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Entropy Solution for a Nonlinear Degenerate Parabolic Problem in Weighted Sobolev Space via Rothe's Time-discretization Approach

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ABSTRACT: In this paper, we prove the existence and uniqueness results of an entropy solution to a class of nonlinear degenerate parabolic problem with Dirichlet-type boundary condition and L1 data. The main tool used here is the Rothe's time-discretization approach combined with the theory of weighted Sobolev spaces.

Key Words: Degenerate parabolic problem, entropy solution, existence, semi-discretization, Rothe's method, uniqueness, weighted Sobolev space.

Contents

1. Introduction

Let $\Omega \subset \mathbb{R}^d, (d \geq 2)$ be an open bounded domain with a connected Lipschitz boundary $\partial\Omega$, $p \in (1, \infty)$, and let T be a fixed positive real number. Our aim of this paper is to prove the existence and uniqueness results of entropy solutions for the nonlinear degenerate parabolic problem

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}\Big(\omega|\nabla u - \Theta(u)|^{p-2}\big(\nabla u - \Theta(u)\big)\Big) = f(x,t) \quad in \quad Q_T := \Omega\times]0,\ T[, \\ u = 0 \quad on \quad \Sigma_T := \partial\Omega\times]0,\ T[, \\ u(.,0) = u_0 \quad in \quad \Omega, \end{cases} \tag{1.1}$$

where ω is a measurable positive function defined on $\mathbb{R}^d$, α is a non decreasing continuous real function defined on $\mathbb{R}$ and Θ is a continuous function defined from $\mathbb{R}$ to $\mathbb{R}^d$, the datum f is in L^1.

In recent years, the study of partial differential equations and variational problems has received considerable attention in many models coming from various branches of mathematical physics, such as elastic mechanics, electrorheological fluid dynamics and image processing. Degenerate phenomena appear in area of oceanography, turbulent fluid flows, induction heating and electrochemical problems (see [7,11,12]).

In general, the Sobolev spaces $W^{k,p}(\Omega)$ without weights occur as spaces of solutions for elliptic and parabolic partial differential equations. For degenerate partial differential equations, i.e., equations with various types of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces(see [5,6,8,15,16]). The notion of entropy solutions was introduced by Bénilan et al in [2], this notion was then adapted by many authors to study some nonlinear elliptic and parabolic problems with a constant or variable exponent and Dirichlet or Neumann boundary conditions (see [1,3,4,16]).

The problem (1.1) is modeling several natural phenomena, we cite for example the following two parabolic models.

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• **Model 1. Filtration in a porous medium.** The filtration phenomena of fluids in porous media are modeled by the following equation,

$$\frac{\partial c(p)}{\partial t} = \nabla a[k(c(p))(\nabla p + e)], \tag{1.2}$$

where p is the unknown pressure, c volumetric moisture content, k the hydraulic conductivity of the porous medium, a the heterogeneity matrix and $-e$ is the direction of gravity.
• **Model 2. Fluid flow through porous media.** This model is governed by the following equation,

$$\frac{\partial \theta}{\partial t} - div\left(|\nabla \varphi(\theta) - K(\theta)e|^{p-2}(\nabla \varphi(\theta) - K(\theta)e)\right) = 0, \tag{1.3}$$

where θ is the volumetric content of moisture, $K(\theta)$ the hydraulic conductivity, $\varphi(\theta)$ the hydrostatic potential and e is the unit vector in the vertical direction.

In this paper, we study the existence and uniqueness question of entropy solutions to the problem (1.1), we apply here a time discretization of the problem (1.1) by Euler forward scheme and we show existence, uniqueness and stability of entropy solutions to the discretized problem. After, we will construct from the entropy solution of the discretized problem a sequence that we show converging to an entropy solution of the nonlinear parabolic problem (1.1). We recall that the Rothe's method was introduced by E. Rothe in 1930 and it has been used and developed by many authors, e.g P.P. Mosolov, K. Rektorys in linear and quasilinear parabolic problems. This method has been used by several authors while studying time discretization of nonlinear parabolic problems, we refer to the works [9,13,14] for some details. The advantage of our method is that we cannot only obtain the existence and uniqueness of weak solutions to the problem (1.1), but also compute the numerical approximations.

2. Preliminaries and notations

In this section, we give some notations and definitions and we state some results which will be used in this work.
Let ω be a measurable positive and a.e finite function defined on $\mathbb{R}^d$, ω is called a weight function. Further, we suppose that the following integrability conditions are satisfied:

(H_1) $\omega \in L^1_{loc}(\Omega)$ and $\omega^{\frac{-1}{p-1}} \in L^1_{loc}(\Omega)$,

(H_2) $\omega^{-s} \in L^1_{loc}(\Omega)$ where $s \in \left(\frac{N}{p}, \infty\right) \cap \left[\frac{1}{p-1}, \infty\right)$.

The weighted Lebesgue space $L^p(\Omega, \omega)$ is defined by

$$L^p(\Omega, \omega) = \{u\ \Omega \to \mathbb{R} \mid u \text{ is measurable and } \int_\Omega \omega(x)|u|^p dx < \infty\},$$

endowed with the norm

$$\|u\|_{p,\omega} := \|u\|_{L^p(\Omega,\omega)} = \left(\int_\Omega \omega(x)|u|^p dx\right)^{\frac{1}{p}}.$$

The weighted Sobolev space $W^{1,p}(\Omega, \omega)$ is defined by

$$W^{1,p}(\Omega, \omega) = \{u \in L^p(\Omega, \omega) \mid |\nabla u| \in L^p(\Omega, \omega)\},$$

with the norm

$$\|u\|_{1,p,\omega} = \|u\|_p + \|\nabla u\|_{p,\omega},\ \forall\ u \in W^{1,p}(\Omega, \omega).$$

In the following, the space $W^{1,p}_0(\Omega, \omega)$ denote the closure of C^∞_0 in $W^{1,p}(\Omega, \omega)$ endowed by the norm

$$\|u\|_{W^{1,p}_0(\Omega,\omega)} = \left(\int_\Omega |\nabla u|^p \omega(x) dx\right)^{\frac{1}{p}}.$$

Let s be a real number satisfying hypothesis (H_2), we define the following critical exponents

$$p^* = \frac{dp}{d-p} \quad \textit{for} \ \ p < d,$$

$$p_s = \frac{ps}{1+s} < p,$$

$$p_s^* = \begin{cases} \dfrac{ps}{(1+s)d - ps} & \text{if } d > p_s, \\ +\infty & \text{if } d \leq p_s. \end{cases}$$

In the following of this work, we need to following results.

Proposition 2.1 ([8]). *Let $\Omega \subset \mathbb{R}^d$ be an open set of $\mathbb{R}^d$ and let hypothesis (H_1) be satisfied, we have*

$$L^p(\Omega, \omega) \hookrightarrow L^1_{loc}(\Omega).$$

Proposition 2.2 ([8]). *Let hypothesis (H_1) be satisfied, the space $(W^{1,p}(\Omega, \omega), \|u\|_{1,p,\omega})$ is a separable and reflexive Banach space.*

Proposition 2.3 ([8]). *Assume that hypotheses (H_1) and (H_2) hold, we have the continuous embedding*

$$W^{1,p}(\Omega, \omega) \hookrightarrow W^{1,p_s}(\Omega, \omega).$$

Moreover, we have the compact embedding

$$W^{1,p}(\Omega, \omega) \hookrightarrow\hookrightarrow L^r(\Omega),$$

where $1 \leq r < p_s^*$.

Proposition 2.4 ([8]). *(Hardy-type inequality) There exist a weight function ω defined on Ω and a parameter q, $1 < q < \infty$ such that the inequality*

$$\left(\int_\Omega \omega(x) |u(x)|^q dx \right)^{\frac{1}{q}} \leqslant C_H \left(\int_\Omega \omega(x) |\nabla u|^p dx \right)^{\frac{1}{p}} \tag{2.1}$$

holds for every $u \in W_0^{1,p}(\Omega, \omega)$, C_H is a strictly positive constant independent of u. Moreover, the embedding

$$W_0^{1,p}(\Omega, \omega) \hookrightarrow L^q(\Omega, \omega)$$

expressed by inequality (2.1) is compact.

Let X be a Banach space and let $T > 0$. For $1 \leq p \leq \infty$, the space $L^p(0, T; X)$ consists of all measurable functions $u : [0, T] \to X$ such that

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p dt \right)^{\frac{1}{p}} < \infty \quad \text{if } 1 \leq p < \infty$$

and

$$\|u\|_{L^\infty(0,T;X)} = \text{esssup}_{t \in [0,T]} \|u(t)\|_X < \infty.$$

The space $C(0, T; X)$ is a space of all continuous functions $u : [0, T] \to X$ such that

$$\|u\|_{C(0,T;X)} = \max_{t \in [0,T]} \|u(t)\|_X < \infty.$$

The spaces $L^p(0, T; X)$ and $C(0, T; X)$ equipped with the norms from the above definitions are the Banach spaces.

Given a constant $k > 0$, we define the cut function $T_k : \mathbb{R} \to \mathbb{R}$ as

$$T_k(s) = \min(k, \max(s, -k)) = \begin{cases} s & \text{if } |s| \leq k, \\ k & \text{if } s > k, \\ -k & \text{if } s < -k. \end{cases}$$

For a function $u = u(x)$ defined on Ω, we define the truncated function $T_k u$ as follows, for every $x \in \Omega$, the value of $(T_k u)$ at x is just $T_k(u(x))$.
Let the function $J_k : \mathbb{R} \to \mathbb{R}^+$ (is the primitive function of T_k) defined by

$$J_k(s) = \int_0^s T_k(t)dt = \begin{cases} \frac{s^2}{2} & \text{if } |s| \leq k \\ k|s| - \frac{k^2}{2} & \text{if } |s| > k. \end{cases}$$

We have as in the paper [10]

$$\langle \frac{\partial s}{\partial t}, T_k(s) \rangle = \frac{d}{dt} \int_\Omega J_k(s) dx \ \text{ in } \ L^1(]0, T[),$$

which implies

$$\int_0^t \langle \frac{\partial s}{\partial t}, T_k(s) \rangle dt = \int_\Omega J_k(s(t))dx - \int_\Omega J_k(s(0))dx.$$

We define also the space

$$\mathcal{T}_0^{1,p}(\Omega, \omega) = \Big\{ u : \Omega \to \mathbb{R}, \ \ u \text{ is measurable and } \ T_k(u) \in W_0^{1,p}(\Omega, \omega) \ \text{ for all } \ k > 0 \Big\}.$$

By [2, Lemma 2.1], the very weak gradient of a measurable function $u \in \mathcal{T}_0^{1,p}(\Omega, \omega)$ is defined as

Proposition 2.5. *For every function $u \in \mathcal{T}_0^{1,p}(\Omega, \omega)$, there exists a unique measurable function $v : \Omega \to \mathbb{R}^d$, which we call the very weak gradient of u (if there is any confusion, we denote $v = \nabla u$)such that*

$$\nabla T_k(u) = v\chi_{\{|u| \leq k\}} \text{ for a.e } x \in \Omega \text{ and for all } k > 0,$$

where χ_B is the characteristic function of the measurable set $B \subset \mathbb{R}^d$. Moreover, if u belongs to $W_0^{1,p}(\Omega, \omega)$, the very weak gradient of u coincides to its weak gradient.

Lemma 2.6 ([1])**.** *For $\xi, \ \eta \in \mathbb{R}^d$ and $1 < p < \infty$, we have*

$$\frac{1}{p}|\xi|^p - \frac{1}{p}|\eta|^p \leq |\xi|^{p-2}\xi(\xi - \eta).$$

Lemma 2.7 ([1])**.** *For $a \geq 0$, $b \geq 0$ and $1 \leq p < +\infty$, we have*

$$(a + b)^p \leq 2^{p-1}(a^p + b^p).$$

Lemma 2.8. ***(Dominated Convergence Theorem)*** *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions on $L^p(\Omega)$ $(1 \leq p < \infty)$ converging almost everywhere to a function f. Assume that there exists a function $g \in L^p(\Omega)$ such that for all $n \in \mathbb{N}$, we have $|f_n(x)| \leq |g(x)|$ almost everywhere in Ω. Then $f \in L^p(\Omega)$ and*

$$\lim_{n\to\infty} \|f_n - f\|_{L^p(\Omega)} = 0.$$

Remark 2.9. *Hereinafter k, τ, T are strictly positive real numbers, N is a strictly positive number and $C(X), C_i(X)$ $(i \in \mathbb{N})$ are positive constants depending only on X.*

3. Assumptions and main result

In this section, we will introduce the concept of entropy solution for the problem (1.1) and we will state the existence and the uniqueness results for this type of solution. Firstly and in addition of the hypotheses (H_1) and (H_2) listed earlier, we suppose the following assumptions.

(H_3) α is a non-decreasing continuous real function defined on $\mathbb{R}$, surjective such that $\alpha(0) = 0$.

(H_4) Θ is a continuous function from $\mathbb{R}$ to $\mathbb{R}^d$ such that $\Theta(0) = 0$, and for all real numbers x, y, we have $|\Theta(x) - \Theta(y)| \leq \lambda|x - y|$, where λ is a real constant such that $0 < \lambda < \dfrac{1}{2}C_H^{-1}$, and C_H is the constant given in Proposition 2.4.

(H_5) $f \in L^1(Q_T)$ and $u_0 \in L^1(\Omega)$.

Definition 3.1. *A measurable function $u : Q_T \to \mathbb{R}$ is an entropy solution of the parabolic problem (1.1) in Q_T if $u(.,0) = u_0$ in Ω, $u \in C(0,T;L^1(\Omega))$, $T_k(u) \in L^p(0,T;W_0^{1,p}(\Omega,\omega))$ for all $k > 0$ and*

$$\int_0^t \left\langle \frac{\partial \varphi}{\partial s},\ T_k(u-\varphi) \right\rangle ds + \int_0^t \int_\Omega \omega(x)\Phi(\nabla u - \Theta(u))\nabla T_k(u-\varphi)dxds + \int_0^t \int_\Omega \alpha(u) T_k(u-\varphi)dxds$$

$$\leq \int_\Omega J_k\big(u(0)-\varphi(0)\big)dx - \int_\Omega J_k\big(u(t)-\varphi(t)\big)dx + \int_0^t \int_\Omega f T_k(u-\varphi)dxds \tag{3.1}$$

for all $\varphi \in L^\infty(Q_T) \cap L^p(0,T;W^{1,p}(\Omega,\omega)) \cap W^{1,1}(0,T;L^1(\Omega))$ and $t \in [0,\ T]$, where $Q_T := \Omega \times]0,\ T[$ and

$$\Phi(\xi) = |\xi|^{p-2}\xi, \quad \forall\, \xi \in \mathbb{R}^d.$$

Our main result is the following Theorem.

Theorem 3.2. *Assume that hypotheses (H_1), (H_2), (H_3), (H_4) and (H_5) hold, then the nonlinear degenerate parabolic problem (1.1) has a unique entropy solution.*

4. Proof of the main result

The proof of our main result is divided into three steps, in the first step, thanks to Euler forward scheme, we discretize the continuous problem (1.1) and we study the existence and uniqueness of entropy solutions to the discretized problems. In the second step, we give some stability results for the discrete entropy solutions. Finally and by Rothe's function, we construct a sequence of functions that we show that this sequence converges to an entropy solution of the nonlinear degenerate parabolic problem (1.1). We finish this step by proving the uniqueness result of entropy solutions.

Step 1. The semi-discrete problem.

By Euler forward scheme, we discretize the problem (1.1) and obtain the following problems

$$\begin{cases} U_n - \tau \text{div}\Big(\omega\Phi(\nabla U_n - \Theta(U_n))\Big) + \tau\alpha(U_n) = \tau f_n + U_{n-1} \textit{ in } \Omega, \\ U_n = 0 \quad \textit{on } \partial\Omega, \\ U_0 = u_0 \quad \textit{in } \Omega, \end{cases} \tag{4.1}$$

where $N\tau = T$, $0 < \tau < 1$, $1 \leq n \leq N$, $t_n = n\tau$ and

$$f_n(.) = \frac{1}{\tau}\int_{t_{n-1}}^{t_n} f(s,.)ds \text{ in } \Omega.$$

Definition 4.1. *An entropy solution to the discretized problem (4.1) is a sequence $(U_n)_{0\leq n\leq N}$ such that $U_0 = u_0$ and for $n = 1, 2, ..., N$, U_n is defined by induction as an entropy solution to the problem*

$$\begin{cases} u - div\Big(\omega\Phi(\nabla U_n - \Theta(U_n))\Big) + \tau\alpha(u) = \tau f_n + U_{n-1} \textit{ in } \Omega, \\ u = 0 \quad \textit{on } \partial\Omega. \end{cases}$$

i.e. for all $n \in \{1, 2, ..., N\}$, $U_n \in \mathcal{T}_0^{1,p}(\Omega,\omega)$ and for all $\varphi \in W_0^{1,p}(\Omega,\omega) \cap L^\infty(\Omega)$, $k > 0, \tau > 0$, we have

$$\int_\Omega U_n T_k(U_n-\varphi)dx + \tau\int_\Omega \omega(x)\Phi(\nabla U_n - \Theta(U_n))\nabla T_k(U_n-\varphi)dx + \int_\Omega \alpha(U_n)T_k(U_n-\varphi)dx$$

$$\leq \int_\Omega (\tau f_n + U_{n-1})T_k(U_n-\varphi)dx. \tag{4.2}$$

Lemma 4.2. *Let hypotheses (H_1), (H_2) and (H_3) be satisfied. If $(U_n)_{0\leq n\leq N}$ is an entropy solution of the discretized problem (4.1), then we have $U_n \in L^1(\Omega)$ for all $n = 1, ..., N$.*

Proof: For $n=1$, if we take $\varphi=0$ in (4.2), we get

$$\int_\Omega U_1T_k(U_1)dx+\tau\int_\Omega \omega(x)\Phi(\nabla U_1-\Theta(U_1))\nabla T_k(U_1)dx+\int_\Omega \alpha(U_1)T_k(U_1)dx\le\int_\Omega \tau f_1T_k(U_1)dx+\int_\Omega u_0T_k(U_1)dx. \tag{4.3}$$

On the one hand, using the Lemma 2.6, we get

$$\int_\Omega \omega(x)\Phi(\nabla U_1-\Theta(U_1))\nabla T_k(U_1)dx+\frac{1}{p}\int_\Omega \omega(x)\mid\Theta(T_k(U_1))\mid^p dx\geqslant 0.$$

On the other hand, by assumption (H_3), we deduce that

$$\int_\Omega \alpha(U_1)T_k(U_1)dx\geqslant 0.$$

Therefore, the inequality (4.3) becomes

$$\int_\Omega U_1T_k(U_1)dx+\le\int_\Omega \tau f_1T_k(U_1)dx+\int_\Omega u_0T_k(U_1)dx+\frac{\tau}{p}\int_\Omega \omega(x)\mid\Theta(T_k(U_1))\mid^p dx.$$

By using hypothesis (H_3) we get

$$\int_\Omega U_1T_k(U_1)dx+\le\int_\Omega \tau f_1T_k(U_1)dx+\int_\Omega u_0T_k(U_1)dx+\frac{\tau}{p}(k\lambda)^p\int_\Omega \omega(x)dx.$$

This implies

$$0\le\int_\Omega U_1\frac{T_k(U_1)}{k}dx\le\|f\|_{L^1(Q_T)}+\|u_0\|_{L^1(\Omega)}+\frac{k^{p-1}\lambda^p}{p}\|w\|_{L^1(\Omega)}.$$

For each $x\in\Omega$, we have

$$\lim_{k\to 0}U_1(x)\frac{T_k(U_1(x))}{k}=|U_1(x)|.$$

Then by Fatou's Lemma, we deduce that $U_1\in L^1(\Omega)$ and

$$\|U_1\|_{L^1(\Omega)}\le\|f\|_{L^1(Q_T)}+\|u_0\|_{L^1(\Omega)}.$$

By induction, we deduce in the same manner that $U_n\in L^1(\Omega)$ for all $n=1,...,N$. □

Theorem 4.3. *Assume that hypotheses (H_1), (H_2) and (H_3) hold. Then the discretized problem (4.1) has a unique entropy solution $(U_n)_{0\le n\le N}$ and $U_n\in L^1(\Omega)\cap\mathcal{T}_0^{1,p}(\Omega,\omega)$ for all $n=1,...,N$.*

Proof:

For $n=1$, we rewrite the discretized problem (4.1) as

$$\begin{cases} -\tau\mathrm{div}\Big(\omega\Phi(\nabla u-\Theta(u))\Big)+\bar{\alpha}(u)=F \ in\ \Omega,\\ u=0\ \ on\ \partial\Omega,\end{cases} \tag{4.4}$$

where $u=U_1$ and $F=\tau f_1+u_0$. According to the hypothesis (H_5), we have $F\in L^1(\Omega)$ and, by hypothesis (H_3), the function defined by $\bar{\alpha}(s):=\tau\alpha(s)+s$ is non-decreasing, continuous and satisfies $\bar{\alpha}(0)=0$. Then, the problem (4.1) has a unique entropy solution U_1 in $L^1(\Omega)\cap\mathcal{T}_0^{1,p}(\Omega,\omega)$ (see [16, Theorem 3.2]). By induction, using the same argument above, we prove that the problem (4.1) has a unique entropy solution $(U_n)_{0\le n\le N}$ and $U_n\in L^1(\Omega)\cap\mathcal{T}_0^{1,p}(\Omega,\omega)$ for all $n=1,...,N$. □

Step 2. Stability results.

Theorem 4.4. *Assume that hypotheses* (H_1), $(H_2), (H_3), (H_4)$ *and* (H_5) *hold. If* $(U_n)_{1\leq n\leq N}$ *is an entropy solution of the discretized problem* (4.1)*, then for all* $n = 1, ..., N$*, we have*

$$\textbf{(a)} \quad \|U_n\|_{L^1(\Omega)} \leq C(u_0, f),$$

$$\textbf{(b)} \quad \tau\sum_{i=1}^{n}\|\alpha(U_i)\|_{L^1(\Omega)} \leq C(u_0, f),$$

$$\textbf{(c)} \quad \sum_{i=1}^{n}\|U_i - U_{i-1}\|_{L^1(\Omega)} \leq C(u_0, f),$$

$$\textbf{(d)} \quad \tau\sum_{i=1}^{n}\|T_k(U_i)\|^p_{W_0^{1,p}(\Omega,\omega)} \leq C(u_0, f, T, k),$$

where $C(u_0, f)$ *and* $C(u_0, f, T, k)$ *are positive constants independents of* N*.*

Proof: For (a) and (b). Let $i \in \{1, 2, ..., N\}$, we take $\varphi = 0$ as a test function in entropy formulation of the discretized problem (4.1), we get

$$\int_\Omega U_i T_k(U_i)dx + \tau\int_\Omega \omega(x)\Phi(\nabla U_i - \Theta(U_i))\nabla T_k(U_i)dx + \frac{\tau}{p}\int_\Omega \omega(x) \mid \Theta(T_k(U_i)) \mid^p dx + \tau\int_\Omega \alpha(U_i)T_k(U_i)dx$$
$$\leq \tau\int_\Omega f_i T_k(U_i)dx + \int_\Omega U_{i-1}T_k(U_i)dx.$$

This inequality implies that

$$\int_\Omega U_i\frac{T_k(U_i)}{k}dx + \int_\Omega \alpha(U_i)\frac{T_k(U_i)}{k}dx \leq \tau\|f_i\|_{L^1(\Omega)} + \|U_{i-1}\|_{L^1(\Omega)} + \frac{1}{kp}\|\Theta(T_k(U_i))\|^p_{L^p(\Omega,\omega)}. \tag{4.5}$$

Note that

$$\lim_{k\to 0}\frac{T_k(s)}{k} = sign(s), \tag{4.6}$$

where

$$\text{sign}(s) := \begin{cases} 1 & \text{if } s > 0 \\ 0 & \text{if } s = 0 \\ -1 & \text{if } s < 0 \end{cases}.$$

Therefore, passing to limit in (4.5), using Fatou's lemma and hypothesis (H_4) , we deduce that

$$\|U_i\|_{L^1(\Omega)} + \tau\|\alpha(U_i)\|_{L^1(\Omega)} \leq \tau\|f_i\|_{L^1(\Omega)} + \|U_{i-1}\|_{L^1(\Omega)}.$$

Summing the above inequality from $i = 1$ to n, we deduce that

$$\|U_n\|_{L^1(\Omega)} + \tau\sum_{i=1}^{n}\|\alpha(U_i)\|_{L^1(\Omega)} \leq \|f\|_{L^1(Q_T)} + \|u_0\|_{L^1(\Omega)}.$$

Hence, the stability results (a) and (b) are then proved.
For (c). Let $i \in \{1, 2, ..., N\}$. Taking $\varphi = T_h\big(U_i - sign(U_i - U_{i-1})\big)$ as a test function in entropy formulation of the discretized problem (4.1), and letting $h \to \infty$, we get, for $k \geqslant 1$

$$\tau\lim_{h\to\infty}\mathcal{I}(k, h) + \|U_i - U_{i-1}\|_{L^1(\Omega)} \leq \tau\left[\|f_i\|_{L^1(\Omega)} + \|\alpha(U_i)\|_{L^1(\Omega)}\right], \tag{4.7}$$

where

$$\begin{aligned}\mathcal{I}(k, h) &:= \int_\Omega \omega(x)\Phi(\nabla U_i - \Theta(U_i))\nabla T_k(U_i - T_h(U_i - \text{sign}(U_i - U_{i-1})))dx \\ &= \int_{\Omega(k,h)\cap\overline{\Omega}(h)} \omega(x)\Phi(\nabla U_i - \Theta(U_i))\nabla U_i dx\end{aligned}$$

and

$$\begin{array}{rcl}\Omega(k,h) & := & \{|U_i - T_h(U_i - \mathrm{sign}(U_i - U_{i-1}))| \leq k\} \\ \overline{\Omega}(h) & := & \{|U_i - \mathrm{sign}(U_i - U_{i-1})| > h\}.\end{array}$$

As

$$\Omega(k,h) \cap \overline{\Omega}(h) \subset \{k-1 \leq |U_i| \leq k+h\},$$

we conclude by using [16, Lemma 4.4] that

$$\lim_{h\to\infty} \mathcal{I}(k,h) = 0$$

This follows by (4.7) that

$$\|U_i - U_{i-1}\|_{L^1(\Omega)} \leq \tau\left[\|f_i\|_{L^1(\Omega)} + \|\alpha(U_i)\|_{L^1(\Omega)}\right].$$

Summing up the above inequality from $i=1$ to n and using the stability result (b), we obtain the stability result (c).

For (d). Let $i \in \{1,2,...,N\}$. Taking $\varphi = 0$ as a test function in (4.2), and using hypothesis (H_3), we get

$$\tau \int_\Omega \omega(x)\Phi(\nabla U_i - \Theta(U_i))\nabla T_k(U_i)dx \leq \tau k\|f_i\|_{L^1(\Omega)} + k\|U_i - U_{i-1}\|_{L^1(\Omega)}.$$

Using Lemmas 2.6 and 2.7 and hypothesis (H_4) , we deduce that

$$\tau\|\nabla T_k(U_i)\|^p_{L^p(\Omega,\omega)} \leq \tau k\|f_i\|_{L^1(\Omega)} + k\|U_i - U_{i-1}\|_{L^1(\Omega)}. \tag{4.8}$$

Summing (4.8) from $i=1$ to n and using the stability result (c), we obtain

$$\tau\sum_{i=1}^{n}\|T_k(U_i)\|^p_{W_0^{1,p}(\Omega,\omega)} \leq C(u_0,f,T,k).$$

Hence the stability result (d) is established. □

Step 3. Entropy solution of the continuous problem.

Let us introduce the following piecewise linear extension (called Rothe function)

$$\begin{cases} u_N(0) := u_0, \\ u_N(t) := U_{n-1} + (U_n - U_{n-1})\frac{(t-t_{n-1})}{\tau}, \ \forall\, t \in\,]t_{n-1},\ t_n], \quad n = 1,...,N \quad in\ \Omega. \end{cases} \tag{4.9}$$

And the following piecewise constant function

$$\begin{cases} \overline{u}_N(0) := u_0, \\ \overline{u}_N(t) := U_n \ \forall\, t \in\,]t_{n-1},\ t_n], \quad n = 1,...,N \quad in\ \Omega. \end{cases} \tag{4.10}$$

We have by Theorem 4.3 that for any $N \in \mathbb{N}$, the entropy solution $(U_n)_{1\leq n\leq N}$ of problems (4.1) is unique, thus, the two sequences $(u_N)_{N\in\mathbb{N}}$ and $(\overline{u}_N)_{N\in\mathbb{N}}$ are uniquely defined.

Lemma 4.5. *Let hypotheses (H_1), (H_2), (H_3), (H_4) and (H_5) be satisfied, then for all $N \in \mathbb{N}$, we have*

(1) $\|\overline{u}_N - u_N\|_{L^1(Q_T)} \leq \frac{1}{N}C(T,u_0,f)$,

(2) $\|\frac{\partial u_N}{\partial t}\|_{L^1(Q_T)} \leq C(T,u_0,f)$,

(3) $\|u_N\|_{L^1(Q_T)} \leq C(T,u_0,f)$,

(4) $\|\overline{u}_N\|_{L^1(Q_T)} \leq C(T,u_0,f)$,

(5) $\|\alpha(\overline{u}_N)\|_{L^1(Q_T)} \leq C(T,u_0,f)$,

(6) $\|T_k(\overline{u}_N)\|_{L^p(0,T,W_0^{1,p}(\Omega,\omega))} \leq C(T,u_0,f,k)$,

where $C(T,u_0,f)$ and $C(T,u_0,f,k)$ are positive constants independents of N.

Proof: For (1). For $N \in \mathbb{N}$, we have

$$\begin{aligned}
\|\overline{u}_N - u_N\|_{L^1(Q_T)} &= \int_0^T \int_\Omega |\overline{u}_N - u_N| dxdt \\
&= \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \|U_n - U_{n-1}\|_{L^1(\Omega)} \frac{(t_n - t)}{\tau} dt \\
&= \frac{\tau}{2} \sum_{n=1}^{N} \|U_n - U_{n-1}\|_{L^1(\Omega)} \\
&\leq \frac{T}{2N} \sum_{n=1}^{N} \|U_n - U_{n-1}\|_{L^1(\Omega)}.
\end{aligned}$$

By using Theorem 4.4, we conclude the result (1).
For (2). We have

$$\frac{\partial u_N(t)}{\partial t} = \frac{(U_n - U_{n-1})}{\tau}.$$

for $n = 1, ..., N$ and $t \in]t_{n-1},\ t_n]$. This implies that

$$\begin{aligned}
\|\frac{\partial u_N}{\partial t}\|_{L^1(Q_T)} &= \int_0^T \int_\Omega |\frac{\partial u_N}{\partial t}| dxdt \\
&= \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \frac{1}{\tau} \|U_n - U_{n-1}\|_{L^1(\Omega)} dt \\
&= \sum_{n=1}^{N} \|U_n - U_{n-1}\|_{L^1(\Omega)}.
\end{aligned}$$

Using Theorem 4.4, we conclude the result (2). We follow the same techniques used above to show the estimates (3), (4), (5) and (6). □

Lemma 4.6. *Let hypotheses* (H_1), $(H_2), (H_3), (H_4)$ *and* (H_5) *be satisfied. The sequence* $(\overline{u}_N)_{N\in\mathbb{N}}$ *converges in measure and a.e. in* Q_T.

Proof: Let ε, r, k be positive real numbers. The $N, M \in \mathbb{N}$, we have the following inclusion

$$\{|\overline{u}_N - \overline{u}_M| > r\} \subset \{\{|\overline{u}_N| > k\} \cup \{|\overline{u}_M| > k\} \cup \{|\overline{u}_N| \leq k, |\overline{u}_M| \leq k, |\overline{u}_N - \overline{u}_M| > r\}\}.$$

By Markov inequality and Lemma 4.5, we deduce

$$\begin{aligned}
meas\{|\overline{u}_N| > k\} &\leq \frac{1}{k} \|\overline{u}_N\|_{L^1(Q_T)} \\
&\leq \frac{1}{k} C(T, u_0, f),
\end{aligned}$$

or equivalently

$$meas\{|\overline{u}_M| > k\} \leq \frac{1}{k} C(T, u_0, f).$$

This implies for k sufficiently large that

$$meas\left(\{|\overline{u}_N| > k\} \cup \{|\overline{u}_M| > k\}\right) \leq \frac{\varepsilon}{2}. \tag{4.11}$$

On the other hand, by Lemma 4.5, the sequence $(T_k(\overline{u}_N))_{N\in\mathbb{N}}$ is bounded in the space $L^p(Q_T, \omega)$. Then, there exists a subsequence, still denoted by $(T_k(\overline{u}_N))_{N\in\mathbb{N}}$ such that $(T_k(\overline{u}_N))_{N\in\mathbb{N}}$ is a Cauchy sequence in $L^p(Q_T, \omega)$ and in measure. Therefore, there exists an $N_0 \in \mathbb{N}$ such that for all $N, M \geq N_0$, we have

$$meas\left(\{|\overline{u}_N| \leq k, |\overline{u}_M| \leq k, |\overline{u}_N - \overline{u}_M| > r\}\right) < \frac{\varepsilon}{2}. \tag{4.12}$$

Consequently, by (4.11) and (4.12), $(\overline{u}_N)_{N\in\mathbb{N}}$ converges in measure and there exists a measurable function on Q_T, u such that

$$\overline{u}_N \to u \text{ a.e. in } Q_T.$$

This finish the proof of lemma 4.6. □

Lemma 4.7. *There exists a function u in $L^1(Q_T)$ such that $T_k(u) \in L^p(0,T;W_0^{1,p}(\Omega,\omega))$ for all $k>0$ and*

(i) *u_N converges to u in $L^1(Q_T)$,*
(ii) *$\overline{u}_N$ converges to u in $L^1(Q_T)$,*
(iii) *$\alpha(\overline{u}_N)$ converges to $\alpha(u)$ in $L^1(Q_T)$,*
(iv) *$\nabla T_k(\overline{u}_N)$ converges to $\nabla T_k(u)$ weakly in $L^p(Q_T,\omega)$,*
(v) *$T_k(\overline{u}_N)$ converges to $T_k(u)$ weakly in $L^p(0,T;W_0^{1,p}(\Omega,\omega))$.*

Proof: For (iv) and (v). By (6) of Lemma 4.5, we have

$$(\nabla T_k(\overline{u}_N))_{N\in\mathbb{N}} \text{ is bounded in } L^p(Q_T,\omega).$$

Then, there exists a subsequence, still denoted $(\nabla T_k(\overline{u}_N))_{N\in\mathbb{N}}$ such that

$$(\nabla T_k(\overline{u}_N))_{N\in\mathbb{N}} \text{ converges weakly to } v \in\ L^p(Q_T,\omega).$$

However

$$T_k(\overline{u}_N) \text{ converges to } T_k(u) \text{ in } L^p(Q_T,\omega).$$

Hence, it follows that

$$\nabla T_k(\overline{u}_N) \text{ converges to } \nabla T_k(u) \text{ weakly in } L^p(Q_T,\omega),$$

and by (6) of Lemma 4.5, we conclude that

$$T_k(\overline{u}_N) \text{ converges to } T_k(u) \text{ weakly in } L^p(0,T;W_0^{1,p}(\Omega,\omega)).$$

□

In order to show that the limit function u is an entropy solution of the problem (1.1), we need the following result.

Lemma 4.8. *The sequence $(u_N)_{N\in\mathbb{N}}$ converges to u in $C\left(0,\ T;\ L^1(\Omega)\right)$.*

Proof: For $\varphi \in L^\infty(Q_T)\cap L^{p_-}(0,T;W_0^{1,p}(\Omega,\omega))\cap W^{1,1}(0,T;L^1(\Omega))$, the inequality (4.2) implies that

$$\int_0^t \left\langle \frac{\partial u_N}{\partial s}, T_k(\overline{u}_N-\varphi)\right\rangle ds+\int_0^t\int_\Omega \omega(x)\Phi(\nabla\overline{u}_N-\Theta(\overline{u}_N))\nabla T_k(\overline{u}_N-\varphi)dxds+\int_0^t\int_\Omega \alpha(\overline{u}_N)T_k(\overline{u}_N-\varphi)dxds$$
$$\le \int_0^t\int_\Omega f_N T_k(\overline{u}_N-\varphi)dxds, \tag{4.13}$$

where $f_N(t,x)=f_n(x)$ for $t\in]t_{n-1},\ t_n]$, $n=1,...,N$.
We consider the two partitions $(t_n=n\tau_N)_{n=1}^N$ and $(t_m=m\tau_M)_{m=1}^M$ of interval $[0,\ T]$ and the corresponding semi-discrete solutions $(u_N(t),\overline{u}_N(t))$, $(u_M(t),\overline{u}_M(t))$ defined by (4.9) and (4.10).
Let $h>0$, for the semi-discrete solution $(u_N(t),\overline{u}_N(t))$ we take $\varphi=T_h(\overline{u}_M)$ and for the semi-discrete solution $(u_M(t),\overline{u}_M(t))$ we take $\varphi=T_h(\overline{u}_N)$. Summing the two inequalities and letting h go to infinity, we have for $k=1$ that

$$\int_0^t \left\langle \frac{\partial(u_N-u_M)}{\partial s},\ T_1(\overline{u}_N-\overline{u}_M)\right\rangle ds+\lim_{h\to\infty} I_{N,M}(h) \le \|f_N-f_M\|_{L^1(Q_T)}+\|\alpha(\overline{u}_N)-\alpha(\overline{u}_M)\|_{L^1(Q_T)}, \tag{4.14}$$

where

$$I_{N,M}(h)=\int_0^t\int_\Omega \omega(x)\Big(\Phi(\nabla\overline{u}_N-\Theta(\overline{u}_N))\nabla T_1(\overline{u}_N-T_h(\overline{u}_M))+\Phi(\nabla\overline{u}_M-\Theta(\overline{u}_M))\nabla T_1(\overline{u}_M-T_h(\overline{u}_N))\Big)dxds.$$

The above inequality (4.14) becomes

$$\int_\Omega J_1\left(u_N(t)-u_M(t)\right)dx+\lim_{h\to\infty} I_{N,M}(h)\leq$$

$$\|f_N-f_M\|_{L^1(Q_T)}+\|\alpha(\overline{u}_N)-\alpha(\overline{u}_M)\|_{L^1(Q_T)}+\left|\int_0^t\left\langle\frac{\partial(u_N-u_M)}{\partial s},\ T_1(\overline{u}_N-\overline{u}_M)-T_1(u_N-u_M)\right\rangle ds\right|. \tag{4.15}$$

By Hölder's inequality

$$\left|\int_0^t\left\langle\frac{\partial(u_N-u_M)}{\partial s},\ T_1(\overline{u}_N-\overline{u}_M)-T_1(u_N-u_M)\right\rangle ds\right|$$

$$\leq\left\|\frac{\partial(u_N-u_M)}{\partial s}\right\|_{L^1(Q_T)}\|T_1(\overline{u}_N-\overline{u}_M)-T_1(u_N-u_M)\|_{L^\infty(Q_T)}.$$

This implies by Lemma 4.5 that

$$\left|\int_0^t\left\langle\frac{\partial(u_N-u_M)}{\partial s}\ T_1(\overline{u}_N-\overline{u}_M)-T_1(u_N-u_M)\right\rangle ds\right|$$

$$\leq C(T,u_0,f)\|T_1(\overline{u}_N-\overline{u}_M)-T_1(u_N-u_M)\|_{L^\infty(Q_T)}.$$

We have

$$\lim_{N,M\to\infty}\|T_1(\overline{u}_N-\overline{u}_M)-T_1(u_N-u_M)\|_{L^\infty(Q_T)}=0.$$

Then

$$\lim_{N,M\to\infty}\left|\int_0^t\left\langle\frac{\partial(u_N-u_M)}{\partial s},\ T_1(\overline{u}_N-\overline{u}_M)-T_1(u_N-u_M)\right\rangle ds\right|=0. \tag{4.16}$$

We also have

$$\lim_{N,M\to\infty}\Big(\|f_N-f_M\|_{L^1(Q_T)}+\|\alpha(\overline{u}_N)-\alpha(\overline{u}_M)\|_{L^1(Q_T)}\Big)=0.$$

Then, the inequality (4.15) becomes

$$\lim_{N,M\to\infty}\int_\Omega J_1\left(u_N(t)-u_M(t)\right)dx+\lim_{N,M\to\infty}\lim_{h\to\infty} I_{N,M}(h)\leq 0. \tag{4.17}$$

Using the same technique used in the proof of uniqueness part of [16, Theorem 3.2], we prove that

$$\lim_{N,M\to\infty}\lim_{h\to\infty} I_{N,M}(h)\geq 0. \tag{4.18}$$

Thus, by inequality (4.17), we get

$$\lim_{N,M\to\infty}\int_\Omega J_1\left(u_N(t)-u_M(t)\right)dx=0. \tag{4.19}$$

By definition of J_1, we have

$$\int_{\{|u_N-u_M|<1\}}|u_N(t)-u_M(t)|^2dx+\frac{1}{2}\int_{\{|u_N-u_M|\geq 1\}}|u_N(t)-u_M(t)|dx\leq\int_\Omega J_1\left(u_N(t)-u_M(t)\right)dx.$$

This implies that

$$\begin{aligned}\int_\Omega |u_N(t)-u_M(t)|dx &= \int_{\{|u_N-u_M|<1\}} |u_N(t)-u_M(t)|dx + \int_{\{|u_N-u_M|\geq 1\}} |u_N(t)-u_M(t)|dx \\ &\leq C(\Omega)\left(\int_{\{|u_N-u_M|<1\}} |u_N(t)-u_M(t)|^2 dx\right)^{\frac{1}{2}} \\ &\quad + \int_{\{|u_N-u_M|\geq 1\}} |u_N(t)-u_M(t)|dx \\ &\leq C(\Omega)\left(\int_\Omega J_1\big(u_N(t)-u_M(t)\big)dx\right)^{\frac{1}{2}} + 2\int_\Omega J_1\big(u_N(t)-u_M(t)\big)dx.\end{aligned}$$

Therefore, by the result (4.19), we conclude that $(u_N)_{N\in\mathbb{N}}$ is a Cauchy sequence in $C(0,\ T; L^1(\Omega))$ and

$$(u_N)_{N\in\mathbb{N}} \textit{ converges to } u \textit{ in } C(0,\ T;\ L^1(\Omega)).$$

□

It remains to prove that the limit function u is an entropy solution of the problem (1.1). Since $u_N(0) = U_0 = u_0$ for all $N \in \mathbb{N}$, then $u(.,0) = u_0$, and by (4.13) we get

$$\int_0^t \left\langle \frac{\partial u_N}{\partial s}, T_k(\overline{u}_N-\varphi) - T_k(u_N-\varphi)\right\rangle ds + \int_0^t\int_\Omega \omega(x)\Phi(\nabla\overline{u}_N - \Theta(\overline{u}_N))\nabla T_k(\overline{u}_N-\varphi)dxds +$$

$$\int_0^t\int_\Omega \alpha(\overline{u}_N)T_k(\overline{u}_N-\varphi)dxds \leqslant -\int_0^t \left\langle \frac{\partial \varphi}{\partial s}, T_k(u_N-\varphi)\right\rangle ds$$

$$+\int_\Omega J_k(u_N(0)-\varphi(0))dx - \int_\Omega J_k(u_N(t)-\varphi(t))dx + \int_0^t\int_\Omega f_N T_k(\overline{u}_N-\varphi)dxds. \tag{4.20}$$

In the same manner, as used for the proof of equality (4.16), we deduce that

$$\lim_{N\to\infty}\int_0^t \left\langle \frac{\partial u_N}{\partial s}, T_k(\overline{u}_N-\varphi) - T_k(u_N-\varphi)\right\rangle ds = 0. \tag{4.21}$$

following the same technique used in the proof of existence part of [16, Theorem 3.2], we show that

$$\lim_{N\to\infty}\int_0^t\int_\Omega \omega(x)\Phi(\nabla\overline{u}_N - \Theta(\overline{u}_N))\nabla T_k(\overline{u}_N-\varphi)dxds = \int_0^t\int_\Omega \omega(x)\Phi(\nabla u - \Theta(u))\nabla T_k(u-\varphi)dxds \tag{4.22}$$

And by Lemma 4.8, we deduce that $u_N(t) \to u(t)$ in $L^1(\Omega)$ for all $t \in [0,\ T]$, which implies that

$$\int_\Omega J_k(u_N(t)-\varphi(t))dx \to \int_\Omega J_k(u(t)-\varphi(t))dx \tag{4.23}$$

Finally, taking limits as N goes to infinity, and using the above results, the continuity of α and Θ, the facts that $f_N \to f$ in $L^1(Q_T)$, and $T_k(\overline{u}_N-\varphi) \to T_k(u-\varphi)$ in $L^\infty(Q_T)$, we deduce that u is an entropy solution of the nonlinear parabolic problem (1.1).

Uniqueness. Let v another entropy solution of the nonlinear parabolic problem (1.1). Taking $\varphi = T_h(u_N)$ as a test function in (3.1) and letting h goes to infinity, we get

$$\int_\Omega J_k(v(t)-u_N(t))dx + \int_0^t \left\langle \frac{\partial u_N}{\partial s},\ T_k(v-u_N)\right\rangle ds + \lim_{h\to\infty} \mathcal{II}_1^N(k,h)$$

$$+\int_0^t \int_\Omega \alpha(v) T_k(v-u_N) dxds \leq \int_0^t \int_\Omega f T_k(v-u_N) dxds, \tag{4.24}$$

where

$$\mathcal{I}\mathcal{I}_1^N(k,h) = \int_0^t \int_\Omega \omega(x)\Phi(\nabla v - \Theta(v))\nabla T_k(v - T_h(u_N)) dxds.$$

On the other hand, taking $\varphi = T_h(v)$ as a test function in the inequality (4.13) and taking h goes to infinity, we get

$$\int_0^t \left\langle \frac{\partial u_N}{\partial s},\ T_k(\overline{u}_N - v)\right\rangle ds + \lim_{h\to\infty} \mathcal{I}\mathcal{I}_2^N(k,h) + \int_0^t \int_\Omega \alpha(\overline{u}_N) T_k(\overline{u}_N - v) dxds$$

$$\leq \int_0^t \int_\Omega f_N T_k(\overline{u}_N - v) dxds \tag{4.25}$$

where

$$\mathcal{I}\mathcal{I}_2^N(k,h) = \int_0^t \int_\Omega \omega(x)\Phi(\nabla \overline{u}_N - \Theta(\overline{u}_N))\nabla T_k(\overline{u}_N - T_h(v)) dxds.$$

Adding (4.24) and (4.25), we get

$$\int_\Omega J_k(v(t) - u_N(t)) dx + \int_0^t \left\langle \frac{\partial u_N}{\partial s},\ T_k(v - u_N) + T_k(\overline{u}_N - v)\right\rangle ds + \lim_{h\to\infty} \mathcal{I}\mathcal{I}^N(k,h)$$

$$+\int_0^t \int_\Omega \left[\alpha(v) T_k(v-u_N) + \alpha(\overline{u}_N) T_k(\overline{u}_N - v)\right] dxds \leq \int_0^t \int_\Omega \left[f T_k(v-u_N) + f_N T_k(\overline{u}_N - v)\right] dxds,$$

where

$$\mathcal{I}\mathcal{I}^N(k,h) = \mathcal{I}\mathcal{I}_1^N(k,h) + \mathcal{I}\mathcal{I}_2^N(k,h).$$

Taking N goes to infinity, using the above convergence results, and the hypothesis (H_3), we get

$$\int_\Omega J_k(v(t) - u(t)) dx + \lim_{N\to\infty} \lim_{h\to\infty} \mathcal{I}\mathcal{I}^N(k,h) \leq 0. \tag{4.26}$$

Applying the technique used in (4.18), we deduce that

$$\lim_{N\to\infty} \lim_{h\to\infty} \mathcal{I}\mathcal{I}^N(k,h) \geq 0. \tag{4.27}$$

Therefore, the inequality (4.26) implies that

$$\int_\Omega J_k(v(t) - u(t)) dx \leq 0,$$

i.e.

$$\int_\Omega \frac{J_k(v(t) - u(t))}{k} dx \leq 0.$$

However

$$\lim_{k\to 0} \frac{J_k(x)}{k} = |x|.$$

Then, by Dominated Convergence Theorem, we get

$$\|v(t) - u(t)\|_{L^1(\Omega)} \leq 0, \quad \text{for } t \in [0,\ T].$$

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