,1}$ such that $\|u_{\alpha,1}\|_\infty<c$.*

Corollary 1.3. *Assume that $(H1)$ hold. Let c be a positive constant. Then, for every*
$$\alpha\in\Lambda_c^*=\left]0,\frac{c^p}{Np(N+1)^{p-1}\max\limits_{|x|\le c}F(x)}\right[,$$
the problem (P_α) has at least one nontrivial solution u^* such that $\|u^*\|_\infty<c$.

Theorem 1.4. *Assume that $(H1)$-$(H4)$ hold. Let c be a positive constant. Then, for each*
$$\alpha\in\Lambda_c^{**}=\left]\frac{\lambda_N}{L_\infty},\frac{c^p}{Np(N+1)^{p-1}\max\limits_{|x|\le c}F(x)}\right[,$$
the problem (P_α) has at least two nontrivial solutions in E_N.

The paper is organized as follows. In Section 2, contains some preliminary lemmas. Section 3, we introduce the eigenvalue problem (1.1) associated to (P_α). The main results will be proved in Section 4.

2. Preliminary lemmas

In the present paper, we define a vector space E_N by

$$E_N = \{u : [-1, N+2]_{\mathbb{Z}} \longrightarrow \mathbb{R} \mid u(0) = u(N+1) = \Delta u(-1) = \Delta u(N+1) = 0 \}, \tag{2.1}$$

and for any $u \in E_N$, define

$$\|u\|_p = \left(\sum_{t=1}^{N} |u(t)|^p \right)^{1/p}.$$

So, $(E_N, \|.\|_p)$ is an N-dimensional reflexive Banach space. In fact, E_N is isomorphic to $\mathbb{R}^N$.
We also put, for every $u \in E_N$,

$$\|u\|_\infty = \max_{t \in [1,N]_{\mathbb{Z}}} |u(t)| \quad \text{and} \quad \|u\| = \left(\sum_{t=1}^{N+1} |\Delta u(t-1)|^p \right)^{1/p}.$$

Lemma 2.1. (*see Lemma* 2.2 *[5]*) *For any $u \in E_N$, we have*

$$\|u\|_\infty \leq \frac{(N+1)^{\frac{p-1}{p}}}{2} \|u\|. \tag{2.2}$$

Define $I_\alpha : E_N \longrightarrow \mathbb{R}$ by

$$I_\alpha(u) = \Phi(u) - \alpha \Psi(u), \quad \forall u \in E_N,$$

where

$$\Phi(u) = \frac{1}{p} \sum_{t=1}^{N+2} |\Delta^2 u(t-2)|^p + |\Delta u(t-1)|^p \quad \text{and} \quad \Psi(u) = \sum_{t=1}^{N} F(u(t)).$$

It is easy to see that Φ and Ψ are continuously differentiable and for any $u, v \in E_N$, we obtain

$$\Phi'(u).v = \sum_{t=1}^{N+2} \varphi_p(\Delta^2 u(t-2)) \Delta^2 v(t-2) + \varphi_p(\Delta u(t-1)) \Delta v(t-1),$$

$$\Psi'(u).v = \sum_{t=1}^{N} f(u(t)) v(t),$$

and

$$I'_\alpha(u).v = \sum_{t=1}^{N+2} \varphi_p(\Delta^2 u(t-2)) \Delta^2 v(t-2) + \varphi_p(\Delta u(t-1)) \Delta v(t-1) - \sum_{t=1}^{N} \alpha f(u(t)) v(t).$$

Lemma 2.2. *For any $u, v \in E_N$, we have*

1) $\sum_{t=1}^{N+2} \varphi_p(\Delta u(t-1)) \Delta v(t-1) = -\sum_{t=1}^{N} \Delta \left(\varphi_p(\Delta u(t-1)) \right) v(t).$

2) $\sum_{t=1}^{N+2} \varphi_p(\Delta^2 u(t-2)) \Delta^2 v(t-2) = \sum_{t=1}^{N} \Delta^2 \left(\varphi_p(\Delta^2 u(t-2)) \right) v(t).$

Proof.

1) Let $u, v \in E_N$, by the summation by parts formula and the fact that $v(0) = v(N+1) = 0$, it follows that

$$\begin{aligned}\sum_{t=1}^{N} \Delta(\varphi_p(\Delta u(t-1)))v(t) &= \left[\varphi_p(\Delta u(t-1))v(t)\right]_1^{N+1} - \sum_{t=1}^{N} \varphi_p(\Delta u(t))\Delta v(t) \\ &= -|\Delta u(0)|^{p-2}\Delta u(0)v(1) - \sum_{t=2}^{N+1} \varphi_p(\Delta u(t-1))\Delta v(t-1) \\ &= -\sum_{t=1}^{N+2} \varphi_p(\Delta u(t-1))\Delta v(t-1).\end{aligned}$$

2) By the summation by parts formula and the fact that $v(0) = v(N+1) = 0$, it follows that

$$\begin{aligned}\sum_{t=1}^{N} \Delta^2\left(\varphi_p(\Delta^2 u(t-2))\right)v(t) &= \left[\Delta(\varphi_p(\Delta^2 u(t-2)))v(t)\right]_1^{N+1} - \sum_{t=1}^{N} \Delta(\varphi_p(\Delta^2 u(t-1)))\Delta v(t) \\ &= -\Delta(\varphi_p(\Delta^2 u(-1)))v(1) - \sum_{t=1}^{N} \Delta(\varphi_p(\Delta^2 u(t-1)))\Delta v(t) \\ &= -\sum_{t=0}^{N} \Delta(\varphi_p(\Delta^2 u(t-1)))\Delta v(t) \\ &= -\sum_{t=1}^{N+1} \Delta(\varphi_p(\Delta^2 u(t-2)))\Delta v(t-1).\end{aligned}$$

Similarly, using the summation by parts formula and the fact that $\Delta v(N+1) = \Delta v(-1) = 0$, we have

$$\begin{aligned}\sum_{t=1}^{N} \Delta^2\left(\varphi_p(\Delta^2 u(t-2))\right)v(t) = &-\left[\varphi_p(\Delta^2 u(t-2)))\Delta v(t-1)\right]_1^{N+2} \\ &+ \sum_{t=1}^{N+1} \varphi_p(\Delta^2 u(t-1)))\Delta^2 v(t-1) \\ =&\varphi_p(\Delta^2 u(-1))\Delta v(0) + \sum_{t=1}^{N+1} \varphi_p(\Delta^2 u(t-1)))\Delta^2 v(t-1) \\ =&\sum_{t=0}^{N+1} \varphi_p(\Delta^2 u(t-1))\Delta^2 v(t-1) \\ =&\sum_{t=1}^{N+2} \varphi_p(\Delta^2 u(t-2))\Delta^2 v(t-2).\end{aligned}$$

This completes the proof of Lemma 2.2.

□

By Lemma 2.2, I'_α can be written as

$$I'_\alpha(u).v = \sum_{t=1}^{N} \left[\Delta^2\left(\varphi_p(\Delta^2 u(t-2))\right) - \Delta\left(\varphi_p(\Delta u(t-1))\right) - \alpha f(u(t))\right]v(t),$$

for any $v \in E_N$.
Thus, finding solutions of (P_α) is equivalent to finding critical point of the functional I_α.

Theorem 2.3. (*see Theorem* 3.3 *[2]*) *Let E be a finite dimensional Banach space and let $I_\alpha : E \longrightarrow \mathbb{R}$ be a function satisfying the following structure hypothesis:*
$(H^\star)$ $I_\alpha(u) = \varphi(u) - \alpha\psi(u)$ for all $u \in E$, where $\varphi, \psi : E \longrightarrow \mathbb{R}$ are two functions of class C^1 on E with φ coercive, such that

$$\inf_{u\in E}\varphi = \varphi(0) = \psi(0) = 0,$$

and α is a real positive parameter.

Then, let $r > 0$, for each $\alpha \in \left]0, \dfrac{r}{\sup\limits_{u\in\varphi^{-1}([0,r])}\psi(u)}\right[$, the function I_α admits at least a local minimum $\overline{u} \in E$ such that $\varphi(\overline{u}) < r$, $I_\alpha(\overline{u}) \le I_\alpha(u)$ for all $u \in \varphi^{-1}([0,r])$ and $I'_\alpha(\overline{u}) = 0$.

3. Eigenvalue problem

We consider the nonlinear eigenvalue problem (1.1) corresponding to the problem (P_α).

Definition 3.1. *$\lambda \in \mathbb{R}$ is called eigenvalue of (1.1) if there exists $u \in E_N \setminus \{0\}$ such that*

$$\sum_{t=1}^{N+2}\varphi_p(\Delta^2 u(t-2))\Delta^2 v(t-2) + \varphi_p(\Delta u(t-1))\Delta v(t-1) = \lambda\sum_{t=1}^{N}\varphi_p(u(t))v(t), \quad \forall v \in E_N.$$

Proposition 3.2. (*see [6]*) *Let E be a real Banach space, $G, J \in C^1(E,\mathbb{R})$ and a set of constraints $S = \{u \in E \mid \; G(u) = 0\}$. Suppose that for any $u \in S$, $G'(u) \neq 0$ and there exists $u_0 \in S$ such that $J(u_0) = \inf\limits_{u\in S} J(u)$.*
Then, there is $\lambda \in \mathbb{R}$ such that $J'(u_0) = \lambda G'(u_0)$.

Proof of Theorem 1.1.
Put

$$J(u) = \sum_{t=1}^{N+2}|\Delta^2 u(t-2)|^p + |\Delta u(t-1)|^p, \; G(u) = \sum_{t=1}^{N}|u(t)|^p - 1 = \|u\|_p^p - 1,$$

and

$$S = \{u \in E_N \mid \; G(u) = 0\} = \{u \in E_N \mid \; \|u\|_p = 1\}.$$

It is easy to see that $G'(u) \neq 0$ for any $u \in S$.
The set S is compact and J is continuous on S, then there exists $u_N \in S$ such that

$$J(u_N) = \max_{u\in S} J(u) = \lambda'.$$

Thus,

$$-J(u_N) = \min_{u\in S}(-J(u)) = -\lambda'.$$

Clearly $\lambda' > 0$. From the Proposition 3.2, there exists λ_N such that

$$J'(u_N) = \lambda_N G'(u_N). \tag{3.1}$$

Witch mains that $\Delta^2(\varphi_p(\Delta^2 u_N(t-2))) - \Delta(\varphi_p(\Delta u_N(t-1))) = \lambda_N\varphi_p(u_N(t))$, $t \in [1,N]_{\mathbb{Z}}$.
Multiplying (3.1) by u_N in the sens of inner product, we obtain

$$\sum_{t=1}^{N+2}|\Delta^2 u_N(t-2)|^p + |\Delta u_N(t-1)|^p = \lambda_N\sum_{t=1}^{N}|u_N(t)|^p,$$

i.e.,

$$J(u_N) = \lambda_N\|u_N\|_p^p = \lambda_N.$$

Therefore, $\lambda' = \lambda_N$ is an eigenvalue of the problem (1.1).
Moreover, we have

$$\begin{aligned}\lambda_N &= \max_{u\in S}\sum_{t=1}^{N+2}|\Delta^2 u(t-2)|^p+|\Delta u(t-1)|^p\\ &= \max_{u\in E_N\setminus\{0\}}\sum_{t=1}^{N+2}|\Delta^2\frac{u(t-2)}{\|u\|_p}|^p+|\Delta\frac{u(t-1)}{\|u\|_p}|^p\\ &= \max_{u\in E_N\setminus\{0\}}\frac{\sum_{t=1}^{N+2}|\Delta^2 u(t-2)|^p+|\Delta u(t-1)|^p}{\sum_{t=1}^{N}|u(t)|^p}.\end{aligned}$$

On the other hand, if λ is an eigenvalue of the problem (1.1), then there exists $u\in E_N\setminus\{0\}$ such that:

$$\sum_{t=1}^{N+2}\varphi_p(\Delta^2 u(t-2))\Delta^2 v(t-2)+\varphi_p(\Delta u(t-1))\Delta v(t-1)=\lambda\sum_{t=1}^{N}\varphi_p(u(t))v(t),\quad \forall v\in E_N.$$

In particular for $v=u$, we get $\lambda = \dfrac{\sum_{t=1}^{N+2}|\Delta^2 u(t-2)|^p+|\Delta u(t-1)|^p}{\sum_{t=1}^{N}|u(t)|^p}$.

So, we deduce that $\lambda \le \lambda_N$.
Then, λ_N is the last eigenvalue of the problem (1.1).
The proof of Theorem 1.1 is complete. □

Remark 3.3. *Similarly of the proof of Theorem 1.1, we show that λ_1 is the first eigenvalue of the problem (1.1), where*

$$\lambda_1 = \min_{v\in E_N\setminus\{0\}}\frac{\sum_{t=1}^{N+2}|\Delta^2 u(t-2)|^p+|\Delta u(t-1)|^p}{\sum_{t=1}^{N}|u(t)|^p}.$$

It is clear to see that

$$\frac{\lambda_1}{p}\|u\|_p^p \le \Phi(u) \le \frac{\lambda_N}{p}\|u\|_p^p,\quad \forall u\in E_N. \tag{3.2}$$

4. Proofs of the main results

Proof of Theorem 1.2.
Let $\alpha>0$ and $c>0$. Our aim is to apply Theorem 2.3, with $\phi=\Phi$ and $\psi=\Psi$.
An easy computation ensures the Φ and Ψ satisfy condition $(H^\star)$.
From (H1), there exists $d>0$ enghout small with $d<\dfrac{c}{6^{\frac{1}{p}}(N+1)^{\frac{p-1}{p}}}$ such that

$$\alpha > \frac{6d^p}{NpF(d)} \quad\text{and}\quad F(d)>0. \tag{4.1}$$

Put $r=\dfrac{c^p}{p(N+1)^{p-1}}$ and choose $\omega\in E_N$ defined by $\omega(t)=\begin{cases} d & ,\ t\in[1,N]_{\mathbb{Z}},\\ 0 & ,\ \text{otherwise.}\end{cases}$

It is easy to see that $\Phi(\omega)=\dfrac{6}{p}d^p\le r$, hence $\omega\in\Phi^{-1}([0,r])$. So,

$$\sup_{u\in\Phi^{-1}([0,r])}\Psi(u)\ge\Psi(\omega)=NF(d)>0.$$

Consequently, from Theorem 2.3, for each

$$\alpha \in \Lambda_c = \left]0, \frac{r}{\sup\limits_{u\in\Phi^{-1}([0,r])} \Psi(u)}\right[,$$

the functional I_α admits a critical point $u_{\alpha,1} \in E_N$ such that

$$I_\alpha(u_{\alpha,1}) \leq I_\alpha(u) \text{ for all } u \in \Phi^{-1}([0,r]) \text{ and } \Phi(u_{\alpha,1}) < r. \tag{4.2}$$

In particular for $u = \omega$, and by (4.1), we have

$$I_\alpha(u_{\alpha,1}) \leq I_\alpha(\omega) = \frac{6}{p}d^p - \alpha N F(d) < 0.$$

Then, the problem (P_α) has at least one nontrivial solution $u_{\alpha,1} \in E_N$.
Now, we prove that $\|u_{\alpha,1}\|_\infty < c$. Since $\Phi(u_{\alpha,1}) < r$, then

$$\begin{aligned}
&\frac{1}{p}\sum_{t=1}^{N+2} |\Delta^2 u_{\alpha,1}(t-2)|^p + |\Delta u_{\alpha,1}(t-1)|^p < r,\\
&\Rightarrow \frac{1}{p}\|u_{\alpha,1}\|^p < r,\\
&\Rightarrow \|u_{\alpha,1}\| < (pr)^{\frac{1}{p}}\\
&\Rightarrow \|u_{\alpha,1}\| < \frac{c}{(N+1)^{\frac{p-1}{p}}}.
\end{aligned}$$

Using Lemma 2.1, we get

$$\begin{aligned}
\|u_{\alpha,1}\|_\infty &\leq \frac{(N+1)^{\frac{p-1}{p}}}{2}\|u_{\alpha,1}\|\\
&< \frac{c}{2}\\
&< c.
\end{aligned}$$

The proof is complete. □

Proof of Corollary 1.2A Let $\alpha > 0$ and $c > 0$. From the proof of Theorem 1.2, $\sup\limits_{u\in\Phi^{-1}([0,r])} \Psi(u) > 0$ where $r = \dfrac{c^p}{p(N+1)^{p-1}}$, and if $\Phi(u) \leq r$ for any $u \in E_N$, we get $\|u\|_\infty \leq c$. Therefore

$$0 < \frac{\sup\limits_{u\in\Phi^{-1}([0,r])} \Psi(u)}{r} = \frac{\sup\limits_{u\in\Phi^{-1}([0,r])} \sum\limits_{t=1}^{N} F(u(t))}{r} \leq \frac{Np(N+1)^{p-1}\max\limits_{|x|\leq c} F(x)}{c^p}.$$

So, we have

$$\left]0, \frac{c^p}{Np(N+1)^{p-1}\max\limits_{|x|\leq c} F(x)}\right[\subseteq \left]0, \frac{r}{\sup\limits_{u\in\Phi^{-1}([0,r])} \Psi(u)}\right[.$$

Hence, from Theorem 1.2 for each $\alpha \in \left]0, \dfrac{c^p}{Np(N+1)^{p-1}\max\limits_{|x|\leq c} F(x)}\right[$, the problem (P_α) has at least one nontrivial solution u^* such that $\|u^*\|_\infty < c$. The proof is complete. □

Proof of Theorem 1.3. Fix $\alpha \in \Lambda_c^{**}$. Since $L_\infty > 0$ then there exists an $\delta \in \mathbb{R}$ such that $L_\infty > \delta > \dfrac{\lambda_N}{\alpha}$. From the definition of L_∞, there is $R > 0$ such that

$$\frac{pF(x)}{|x|^p} \geq \delta - \varepsilon \quad \text{for} \quad (t, |x|) \in [1, N]_{\mathbb{Z}} \times]R, +\infty[,$$

where $0 < \varepsilon < \delta - \dfrac{\lambda_N}{\alpha}$, i.e.,

$$F(x) \geq \frac{1}{p}(\delta - \varepsilon)|x|^p \qquad \text{for } (t, |x|) \in [1, N]_{\mathbb{Z}} \times]R, +\infty[. \tag{4.3}$$

Then, by (4.3) and the continuity of $x \longrightarrow F(x)$, there exists $M > 0$ such that

$$F(x) \geq \frac{1}{p}(\delta - \varepsilon)|x|^p - M, \quad \forall (t, x) \in [1, N]_{\mathbb{Z}} \times \mathbb{R}. \tag{4.4}$$

Therefore,

$$-\alpha\Psi(u) \leq \frac{-\alpha}{p}(\delta - \varepsilon)\|u\|_p^p + \alpha M N \quad \text{for any } u \in E_N. \tag{4.5}$$

According to (3.2) and (4.5), we have

$$\begin{aligned} I_\alpha(u) = \Phi(u) - \alpha\Psi(u) &\leq \frac{1}{p}\lambda_N\|u\|_p^p - \frac{\alpha}{p}(\delta - \varepsilon)\|u\|_p^p + \alpha M N \\ &\leq \frac{1}{p}\left[\lambda_N - \alpha(\delta - \varepsilon)\right]\|u\|_p^p + \alpha M N. \end{aligned}$$

Since $\varepsilon < \delta - \dfrac{\lambda_N}{\alpha}$, we obtain

$$\lim_{\|u\|_p \to \infty} I_\alpha(u) = -\infty.$$

So I_α is anti-coercive and bounded from above, then there is a maximum of I_α at some $u_{\alpha,2} \in E_N$, i.e., $I_\alpha(u_{\alpha,2}) = \sup\limits_{u \in E_N} I_\alpha(u)$, witch is a critical point of I_α and in turn is a solution of the problem (P_α).
On the other hand, it is clear to see that $I_\alpha(u_{\alpha,2}) \geq I_\alpha(0) = 0$.
Now, if $I_\alpha(u_{\alpha,2}) = 0$, we get $I_\alpha(u) = \Phi(u) - \alpha\Psi(u) \leq 0$ for any $u \in E_N$.
Since $\alpha > 0$ and $\Phi(u) \geq 0$, then $\Psi(u) \geq 0$ for any $u \in E_N$.
In particular, for $u = \omega^\star$ with $\omega^\star \in E_N$ defined by $\omega^\star(t) = \begin{cases} d^\star, & t \in [1, N]_{\mathbb{Z}}, \\ 0, & \text{otherwise}, \end{cases}$
we obtain, $\Psi(\omega^\star) = NF(d^\star) < 0$. This is contradiction. Hence, $I_\alpha(u_{\alpha,2}) > 0$. Therefore $u_{\alpha,2}$ is nontrivial solution of the problem (P_α).
By $(H2)$ and the Corollary 1.3, the problem (P_α) has a solution nontrivial $u_{\alpha,1}$ such that $I_\alpha(u_{\alpha,1}) < 0$.
Thus, the problem (P_α) has at least two nontrivial solution $u_{\alpha,1}$ and $u_{\alpha,2}$ such that $I_\alpha(u_{\alpha,1}) < 0 < I_\alpha(u_{\alpha,2})$. The proof is complete. $\square$

Example 4.1. *Let $p = 3$ and f be a function as follows:*

$$f(x) = \begin{cases} -2x^4 - 1, & \textit{if } x < -1, \\ 3x, & \textit{if } -1 \leq x < 1, \\ -5x^2 + 8, & \textit{if } 1 \leq x < 2, \\ x^3 + 2x - 24, & \textit{if } x \geq 2. \end{cases}$$

It is easy to see that

$$F(x) = \begin{cases} \dfrac{-2}{5}x^5 - x + \dfrac{1}{10}, & \textit{if } x < -1, \\ \dfrac{3}{2}x^2, & \textit{if } -1 \leq x < 1, \\ \dfrac{-5}{3}x^3 + 8x - \dfrac{29}{6}, & \textit{if } 1 \leq x < 2, \\ \dfrac{1}{4}x^4 + x^2 - 24x + \dfrac{227}{6}, & \textit{if } x \geq 2. \end{cases}$$

Clearly

$$L_\infty = \infty \quad \textit{and} \quad \lim_{x\to 0}\frac{F(x)}{|x|^3} = \infty.$$

We can choose $c = 1$ *and* $d^\star = 2$. *After a simple calculation, we get*

$$F(d^\star) = -\frac{13}{6} < 0 \textit{ and } \frac{c^p}{Np(N+1)^{p-1}\max\limits_{|x|\le c}F(x)} = \frac{2}{9N(N+1)^2},$$

then one has F *satisfies the conditions of Theorem 1.4.*
Thus for each $\alpha \in \left]0, \dfrac{2}{9N(N+1)^2}\right[$, *the problem*

$$\begin{cases} \Delta^2(|\Delta^2 u(t-2)|\Delta^2 u(t-2)) - \Delta(|\Delta u(t-1)|\Delta u(t-1)) = \alpha f(u(t)),\ t\in[1,N]_{\mathbb{Z}}, \\ \quad u(0) = u(N+1) = \Delta u(-1) = \Delta u(N+1) = 0, \end{cases}$$

has at least two nontrivial solutions.

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References

1. R. P. Agarwal, *Difference Equations and Inequalities: Theory, Methods and Applications, Marcel Dekker, New York, NY, USA, 1992.*
2. G. Bonanno, P. Candito, G. D'Agui, *Variational methods on finite dimensional Banach spaces and discrete problems, Adv. Nonlinear Stud. 14, pp. 915-939, 2014.*
3. G. Bonanno, P. Jebelean, C. Serban, *Superlinear discrete problems, Appl. Mth. Letters, vol. 52, pp. 162-168, 2016.*
4. S. N. Elaydi, *An Introduction to Difference Equations, Springer, New York, NY, USA, 1996.*
5. L. Jiang, Z. Zhou, *Three solutions to Dirichlet boundary value problems for p-Laplacian difference equations, Adv. Difference Equ., pp. 1-10, 2008.*
6. O. Kavian, *Introduction à la théorie des points critiques, Springer-Verlag France, Paris, 1993.*
7. W. G. Kelley and A. C. Peterson, *Difference Equations: An Introduction with Applications, Academic Press, New York, NY, USA, 1991.*
8. V. Lakshmikantham and D. Trigiante, *Theory of Difference Equations: Numerical Methods and Applications, Academic Press, New York, NY, USA, 1988.*
9. R. E. Mickens, *Difference Equations: Theory and Application, Van Nostrand Reinhold, New York, NY, USA, 1990.*
10. A. N. Sharkovsky, Yu. L. Maıstrenko, and E. Yu. Romanenko, *Difference Equations and Their Applications, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1993.*

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