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On a New Nonlinear Integro-Differential Fredholm-Chandrasekhar Equation *

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Abstract: This paper presents an analytical and numerical study of a new integro-differential Fredholm-Chandrasekhar equation of the second type. We suggest the conditions that ensure the existence and uniqueness of the nonlinear problem's solution. Then, we create a numerical technique based on the Nyström's method. The numerical application illustrates the efficiency of the proposed process.

Key Words: Nonlinear Fredholm integral equation, integro-differential equation, Chandrasekhar integral equation, fixed point, Nyström method.

Contents

1. Introduction

In many fields of applied mathematics, the equations of various phenomena of chemical, physical or medical natures, are generally modeled with an integro-differential equation. Therefore, many analytical studies and numerical applications have been carried out in these fields (see c.f [10,15,14,8]). In a recent paper [3], the authors treated the following nonlinear integro-differential Fredholm equation of the second type:

$$\varphi(t) - \int_a^b K_2(t, s, \varphi(s), \varphi'(s))ds = f(t), \quad t \in [a, b]. \tag{1.1}$$

In addition, the nonlinear integral equation given by

$$\varphi(t) - K_1(t, \varphi(t)) \int_a^b K_2(t, s, \varphi(s))ds = f(t), \quad t \in [a, b], \tag{1.2}$$

is recently analyzed analytically and numerically in [12,13]. This equation is known in the literature as, the Chandrasekhar quadratic integral equation of the second type (see [5,4]). On the other hand, in [7], the authors studied an integro-differential Volterra equation, and in [1], they investigated a general format, an integro-differential Volterra-Fredholm equation.

In this paper, we study a general format of the equations (1.1) and (1.2) which takes the form

$$\begin{cases} \varphi(t) = K_1(t, \varphi(t)) \displaystyle\int_a^b K_2(s, \varphi(s), \varphi'(s))ds + f(t), & t \in [a, b], \\ \left|\varphi'(t)\right| \leq d, \quad \forall t \in [a, b], \quad \text{where}, & d \in \mathbb{R}_+^*. \end{cases} \tag{1.3}$$

We identify (1.3) as the second type's nonlinear integro-differential Fredholm-Chandrasekhar equation. The data f, K_1, K_2 and d are given. The parameter d, will be considered as a large number which ensures

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the boundedness of the derivative. Our study relates to the originality of the type of this equation which concretizes the fast development and the interest of the researchers in this field. Finally, for a survey on the Fredholm integral equations of the second type, see [6].

In this work, we analyze (1.3) analytically and numerically. First, we demonstrate the existence and uniqueness of the solution by employing ideas based on Picard's successive method [12,13,3]. After that, in the numerical sense we use the Nyström's method [8,3] to approximate the exact solution where, we present a numerical application to show the accuracy of our approach.

2. Analytical study

In this section, we investigate the existence and uniqueness of the solution of (1.3). We consider the Banach space $C^1([a,b],\mathbb{R})$, consisting of all continuously differentiable real-valued functions defined on $[a,b]$, which is equipped with the following norm

$$\|u\|_{C^1([a,b],\mathbb{R})} = \max_{t\in[a,b]} |u(t)| + \max_{t\in[a,b]} |u'(t)|.$$

We define the set $\mathcal{F}$ as follow

$$\mathcal{F} = \Big\{ u \in C^1([a,b],\mathbb{R}) : \max_{t\in[a,b]} |u'(t)| < d \Big\}.$$

The examination of φ, the solution of (1.3), largely depends on the properties of the data f, K_1 and K_2. Let be K_1 and K_2:

$$K_1 : [a,b]\times\mathbb{R} \to \mathbb{R}, (t,x) \mapsto K_1(t,x),$$

$$K_2 : [a,b]\times\mathbb{R}^2 \to \mathbb{R}, (t,x,y) \mapsto K_2(t,x,y).$$

In the following, we are going to assume that $a = 0$ and $b = 1$. The next hypothesis fixes the framework such that the functions K_1, K_2 and f are regular:

$$(H1) \left\| K_2, \frac{\partial K_1}{\partial t}, \frac{\partial K_1}{\partial x} \in C^0([0,1]\times\mathbb{R},\mathbb{R}), \quad K_2 \in C^0([0,1]\times\mathbb{R}^2,\mathbb{R}) \quad f \in C^1([0,1],\mathbb{R}). \right.$$

If we differentiate the equation in (1.3), we obtain

$$\varphi'(t) = H(t,\varphi(t),\varphi'(t)) \int_0^1 K_2(s,\varphi(s),\varphi'(s))ds + f'(t), \quad \forall t \in [0,1],$$

where,

$$H(t,x,y) = \frac{\partial K_1}{\partial t}(t,x) + y\frac{\partial K_1}{\partial x}(t,x).$$

We assume that K_1, K_2 and H satisfy the following hypotheses:

$$(H2) \left\| \begin{array}{l} \text{i) } \exists M_1, M_2 \in \mathbb{R}_+^*, \forall t\in[0,1], \forall x,y\in\mathbb{R} : |K_1(t,x)| \le M_1,\ |K_2(t,x,y)| \le M_2, \\ \text{ii) } \exists M_3 \in \mathbb{R}_+^*, \forall t\in[0,1], \forall x\in\mathbb{R}, \forall y \in [-d,d] : |H(t,x,y)| \le M_3. \end{array} \right.$$

Now, we define the following operator G, as

$$G(\tilde\varphi)(t) = K_1(t,\tilde\varphi(t)) \int_0^1 K_2(s,\tilde\varphi(s),\tilde\varphi'(s))ds + f(t), \quad t\in[0,1],$$

for $\tilde\varphi \in \mathcal{F}$ and $f \in C^1([0,1],\mathbb{R})$. Moreover, we can find by differentiating that

$$[G(\tilde\varphi)]'(t) = H(t,\tilde\varphi(t),\tilde\varphi'(t)) \int_0^1 K_2(s,\tilde\varphi(s),\tilde\varphi'(s))ds + f'(t), \quad t\in[0,1].$$

The next lemma shows the solution space containing φ, the solution of (1.3).

Lemma 2.1. *Under the hypotheses* $(H1)$ *and* $(H2)$, *the operator* G *is continuous from* $\mathcal{F}$ *into* $C^1([0,1],\mathbb{R})$.

Proof. Let $\tilde{\varphi} \in \mathcal{F}$. It is clear that $G(\tilde{\varphi})(\cdot)$ and $[G(\tilde{\varphi})]'(\cdot)$ are continuous on $[0,1]$. In addition, we can find according to $(H2)$ that,

$$|G(\tilde{\varphi})(t)| + |[G(\tilde{\varphi})]'(t)| \leq (M_1 + M_3)M_2 + \|f\|_{C^1([0,1],\mathbb{R})}, \quad t \in [0,1].$$

Now, let be $(\tilde{\varphi}_n)_{n\in\mathbb{N}}$ a sequence in $\mathcal{F}$ which converges to $\tilde{\varphi} \in C^1([0,1],\mathbb{R})$. Using the hypothesis $(H1)$, we obtain that, for all $t \in [0,1]$

$$\lim_{n\to\infty} G(\tilde{\varphi}_n)(t) = G(\tilde{\varphi})(t), \quad \lim_{n\to\infty} [G(\tilde{\varphi}_n)]'(t) = G(\tilde{\varphi})(t).$$

□

Our aim now is to construct a sequence which converges to φ, the solution of (1.3). We define the sequence $(\varphi_n)_{n\in\mathbb{N}}$ by

$$\begin{cases} \varphi_0(t) = g(t), \quad t \in [0,1], \\ \varphi_{n+1}(t) = K_1(t, \varphi_n(t)) \displaystyle\int_0^1 K_2(s, \varphi_n(s), \varphi_n'(s))ds + f(t), \quad t \in [0,1], \end{cases}$$

where $g \in C^1([0,1],\mathbb{R})$. Thus, we can easily find that

$$\begin{cases} \varphi_0'(t) = g'(t), \quad t \in [0,1], \\ \varphi_{n+1}'(t) = H(t, \varphi_n(t), \varphi_n'(t)) \displaystyle\int_0^1 K_2(s, \varphi_n(s), \varphi_n'(s))ds + f'(t), \quad t \in [0,1]. \end{cases}$$

We add the following hypotheses to demonstrate the existence and uniqueness of the solution of (1.3).

$$(H3) \left\| \begin{array}{l} \text{i) } \exists A_1 \in \mathbb{R}_+^*, \quad \forall t \in [0,1] \quad \forall x, x' \in \mathbb{R} : \\ |K_1(t,x) - K_1(t,x')| \leq A_1|x - x'|, \\ \text{ii) } \exists A_2, B_2 \in \mathbb{R}_+^*, \quad \forall s \in [0,1] \quad \forall x, y, x', y' \in \mathbb{R} : \\ |K_2(s,x,x') - K_2(s,y,y')| \leq A_2|x-y| + B_2|x'-y'|, \\ \text{iii) } \exists A_3, B_3 \in \mathbb{R}_+^*, \quad \forall t \in [0,1] \quad \forall x, x' \in \mathbb{R}, \forall y, y' \in [-d,d] : \\ |H(t,x,y) - H(t,x',y')| \leq A_3|x-x'| + B_3|y-y'|. \end{array} \right.$$

These hypotheses are also similar to assumptions stated in [3]. The next theorem proves that (1.3) has a unique solution φ in $C^1([0,1],\mathbb{R})$.

Theorem 2.2. *Assume that $(H1)$, $(H2)$ and $(H3)$ are satisfied, if $g \in \mathcal{F}$ and the condition*

$$\begin{cases} \Big(M_1 \max\{A_2, B_2\} + A_1 M_2\Big) \leq \gamma < \dfrac{1}{2}, \\ \Big(M_3 \max\{A_2, B_2\} + M_2 \max\{A_3, B_3\}\Big) \leq \gamma < \dfrac{1}{2}. \end{cases} \tag{2.1}$$

Then the equation (1.3) has a unique solution $\varphi \in C^1([0,1],\mathbb{R})$, such that

$$\|\varphi_{n+1} - \varphi_n\|_{C^1([0,1],\mathbb{R})} \leq C(2\gamma)^{n+1},$$

$$\|\varphi_n - \varphi\|_{C^1([0,1],\mathbb{R})} \to 0, \quad n \to +\infty,$$

where, C is a positive constant.

Proof. It is clear that, for all $t \in [0,1]$ and for $n \geq 1$

$$\begin{cases} \varphi_{n+1}(t) = G(\varphi_n)(t), \\ \varphi_{n+1}'(t) = [G(\varphi_n)]'(t). \end{cases}$$

Now, let $t \in [0,1]$ and $n \geq 1$:

$$|\varphi_{n+1}(t) - \varphi_n(t)| =$$

$$\Big|K_1\big(t,\varphi_n(t)\big)\int_0^1 K_2\big(s,\varphi_n(s),\varphi_n'(s)\big)ds - K_1\big(t,\varphi_{n-1}(t)\big)\int_0^1 K_2\big(s,\varphi_{n-1}(s),\varphi_{n-1}'(s)\big)ds\Big| =$$

$$\Big|K_1\big(t,\varphi_n(t)\big)\left(\int_0^1 K_2\big(s,\varphi_n(s),\varphi_n'(s)\big)ds - \int_0^1 K_2\big(s,\varphi_{n-1}(s),\varphi_{n-1}'(s)\big)ds\right) +$$

$$\Big(K_1\big(t,\varphi_n(t)\big) - K_1\big(t,\varphi_{n-1}(t)\big)\Big)\int_0^1 K_2\big(s,\varphi_{n-1}(s),\varphi_{n-1}'(s)\big)ds\Big| \leq$$

$$M_1\int_0^1 \big(A_2|\varphi_n(s)-\varphi_{n-1}(s)| + B_2|\varphi_n'(s)-\varphi_{n-1}'(s)|\big)ds + A_1|\varphi_n(t)-\varphi_{n-1}(t)|\int_0^1 M_2 ds \leq$$

$$\Big(M_1\max\{A_2,B_2\} + A_1M_2\Big)\|\varphi_n-\varphi_{n-1}\|_{C^1([0,1],\mathbb{R})} \leq \gamma\|\varphi_n-\varphi_{n-1}\|_{C^1([0,1],\mathbb{R})},$$

and also, we have

$$|\varphi_{n+1}'(t)-\varphi_n'(t)| =$$

$$\Big|H\big(t,\varphi_n(t),\varphi_n'(t)\big)\left(\int_0^1 K_2\big(s,\varphi_n(s),\varphi_n'(s)\big)ds - \int_0^1 K_2\big(s,\varphi_{n-1}(s),\varphi_{n-1}'(s)\big)ds\right) +$$

$$\Big(H\big(t,\varphi_n(t),\varphi_n'(t)\big) - H\big(t,\varphi_{n-1}(t),\varphi_{n-1}'(t)\big)\Big)\int_0^1 K_2\big(s,\varphi_{n-1}(s),\varphi_{n-1}'(s)\big)ds\Big| \leq$$

$$M_3\int_0^1 \big(A_2|\varphi_n(s)-\varphi_{n-1}(s)| + B_2|\varphi_n'(s)-\varphi_{n-1}'(s)|\big)ds +$$

$$\big(A_3|\varphi_n(t)-\varphi_{n-1}(t)| + B_3|\varphi_n'(t)-\varphi_{n-1}'(t)|\big)\int_0^1 M_2 ds \leq$$

$$\Big(M_3\max\{A_2,B_2\} + M_2\max\{A_3,B_3\}\Big)\|\varphi_n-\varphi_{n-1}\|_{C^1([0,1],\mathbb{R})} \leq \gamma\|\varphi_n-\varphi_{n-1}\|_{C^1([0,1],\mathbb{R})}.$$

Hence, we find that

$$\|\varphi_{n+1}-\varphi_n\|_{C^1([0,1],\mathbb{R})} \leq 2\gamma\|\varphi_n-\varphi_{n-1}\|_{C^1([0,1],\mathbb{R})}.$$

Then,

$$\begin{aligned}\|\varphi_{n+1}-\varphi_n\|_{C^1([0,1],\mathbb{R})} &\leq (2\gamma)^n\left[\|g\|_{C^1([0,1],\mathbb{R})} + \|G(g)\|_{C^1([0,1],\mathbb{R})}\right]\\ &\leq (2\gamma)^n\left[(M_1+M_3)M_2 + 2\|g\|_{C^1([0,1],\mathbb{R})}\right].\end{aligned}$$

Now, for all $n \geq 0$, if $\varphi_n \in \mathcal{F}$ then we prove by induction that $\varphi_{n+1} \in \mathcal{F}$. Further, we have for all $t \in [0,1]$

$$\varphi(t) = \lim_{n\to+\infty}\varphi_{n+1}(t) = \lim_{n\to+\infty} K_1(t,\varphi_n(t))\int_0^1 K_2(s,\varphi_n(s),\varphi_n'(s))ds + f(t) =$$

$$K_1(t,\varphi(t))\int_0^1 K_2(s,\varphi(s),\varphi'(s))ds + f(t).$$

Finally for uniqueness, we use the difference technique. So, the desired result is obtained. □

We remark that the condition (2.1) is sufficient but not necessary to establish the existence and uniqueness of the solution of (1.3). However, we can use the hypotheses $(H1)$ and $(H2)$ to establish only the existence of the solution of (1.3), by employing Schauder's fixed point theorem as used in [3] for their integro-differential equation (1.1). Finally, we can notice that the parameter d has no influence in (1.3), so we can always choose it as big as we want.

3. Numerical approximation

In this section, we define a finite-dimensional system of nonlinear equations by using the Nyström's method (see [2,11]) where, this system will be an approximation approach to the solution of (1.3). Then, we use a successive method to solve this algebraic system.

Let $n \in \mathbb{N}^*$, we define the subdivision L_n of the interval $[0,1]$ as the set

$$L_n = \left\{ t_i = ih : h = \frac{1}{n}, i = 0, 1, \cdots, n \right\}.$$

Now, according to (1.3) and for all $i = 0, 1, \cdots n$, we find that

$$\varphi(t_i) = K_1(t_i, \varphi(t_i)) \int_0^1 K_2(s, \varphi(s), \varphi'(s))ds + f(t_i),$$

and similarly that

$$\varphi'(t_i) = H(t_i, \varphi(t_i), \varphi'(t_i)) \int_0^1 K_2(s, \varphi(s), \varphi'(s))ds + f'(t_i).$$

Then, employing the quadrature formula on the previous equations, we obtain for $i = 0, \cdots, n$

$$\varphi_i = f_i + K_1(t_i, \varphi_i) \sum_{j=0}^{n} w_j K_2(t_j, \varphi_j, \varphi_j') + K_1(t_i, \varphi_i) R(h), \tag{3.1}$$

and

$$\varphi_i' = f_i' + H(t_i, \varphi_i, \varphi_i') \sum_{j=0}^{n} w_j K_2(t_j, \varphi_j, \varphi_j') + H(t_i, \varphi_i, \varphi_i') R(h), \tag{3.2}$$

where, $\varphi(t_i) = \varphi_i$, $\varphi'(t_i) = \varphi_i'$, $f(t_i) = f_i$ and $f'(t_i) = f_i'$ and $R(h)$ is the residual of the quadrature rule used. $R(h)$ also defines the local consistency error (see c.f. [2]). Neglecting then the local consistency error $R(h)$, we find the nonlinear algebraic system of dimension $2n + 1$,

$$\begin{cases} \alpha_i = f_i + K_1(t_i, \alpha_i) \displaystyle\sum_{j=0}^{n} w_j K_2(t_j, \alpha_j, \beta_j), \quad i = 0, \cdots, n \\ \beta_i = f_i' + H(t_i, \alpha_i, \beta_i) \displaystyle\sum_{j=1}^{n} w_j K_2(t_j, \alpha_j, \beta_j), \quad i = 0, \cdots, n. \end{cases} \tag{3.3}$$

The next theorem proves that the system (3.3) has a unique solution $(\alpha, \beta) \in \mathbb{R}^{2n+2}$.

Theorem 3.1. *If the hypotheses $(H1), (H2), (H3)$ and the condition (2.1) are satisfied, then the system (3.3) has a unique vector solution $(\alpha, \beta) \in \mathbb{R}^{2n+2}$.*

Proof. For all $i = 0, \cdots, n$, we set the functions $\Phi : \mathbb{R}^{2n+2} \to \mathbb{R}^{n+1}$ and $\Psi : \mathbb{R}^{2n+2} \to \mathbb{R}^{n+1}$ such that

$$\Phi_i(\alpha, \beta) = f_i + K_1(t_i, \alpha_i) \sum_{j=0}^{n} w_j K_2(t_j, \alpha_j, \beta_j), \tag{3.4}$$

$$\Psi_i(\alpha, \beta) = f_i' + H(t_i, \alpha_i, \beta_i) \sum_{j=0}^{n} w_j K_2(t_j, \alpha_j, \beta_j), \tag{3.5}$$

where, $\alpha = (\alpha_0, \cdots, \alpha_n)$ and $\beta = (\beta_0, \cdots, \beta_n)$. Let α, α^1, β and β^1 be in $\mathbb{R}^{n+1}$, we can therefore conclude that by employing the hypotheses $(H1), (H2)$ and $(H3)$,

$$\left| \Phi(\alpha, \beta) - \Phi(\alpha^1, \beta^1) \right|_{\mathbb{R}^{n+1}} \leq \gamma(|\alpha - \alpha^1|_{\mathbb{R}^{n+1}} + |\beta - \beta^1|_{\mathbb{R}^{n+1}}),$$

and in addition,

$$\left|\Psi(\alpha,\beta)-\Psi(\alpha^1,\beta^1)\right|_{\mathbb{R}^{n+1}} \leq \gamma(|\alpha-\alpha^1|_{\mathbb{R}^{n+1}}+|\beta-\beta^1|_{\mathbb{R}^{n+1}}).$$

Now, using the condition (2.1), we prove that the function Φ and Ψ are contractions, so according to the Banach fixed point theorem, the system (3.3) has a unique solution $(\alpha,\beta)\in\mathbb{R}^{2n+2}$. □

We define now the errors such that

$$\varepsilon_i = |\alpha_i-\varphi(t_i)|+|\beta_i-\varphi'(t_i)|,\ \text{for } i=0,\cdots,n$$

where, φ is the solution of (1.3) and $(\alpha,\beta)\in\mathbb{R}^{2n+2}$ is the vector solution of system (3.3). We also define the local consistency error as,

$$R(h)=\left|\int_0^1 K_2(s,\varphi(s),\varphi'(s))ds-\sum_{j=0}^{n} w_j K_2(t_j,\varphi(t_j),\varphi'(t_j))\right|.$$

We recall that the use of quadrature rule is said to be consistent if the residual satisfies,

$$\lim_{n\to+\infty} R(h)=0.$$

Theorem 3.2. *If the quadrature rule is consistent, then the approximation method given by* (3.3) *is convergent, i.e.*

$$\lim_{n\to+\infty}\max_{0\leq i\leq n}\varepsilon_i=0$$

Proof. Let be $\varphi=(\varphi(t_0),\cdots,\varphi(t_n))$, $\varphi'=(\varphi'(t_0),\cdots,\varphi'(t_n))$. Employing the same strategies (adding and subtracting) as in the proof of the previous theorems, we show that

$$\begin{cases} |\alpha-\varphi|_{\mathbb{R}^{n+1}} \leq \gamma\left(|\alpha-\varphi|_{\mathbb{R}^{n+1}}+|\beta-\varphi'|_{\mathbb{R}^{n+1}}\right)+M_1R(h) \\ |\beta-\varphi'|_{\mathbb{R}^{n+1}} \leq \gamma\left(|\alpha-\varphi|_{\mathbb{R}^{n+1}}+|\beta-\varphi'|_{\mathbb{R}^{n+1}}\right)+M_3R(h). \end{cases} \tag{3.6}$$

So, we find the estimation

$$|\alpha-\varphi|_{\mathbb{R}^{n+1}}+|\beta-\varphi'|_{\mathbb{R}^{n+1}} \leq \left(\frac{M_1+M_3}{1-2\gamma}\right)R(h),$$

thus

$$\max_{0\leq i\leq n}\varepsilon_i \leq \left(\frac{M_1+M_3}{1-2\delta}\right)R(h).$$

So, if the quadrature rule is consistent, then we find the desired result. □

4. Numerical results

In this section, we use the trapezoidal method, since it ensures that the quadrature rule utilized to find the system (3.3), is consistent (see [2]) i.e.

$$\lim_{n\to+\infty} R(h)=0.$$

The terms α_i and β_i will not be exactly calculated for $i=1,\cdots,n$. They will be approximated using Banach's iteration method with stopping criterion of the type:

$$\|\alpha_{new}-\alpha_{old}\|+\|\beta_{new}-\beta_{old}\| \leq \frac{1}{10^N},$$

where N is a given positive number.

Example 1: we consider the nonlinear problem,

$$\begin{cases} u(t) = \sin\left(t^2 + u(t)\right) \displaystyle\int_0^1 \frac{(s+1)}{1+(u(s)+u'(s))^2} ds + f(t) \ t \in [0,1], \\ \left|u'(t)\right| \leq 10^5, \quad t \in [0,1]. \end{cases}$$

where, $f(t) = t - (\sin(t^2+t)) \ln\left(\frac{\sqrt{10}}{2}\right)$ and the exact solution is given by $u(t) = t$. It is clear that the hypotheses $(H1)$-$(H3)$ and the condition (2.1) are well verified. The next table (1) and figure (1) show the numerical results. These results confirm the theoretical study and show the numerical efficiency of the system (3.3) built, where we notice that the efficiency is established from $n = 10$.

Table 1: Numerical Results of Ex.1

n	$E_n = \max\limits_{0\leq i\leq n} \varepsilon_i$
10	2.00E-4
50	8.06E-6
100	2.01E-6
250	3.22E-7
500	8.03E-8

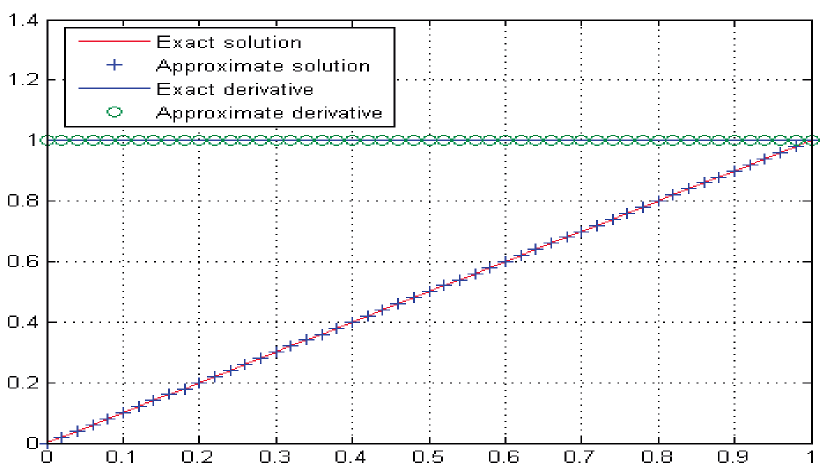

Figure 1: Results of Ex.1 according to $n = 50$.

Example 2: we consider the nonlinear problem,

$$\begin{cases} u(t) = \dfrac{1}{1+t+u(t)^2} \displaystyle\int_0^1 \cos\left(e^s + \frac{\pi}{2}s + u(s) - u'(s)\right) ds + f(t), \quad t \in [0,1], \\ \left|u'(t)\right| \leq e^7, \quad \forall t \in [0,1], \end{cases}$$

where,

$$f(t) = te^t - \frac{2}{\pi + \pi t(1+t)e^{2t}},$$

and the exact solution is $u(t) = te^t$. The next table (2) and the figure (2) show the numerical results. In this example, we notice that the error is better starting from $n = 50$.

Table 2: Numerical Results of Ex.2

n	$E_n = \max_{0\le i\le n} \varepsilon_i$
10	2.64E-3
50	1.05E-4
100	2.63E-5
250	4.21E-6
500	1.05E-6

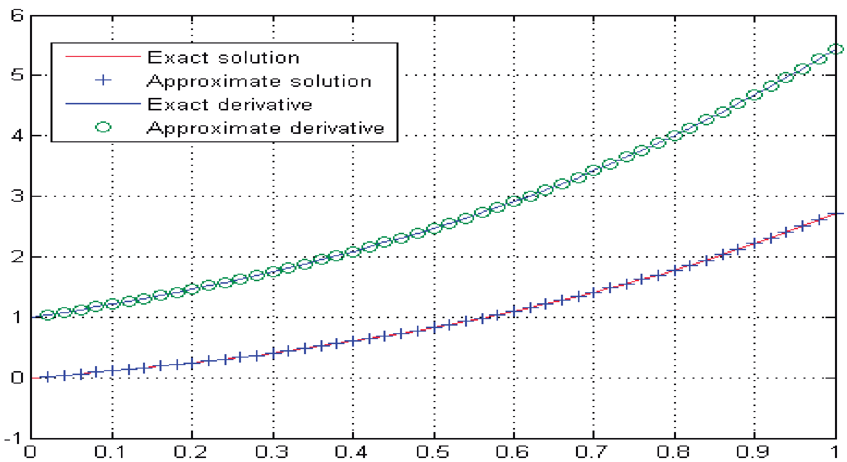

Figure 2: Results of Ex.2 according to $n = 50$.

Conclusion

We build hypotheses and conditions that guarantee the solution's existence and uniqueness for a new generalized integro-differential nonlinear Fredholm equation. The developed numerical example shows the effectiveness of the Nyström method used to approximate the solution of this equation. As perspective, there still are some interesting points but we will first study the case where the kernel is weakly singular using a different method as used in [9].

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