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Explicit Formulas for the Matrix Exponential

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ABSTRACT: In this work, new closed-form formulas for the matrix exponential are provided using certain polynomials which are constructed with the help of a generalization of Hermite's interpolation formula. Our method is direct and elementary, it gives tractable and manageable formulas not current in the extensive literature on this essential subject. Moreover, others are recuperated and generalized. Several particular cases and examples are formulated to illustrate the method presented in this paper.

Key Words: Matrix exponential, Vandermonde matrices, polynomials, eigenvalues.

Contents

1. Introduction

In numerical linear algebra, the computation of the exponential matrix has been one of the most challenging problems. Therefore, many different methods have been proposed for the calculation of such matrix, a large number of them are of pedagogic interest only or of dubious numerical stability.

In their paper [11], the authors have concluded that the exponential matrix could be calculated in many different ways and they have noticed that some of them are preferable to others but none are completely satisfactory. In [14] J. S. Respondek agreed with this opinion and gave a list of reasons and arguments that the set of the preferable ways and methods should be extended.

The present article proposes a new method for computing the matrix exponential using certain polynomials which are constructed with the help of a generalization of Hermite's interpolation formula. We hold the same opinion as the authors of [11,14] and believe that this method can be used efficiently for numerical computation of the exponential of a matrix once the eigenvalues of the given matrix are known.

As we noted above many works deal with the computation of the exponential matrix. Let us mention some of these works. For instance, In [11], the authors presented a careful investigation on various such efforts they attempted to describe all the methods that seem to be practical. J. L. Howland [7] presented a procedure that generalizes a method described in [11]. T. M. Apostol [2] presented an elegant and manageable approach that gave explicit formulas for some special cases. His method does not produce the general case when the characteristic polynomial of the matrix has multiple roots.

Much of the difficulty of the computation of the matrix exponential is bypassed by the algorithm of Putzer [13]. In [8], I. E. Leonard presented an alternative method intending to minimize the mathematical prerequisites. His approach requires exactly the solution of homogeneous linear differential equations with constant coefficients. To avoid solving the initial value problems for these differential equations, E. Liz [9] provided a method which requires only the knowledge of a basis for each solution space of these equations.

For greater detail on some of the methods and algorithms used for computing the exponential matrix, we refer the interested reader to [1,2,3,4,5,6,7,8,9,10,11,14,16]) and the references there.

It is important to mentien that the computation of the matrix exponential requires to multiply matrices. The fast matrix multiplication make the matrix calculations faster, and the intelligible cover of that is given by the survey article of J. S. Respondek [15].

Among the essential tools used in this paper are the Vandermonde matrices and their inverses and certain polynomials. So it is convenient to fix some notations.

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Let $\alpha_1, \alpha_2, \ldots, \alpha_s$ be distinct elements of $\mathbb{C}$ and $m_1, m_2, \ldots, m_s$ be nonnegative integers. For a non-constant polynomial $P(x) = (x-\alpha_1)^{m_1}(x-\alpha_2)^{m_2}\cdots(x-\alpha_s)^{m_s}$ of degree n, we denote by $L_{jk_j}(x)[P]$ the following polynomial

$$L_{jk_j}(x)[P] = P_j(x)(x-\alpha_j)^{k_j}\sum_{i=0}^{m_j-1-k_j}\frac{1}{i!}g_j^{(i)}(\alpha_j)(x-\alpha_j)^i, \tag{1.1}$$

where $1 \le j \le s,\ 0 \le k_j \le m_j - 1$,

$$P_j(x) = \prod_{i=1, i\neq j}^{s}(x-\alpha_i)^{m_i} = \frac{P(x)}{(x-\alpha_j)^{m_j}}, 1 \le j \le s$$

and

$$g_j(x) = (P_j(x))^{-1}.$$

Here and further $L_{jk_j}^{(l)}(x)[P]$ means the lth derivative of $L_{jk_j}(x)[P]$.

According to [17], we can write every polynomial Q of degree less than or equal to $n-1$ as

$$Q = \sum_{j=1}^{s}(\sum_{k_j=0}^{m_j-1}\frac{1}{k_j!}Q^{(k_j)}(\alpha_j)L_{jk_j}(x)[P]). \tag{1.2}$$

This formula is of great importance, it is used in [12] to invert the confluent Vandermonde matrix.

In [12], the confluent Vandermonde matrix associated with the polynomial P of degree $n = m_1 + m_2 + \cdots + m_s$

$$P(x) = (x-\alpha_1)^{m_1}(x-\alpha_2)^{m_2}\cdots(x-\alpha_s)^{m_s},$$

is defined to be the following block matrix

$$V_G(P) = (V_1\ V_2\ \ldots\ V_s), \tag{1.3}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_s$ are distinct elements of $\mathbb{C}$.
The block matrix V_k is of order $n \times m_k, k = 1, \ldots, s$, and defined to be the matrix

$$V_k = V_G((x-\alpha_k)^{m_k})$$

with entries

$$(V_k)_{ij} = \begin{cases} \binom{i-1}{j-1}\alpha_k^{i-j} & \text{if } i \ge j \\ 0, & \text{otherwise,} \end{cases}$$

where $\binom{q}{p}$ denotes the binomial coefficient.

For completeness, we recall the following Theorem and corollary (needed in the sequel) which provide an explicit closed-form for the inverse of the confluent Vandermonde matrices (for more details, see [12]).

Theorem 1.1. *Let $P(x) = (x-\alpha_1)^{m_1}(x-\alpha_2)^{m_2}\cdots(x-\alpha_s)^{m_s}$ be a polynomial of degree n, where $\alpha_1, \alpha_2, \ldots, \alpha_s$ are distinct elements of $\mathbb{C}$. The explicit inverse of the confluent Vandermonde matrix $V_G(P)$ has the form*

$$V_G^{-1}(P) = \begin{pmatrix} \mathcal{L}_{1m_1} \\ \mathcal{L}_{2m_2} \\ \vdots \\ \mathcal{L}_{sm_s} \end{pmatrix}, \tag{1.4}$$

where, for $r = 1, 2, \ldots, s$, the block matrix $\mathcal{L}_{rm_r}$ is of order $m_r \times n$ and given by

$$\mathcal{L}_{rm_r} = \left(\frac{1}{(j-1)!}L_{r(i-1)}^{(j-1)}(0)[P]\right)_{1\le i\le m_r, 1\le j\le n}.$$

More precisely,

$$\mathcal{L}_{rm_r} = \begin{pmatrix} L_{r0}(0)[P] & L_{r0}^{(1)}(0)[P] & \cdots & \frac{1}{(n-1)!}L_{r0}^{(n-1)}(0)[P] \\ L_{r1}(0)[P] & L_{r1}^{(1)}(0)[P] & \cdots & \frac{1}{(n-1)!}L_{r1}^{(n-1)}(0)[P] \\ \vdots & \vdots & \cdots & \vdots \\ L_{rm_r-1}(0)[P] & L_{rm_r-1}^{(1)}(0)[P] & \cdots & \frac{1}{(n-1)!}L_{rm_r-1}^{(n-1)}(0)[P] \end{pmatrix}.$$

The following corollary treats the interesting case of the generalized Pascal matrix

Corollary 1.2. *Let α be a complex number and n be a positive integer. Then the inverse of the Vandermonde matrix*

$$V_G((x-\alpha)^n) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \alpha & 1 & \cdots & 0 \\ \alpha^2 & 2\alpha & \cdots & 0 \\ \alpha^3 & 3\alpha^2 & \cdots & 0 \\ \binom{4}{0}\alpha^4 & \binom{4}{1}\alpha^3 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ \binom{n-1}{0}\alpha^{n-1} & \binom{n-1}{1}\alpha^{n-2} & \cdots & 1 \end{pmatrix} \tag{1.5}$$

is

$$V_G^{-1}((x-\alpha)^n) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -\alpha & 1 & \cdots & 0 \\ \alpha^2 & -2\alpha & \cdots & 0 \\ -\alpha^3 & 3\alpha^2 & \cdots & 0 \\ \binom{4}{0}(-\alpha)^4 & \binom{4}{1}(-\alpha)^3 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ \binom{n-1}{0}(-\alpha)^{n-1} & \binom{n-1}{1}(-\alpha)^{n-2} & \cdots & 1 \end{pmatrix}.$$

2. Explicit formulas for the matrix exponential

Let us consider the following system of differential equations

$$\dot{X} = AX,$$

where $A = (a_{ij})_{1\le i,j\le k}$ is a constant matrix with entries in $\mathbb{C}$ and $X(t)$ the vector column defined by $X(t) = (x_1(t), x_2(t), \ldots, x_k(t))^T$.

Given any square matrix A, the exponential matrix function is

$$\exp(tA) = \sum_{n=0}^{+\infty} \frac{t^n}{n!}A^n.$$

It is well known that the function

$$X(t) = \exp(tA)X_0$$

is the theoretical solution of the equation

$$\dot{X} = AX, \quad X(0) = X_0.$$

The last differential equation has gained much importance in linear and dynamical systems.

Using the results of the previous section, we develop a purely algebraic approach, that requires only the knowledge of eigenvalues of the matrix, to derive more explicit expressions for the exponential of an arbitrary complex matrix.

Let χ be a unital polynomial of degree k

$$\chi(x) = x^k - a_1x^{k-1} - a_2x^{k-2} - \cdots - a_k$$

and consider the following set of differentiable functions mapping $\mathbb{C}$ to the complex matrices of order k

$$F(\chi) = \{f/f^{(k)}(t) = a_1 f^{(k-1)}(t) + a_2 f^{(k-2)}(t) + \cdots + a_k f(t)\},$$

where $f^{(l)}(t)$ denote the lth derivative of $f(t)$. It is well known that $F(\chi)$ is a $\mathbb{C}$-vector space of dimension k.

Let $D(\chi)$ be the vector space of all complex functions satisfying the following linear differential equation

$$y^{(k)}(t) = a_1 y^{(k-1)}(t) + a_2 y^{(k-2)}(t) + \cdots + a_k y(t).$$

Let $y_i, 0 \leq i \leq k-1$, be the elements of $D(\chi)$ with the initial conditions

$$y_i^{(j)}(0) = \delta_{ij}, \text{ for } 0 \leq j \leq k-1.$$

It is well known that $\{y_0(t), y_1(t), \ldots, y_{k-1}(t)\}$ is a basis of $D(\chi)$; it is called the canonical basis of $D(\chi)$. The most important property of this basis is that all the elements of $F(\chi)$ can be expressed as the linear combinations of the y_i's only in terms of their derivatives at $t = 0$. More precisely, each $f \in F(\chi)$ can be written

$$f(t) = \sum_{i=0}^{k-1} y_i(t) f^{(i)}(0). \tag{2.1}$$

As a particular case of Formula (2.1), we find the following well known result.

Proposition 2.1. *For any $k \times k$ matrix A, the exponential matrix function is given by*

$$\exp(tA) = \sum_{i=0}^{k-1} y_i(t) A^i,$$

where $\{y_0(t), y_1(t), \ldots, y_{k-1}(t)\}$ is the canonical basis of $D(\chi_A)$ and χ_A is the characteristic polynomial of A.

Proof. Follows immediately from the fact that $\exp(tA) \in F(\chi_A)$. □

In [2] the author intents to find, in the simplest way, a method for computing the exponential of a matrix but he does not treat all cases. In what follows, we present the general results without any disadvantages. We begin with a proposition that will be useful in the sequel.

Proposition 2.2. *If $\{e_1(t), e_2(t), \ldots, e_k(t)\}$ is a basis of the vector space $D(\chi_A)$, then there exist unique constant matrices $B_1, B_2, \ldots, B_k$ such that*

$$\exp(tA) = e_1(t)B_1 + e_2(t)B_2 + \cdots + e_k(t)B_k.$$

Proof. This result follows directly from Proposition 2.1 and the fact that

$$\{e_1(t), e_2(t), \ldots, e_k(t)\}$$

is a basis of the vector space $D(\chi_A)$. □

In the following, we provide the explicit expressions of the elements

$$y_0(t), y_1(t), \ldots, y_{k-1}(t)$$

of $D(\chi)$ in terms of the roots of χ.

Theorem 2.3. *Let $\chi(x) = (x-\alpha_1)^{m_1}(x-\alpha_2)^{m_2}\cdots(x-\alpha_s)^{m_s}$ be a unital polynomial of degree k. Then the (i+1)th element of the canonical basis of $D(\chi)$ is*

$$y_i(t) = \sum_{p=1}^{s} \frac{e^{\alpha_p t}}{i!}\Big(\sum_{r_p=0}^{m_p-1} \frac{t^{r_p}}{r_p!} L^{(i)}_{pr_p}(0)[\chi]\Big), 0 \le i \le k-1.$$

Proof. It is well known that

$$B = \{e^{\alpha_1 t}, te^{\alpha_1 t}, \dots, \frac{t^{m_1-1}}{(m_1-1)!}e^{\alpha_1 t}, \dots, e^{\alpha_s t}, te^{\alpha_s t}, \dots, \frac{t^{m_s-1}}{(m_s-1)!}e^{\alpha_s t}\}$$

is a basis of $D(\chi)$. By expressing each member of this basis in terms of the canonical basis, we get

$$\frac{t^i}{i!}e^{\alpha_p t} = \sum_{j=0}^{k-1}\Big(\frac{t^i}{i!}e^{\alpha_p t}\Big)^{(j)}(0) y_j(t)$$

for $0 \le i \le m_p - 1$ and $p = 1, 2, \dots, s$. The resulting change of basis matrix from B to the canonical basis is the confluent Vandermonde matrix (1.3)

$$V_G(\chi) = \begin{pmatrix} 1 & 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 \\ \alpha_1 & 1 & \cdots & 0 & \cdots & \alpha_s & 1 & \cdots & 0 \\ \alpha_1^2 & 2\alpha_1 & \cdots & 0 & \cdots & \alpha_s^2 & 2\alpha_s & \cdots & 0 \\ \alpha_1^3 & 3\alpha_1^2 & \cdots & 0 & \cdots & \alpha_s^3 & 3\alpha_s^2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \alpha_1^{k-1} & (k-1)\alpha_1^{k-2} & \cdots & 1 & \cdots & \alpha_s^{k-1} & (k-1)\alpha_s^{k-2} & \cdots & 1 \end{pmatrix}.$$

Using the inverse given by (1.4), we obtain

$$y_i(t) = \sum_{p=1}^{s} \frac{e^{\alpha_p t}}{i!}\Big(\sum_{r_p=0}^{m_p-1} \frac{t^{r_p}}{r_p!} L^{(i)}_{pr_p}(0)[\chi]\Big), 0 \le i \le k-1$$

as desired. □

As a particular case we have the following corollary.

Corollary 2.4. *If $\chi(x) = (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_k)$ has distinct roots $\alpha_1, \alpha_2, \dots, \alpha_k$, then*

$$y_i(t) = \sum_{j=1}^{k} \frac{e^{\alpha_j t}}{i!} L^{(i)}_{j0}(0)[\chi], 0 \le i \le k-1.$$

In the following we state one of our main results.

Theorem 2.5. *Let A be a $k \times k$ matrix, and let $\chi_A(x) = (x-\alpha_1)^{m_1}(x-\alpha_2)^{m_2}\cdots(x-\alpha_s)^{m_s}$ be its characteristic polynomial. Then,*

$$\exp(tA) = \sum_{i=0}^{k-1}\Big[\sum_{p=1}^{s} \frac{e^{\alpha_p t}}{i!}\Big(\sum_{r_p=0}^{m_p-1} \frac{t^{r_p}}{r_p!} L^{(i)}_{pr_p}(0)[\chi_A]\Big)\Big]A^i.$$

Proof. The result is an immediate consequence of Theorem 2.3 and Proposition 2.1. □

As a particular case we have the following corollary.

Corollary 2.6. *If $\chi_A(x) = (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_k)$ has distinct roots $\alpha_1, \alpha_2, \dots, \alpha_k$, then*

$$\exp(tA) = \sum_{i=0}^{k-1}\Big(\sum_{j=1}^{k} \frac{e^{\alpha_j t}}{i!} L_{j0}^{(i)}(0)[\chi_A]\Big)A^i.$$

Proof. Follows immediately from Theorem 2.3 and Theorem 2.5. □

In the case where the matrix A has a single eigenvalue, we have the following result.

Corollary 2.7. *If $\chi_A(x) = (x-\alpha)^k$ has a single root α, then*

$$y_i(t) = \frac{e^{\alpha t}}{i!}\Big(\sum_{l=i}^{k-1} \frac{(-1)^{l+i}}{(l-i)!}\alpha^{l-i}t^l\Big), 0 \le i \le k-1$$

and

$$\exp(tA) = \sum_{i=0}^{k-1} \frac{1}{i!}\Big(\sum_{l=i}^{k-1} \frac{(-1)^{l+i}}{(l-i)!}\alpha^{l-i}t^l\Big)A^i e^{\alpha t}.$$

Proof. The result is an immediate consequence of Theorem 2.3 and Theorem 2.5. □

For illustration purposes, we consider the case of a square matrix of order 3.

Example 2.8. *Let A be a square matrix of order 3 with characteristic polynomial*

$$\chi_A(x) = x^3 - a_1x^2 - a_2x - a_3.$$

1. *If α is a root of multiplicity 3 of $\chi_A(x)$ then, using Corollary 2.7, we have*

$$\begin{aligned}\exp(tA) =& \Big(e^{\alpha t}\sum_{l=0}^{2}\frac{(-1)^l}{l!}\alpha^l t^l\Big)I + \Big(e^{\alpha t}\sum_{l=1}^{2}\frac{(-1)^{l+1}}{(l-1)!}\alpha^{l-1}t^l\Big)A+\\ &\Big(\frac{e^{\alpha t}}{2!}\sum_{l=2}^{2}\frac{(-1)^l}{(l-2)!}\alpha^{l-2}t^l\Big)A^2.\end{aligned}$$

Consequently, we obtain

$$\exp(tA) = (1-\alpha t + (1/2)\alpha^2t^2)e^{\alpha t}I + (t-\alpha t^2)e^{\alpha t}A + (1/2)t^2e^{\alpha t}A^2.$$

2. *If α_1 is a root of multiplicity 2, and α_2 is a simple root, then Theorem 2.5 gives*

$$\begin{aligned}\exp(tA) =& \Big(e^{\alpha_1 t}\big(L_{10}(0)[\chi_A] + tL_{11}(0)[\chi_A]\big) + e^{\alpha_2 t}L_{20}(0)[\chi_A]\Big)I+\\ &\Big(e^{\alpha_1 t}\big(L_{10}^{(1)}(0)[\chi_A] + tL_{11}^{(1)}(0)[\chi_A]\big) + e^{\alpha_2 t}L_{20}^{(1)}(0)[\chi_A]\Big)A+\\ &\frac{1}{2!}\Big(e^{\alpha_1 t}\big(L_{10}^{(2)}(0)[\chi_A] + tL_{11}^{(2)}(0)[\chi_A]\big) + e^{\alpha_2 t}L_{20}^{(2)}(0)[\chi_A]\Big)A^2.\end{aligned}$$

Formula (1.1) yields

$$\begin{cases} L_{10}(x)[\chi_A] &= \dfrac{-x^2+2\alpha_1x-2\alpha_1\alpha_2+\alpha_2^2}{(\alpha_1-\alpha_2)^2},\\ L_{11}(x)[\chi_A] &= \dfrac{x^2-(\alpha_1+\alpha_2)x+\alpha_1\alpha_2}{(\alpha_1-\alpha_2)},\\ L_{20}(x)[\chi_A] &= \dfrac{x^2-2\alpha_1x+\alpha_1^2}{(\alpha_2-\alpha_1)^2}.\end{cases}$$

A simple calculation gives

$$\begin{aligned}\exp(tA) = \\ &\frac{1}{(\alpha_1-\alpha_2)^2}\Big[\Big((\alpha_2^2-2\alpha_1\alpha_2)\mathrm{e}^{\alpha_1 t}+(\alpha_1^2\alpha_2-\alpha_1\alpha_2^2)t\mathrm{e}^{\alpha_1 t}+\alpha_1^2\mathrm{e}^{\alpha_2 t}\Big)I+ \\ &\Big((\alpha_1^2-\alpha_2^2)t\mathrm{e}^{\alpha_1 t}+2\alpha_1(\mathrm{e}^{\alpha_1 t}-\mathrm{e}^{\alpha_2 t})\Big)A+ \\ &\Big(\mathrm{e}^{\alpha_2 t}-\mathrm{e}^{\alpha_1 t}+(\alpha_1-\alpha_2)t\mathrm{e}^{\alpha_1 t}\Big)A^2\Big].\end{aligned}$$

3. If $\alpha_1,\alpha_2,\alpha_3$ are simple roots of $\chi_A(x)$ then, using Corollary 2.4, we have

$$\begin{aligned}\exp(tA) =&\Big(\sum_{j=1}^{3}\mathrm{e}^{\alpha_j t}L_{j0}(0)[\chi_A]\Big)I+\Big(\sum_{j=1}^{3}\mathrm{e}^{\alpha_j t}L_{j0}^{(1)}(0)[\chi_A]\Big)A+ \\ &\Big(\sum_{j=1}^{3}\frac{\mathrm{e}^{\alpha_j t}}{2!}L_{j0}^{(2)}(0)[\chi_A]\Big)A^2.\end{aligned}$$

Formula (1.1) gives

$$\begin{cases} L_{10}(x)[\chi_A]) &= \dfrac{(x-\alpha_2)(x-\alpha_3)}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)} = \dfrac{x^2-(\alpha_2+\alpha_3)x+\alpha_2\alpha_3}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}, \\ L_{20}(x)[\chi_A]) &= \dfrac{(x-\alpha_1)(x-\alpha_3)}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)} = \dfrac{x^2-(\alpha_1+\alpha_3)x+\alpha_1\alpha_3}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)}, \\ L_{30}(x)[\chi_A]) &= \dfrac{(x-\alpha_1)(x-\alpha_2)}{(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)} = \dfrac{x^2-(\alpha_1+\alpha_2)x+\alpha_1\alpha_2}{(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)}. \end{cases}$$

Consequently, the matrix exponential in this case is given by

$$\begin{aligned}\exp(tA) =&\Big[\frac{\alpha_2\alpha_3}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}\mathrm{e}^{\alpha_1 t}+\frac{\alpha_1\alpha_3}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)}\mathrm{e}^{\alpha_2 t}+ \\ &\frac{\alpha_1\alpha_2}{(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)}\mathrm{e}^{\alpha_3 t}\Big]I+\Big[\frac{\alpha_2+\alpha_3}{(\alpha_3-\alpha_1)(\alpha_1-\alpha_2)}\mathrm{e}^{\alpha_1 t}+ \\ &\frac{\alpha_1+\alpha_3}{(\alpha_3-\alpha_2)(\alpha_2-\alpha_1)}\mathrm{e}^{\alpha_2 t}+\frac{\alpha_1+\alpha_2}{(\alpha_1-\alpha_3)(\alpha_3-\alpha_2)}\mathrm{e}^{\alpha_3 t}\Big]A+ \\ &\Big[\frac{\mathrm{e}^{\alpha_1 t}}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}+\frac{\mathrm{e}^{\alpha_2 t}}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)}+\frac{\mathrm{e}^{\alpha_3 t}}{(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)}\Big]A^2.\end{aligned}$$

In the present approach, we provide closed-form formulas for the matrix exponential. The approach makes the determination of the exponential of any square matrix more practical. Using Lagrange polynomials W. A. Harris et al. [5] have derived some explicit formulas for the exponential of a matrix with simple eigenvalues. Here, we generalize this result to any matrix using a generalization of Hermite's interpolation formula given by A. Spitzbart [17].

Theorem 2.9. *Let A be a $k\times k$ matrix, and let $\chi_A(x)=(x-\alpha_1)^{m_1}(x-\alpha_2)^{m_2}\cdots(x-\alpha_s)^{m_s}$ be its characteristic polynomial. Then,*

$$\exp(tA)=\sum_{j=1}^{s}\sum_{k_j=0}^{m_j-1}\frac{t^{k_j}}{k_j!}\mathrm{e}^{\alpha_j t}B_{jk_j}, \tag{2.2}$$

where $B_{jk_j}=L_{jk_j}(A)[\chi_A]$.

Proof. Using Proposition 2.2 with the fact that

$$\{\mathrm{e}^{\alpha_1 t}, t\mathrm{e}^{\alpha_1 t}, \ldots, \frac{t^{m_1-1}}{(m_1-1)!}\mathrm{e}^{\alpha_1 t}, \ldots, \mathrm{e}^{\alpha_s t}, t\mathrm{e}^{\alpha_s t}, \ldots, \frac{t^{m_s-1}}{(m_s-1)!}\mathrm{e}^{\alpha_s t}\}$$

is a basis of $D(\chi_A)$, we can find unique constant matrices B_{jk_j}, $j = 1, 2, \ldots, s$ and $0 \le k_j \le m_j - 1$, such that

$$\exp(tA) = \sum_{j=0}^{m_1-1} \frac{t^j}{j!}\mathrm{e}^{\alpha_1 t} B_{1j} + \sum_{j=0}^{m_2-1} \frac{t^j}{j!}\mathrm{e}^{\alpha_2 t} B_{2j} + \cdots + \sum_{j=0}^{m_s-1} \frac{t^j}{j!}\mathrm{e}^{\alpha_s t} B_{sj}.$$

Applying Proposition 2.2 to this equality yields

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 \\ \alpha_1 & 1 & \cdots & 0 & \cdots & \alpha_s & 1 & \cdots & 0 \\ \alpha_1^2 & 2\alpha_1 & \cdots & 0 & \cdots & \alpha_s^2 & 2\alpha_s & \cdots & 0 \\ \alpha_1^3 & 3\alpha_1^2 & \cdots & 0 & \cdots & \alpha_s^3 & 3\alpha_s^2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \alpha_1^{k-1} & (k-1)\alpha_1^{k-2} & \cdots & 1 & \cdots & \alpha_s^{k-1} & (k-1)\alpha_s^{k-2} & \cdots & 1 \end{pmatrix} \begin{pmatrix} B_{10} \\ B_{11} \\ \vdots \\ B_{1m_1-1} \\ \vdots \\ B_{s0} \\ \vdots \\ B_{sm_s-1} \end{pmatrix} = \begin{pmatrix} I \\ A \\ \vdots \\ A^{m_1-1} \\ \vdots \\ \vdots \\ \vdots \\ A^{k-1} \end{pmatrix}. \tag{2.3}$$

On the other hand, utilizing Formula (1.2) for the canonical basis of $\mathbb{C}_{k-1}[x]$, we obtain the following system

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 \\ \alpha_1 & 1 & \cdots & 0 & \cdots & \alpha_s & 1 & \cdots & 0 \\ \alpha_1^2 & 2\alpha_1 & \cdots & 0 & \cdots & \alpha_s^2 & 2\alpha_s & \cdots & 0 \\ \alpha_1^3 & 3\alpha_1^2 & \cdots & 0 & \cdots & \alpha_s^3 & 3\alpha_s^2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \alpha_1^{k-1} & (k-1)\alpha_1^{k-2} & \cdots & 1 & \cdots & \alpha_s^{k-1} & (k-1)\alpha_s^{k-2} & \cdots & 1 \end{pmatrix} \begin{pmatrix} L_{10}(x)[\chi_A] \\ L_{11}(x)[\chi_A] \\ \vdots \\ L_{1m_1-1}(x)[\chi_A] \\ \vdots \\ L_{s0}(x)[\chi_A] \\ \vdots \\ L_{sm_s-1}(x)[\chi_A] \end{pmatrix} = \begin{pmatrix} 1 \\ x \\ \vdots \\ x^{m_1-1} \\ \vdots \\ \vdots \\ \vdots \\ x^{k-1} \end{pmatrix}.$$

Replacing in this matrix equation x with A yields

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 \\ \alpha_1 & 1 & \cdots & 0 & \cdots & \alpha_s & 1 & \cdots & 0 \\ \alpha_1^2 & 2\alpha_1 & \cdots & 0 & \cdots & \alpha_s^2 & 2\alpha_s & \cdots & 0 \\ \alpha_1^3 & 3\alpha_1^2 & \cdots & 0 & \cdots & \alpha_s^3 & 3\alpha_s^2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \alpha_1^{k-1} & (k-1)\alpha_1^{k-2} & \cdots & 1 & \cdots & \alpha_s^{k-1} & (k-1)\alpha_s^{k-2} & \cdots & 1 \end{pmatrix} \begin{pmatrix} L_{10}(A)[\chi_A] \\ L_{11}(A)[\chi_A] \\ \vdots \\ L_{1m_1-1}(A)[\chi_A] \\ \vdots \\ L_{s0}(A)[\chi_A] \\ \vdots \\ L_{sm_s-1}(A)[\chi_A] \end{pmatrix} = \begin{pmatrix} I \\ A \\ \vdots \\ A^{m_1-1} \\ \vdots \\ \vdots \\ \vdots \\ A^{k-1} \end{pmatrix}.$$

Since the confluent Vandermonde matrix is invertible, we have

$$B_{jk_j} = L_{jk_j}(A)[\chi_A] \text{ for all } 1 \le j \le s,\ 0 \le k_j \le m_j - 1.$$

□

The following example is used to illustrate Theorem 2.5 and Theorem 2.9.

Example 2.10. *Let us consider Example* 1. *of [8]*

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

The characteristic polynomial of A *is* $\chi_A(x) = (x-2)^2(x-3)$*. In this case, by using Formula* (1.1)*, we have*

$$\begin{aligned} L_{10}(x)[\chi_A] &= (x-3)(1-x) = -x^2+4x-3, \\ L_{11}(x)[\chi_A] &= (x-3)(2-x) = -x^2+5x-6, \\ L_{20}(x)[\chi_A] &= (x-2)^2 = x^2-4x+4. \end{aligned}$$

We deduce that

$$\begin{array}{lclllcllcl} L_{10}(0)[\chi_A] & = & -3, & L'_{10}(0)[\chi_A] & = & 4, & L''_{10}(0)[\chi_A] & = & -2, \\ L_{11}(x)[\chi_A] & = & -6, & L'_{11}(0)[\chi_A] & = & 5, & L''_{11}(0)[\chi_A] & = & -2, \\ L_{20}(x)[\chi_A] & = & 4, & L'_{20}(0)[\chi_A] & = & -4, & L''_{20}(0)[\chi_A] & = & 2. \end{array}$$

Using the formula of Theorem 2.5*, we obtain*

$$\begin{aligned} \exp(tA) = &\left(\mathrm{e}^{2t}(-3-6t)+4\mathrm{e}^{3t}\right)I + \left(\mathrm{e}^{2t}(4+5t)-4\mathrm{e}^{3t}\right)A + \\ &(1/2)\left(\mathrm{e}^{2t}(-2-2t)+2\mathrm{e}^{3t}\right)A^2. \end{aligned}$$

Therefore

$$\exp(tA) = \begin{pmatrix} \mathrm{e}^{2t} & 0 & \mathrm{e}^{3t}-\mathrm{e}^{2t} \\ 0 & \mathrm{e}^{2t} & 0 \\ 0 & 0 & \mathrm{e}^{3t} \end{pmatrix}.$$

Now we find the matrix exponential of A*, but using this time Theorem* 2.9*. From Formula* (2.2)*, we get*

$$\exp(tA) = \mathrm{e}^{2t}B_{10} + t\mathrm{e}^{2t}B_{11} + \mathrm{e}^{3t}B_{20},$$

where

$$\begin{aligned} B_{10} &= L_{10}(A)[\chi_A] = -A^2+4A-3I, \\ B_{11} &= L_{11}(A)[\chi_A] = -A^2+5A-6I, \\ B_{20} &= L_{20}(A)[\chi_A] = A^2-4A+4I. \end{aligned}$$

More explicitly,

$$B_{10} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, B_{11} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, B_{20} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus

$$\exp(tA) = \mathrm{e}^{2t}B_{10} + t\mathrm{e}^{2t}B_{11} + \mathrm{e}^{3t}B_{20}.$$

That is

$$\exp(tA) = \begin{pmatrix} \mathrm{e}^{2t} & 0 & \mathrm{e}^{3t}-\mathrm{e}^{2t} \\ 0 & \mathrm{e}^{2t} & 0 \\ 0 & 0 & \mathrm{e}^{3t} \end{pmatrix}.$$

Next, we derive some corollaries of Theorem 2.9.

Corollary 2.11. *If $\chi_A(x) = (x-\alpha_1)^{m_1}(x-\alpha_2)^{m_2}$, α_1 and α_2 are two distinct complex numbers, then*

$$\exp(tA) = \sum_{j=0}^{m_1-1} \frac{t^j}{j!} e^{\alpha_1 t} B_{1j} + \sum_{j=0}^{m_2-1} \frac{t^j}{j!} e^{\alpha_2 t} B_{2j},$$

where

$$\begin{cases} B_{1j} &= (A-\alpha_1 I)^j (A-\alpha_2 I)^{m_2} \displaystyle\sum_{i=0}^{m_1-j-1} \frac{(-1)^i \binom{m_2+i-1}{m_2-1}}{(\alpha_1-\alpha_2)^{m_2+i}} (A-\alpha_1 I)^i, \\ B_{2j} &= (A-\alpha_1 I)^{m_1} (A-\alpha_2 I)^j \displaystyle\sum_{i=0}^{m_2-j-1} \frac{(-1)^i \binom{m_1+i-1}{m_1-1}}{(\alpha_2-\alpha_1)^{m_1+i}} (A-\alpha_2 I)^i. \end{cases}$$

Proof. This follows directly from (2.2) and the fact that

$$\begin{cases} L_{1j}(x)[\chi_A] &= \displaystyle\sum_{i=0}^{m_1-j-1} \frac{(-1)^i \binom{m_2+i-1}{m_2-1}}{(\alpha_1-\alpha_2)^{m_2+i}} (x-\alpha_1)^{i+j} (x-\alpha_2)^{m_2}, \\ L_{2j}(x)[\chi_A] &= \displaystyle\sum_{i=0}^{m_2-j-1} \frac{(-1)^i \binom{m_1+i-1}{m_1-1}}{(\alpha_2-\alpha_1)^{m_1+i}} (x-\alpha_1)^{m_1} (x-\alpha_2)^{i+j}. \end{cases}$$

□

Corollary 2.12. *If $\chi_A(x) = (x-\alpha_1)^{m_1}(x-\alpha_2)$, α_1 and α_2 are distinct two complex numbers, then*

$$\exp(tA) = \sum_{j=0}^{m_1-1} \frac{t^j}{j!} e^{\alpha_1 t} B_{1j} + e^{\alpha_2 t} B_{20},$$

where

$$\begin{cases} B_{1j} &= (A-\alpha_1 I)^j (A-\alpha_2 I) \displaystyle\sum_{i=0}^{m_1-j-1} \frac{(-1)^i}{(\alpha_1-\alpha_2)^{i+1}} (A-\alpha_1 I)^i, \\ B_{20} &= \dfrac{1}{(\alpha_2-\alpha_1)^{m_1}} (A-\alpha_1 I)^{m_1}. \end{cases}$$

More generally, we have the following result.

Corollary 2.13. *If $\chi_A(x) = (x-\alpha_1)^{m_1} \prod_{j=2}^{s} (x-\alpha_j)$, $\alpha_1, \dots, \alpha_s$ are distinct complex numbers, then*

$$\exp(tA) = \sum_{j=0}^{m_1-1} \frac{t^j}{j!} e^{\alpha_1 t} B_{1j} + \sum_{j=2}^{s} e^{\alpha_j t} \frac{1}{P_j(\alpha_j)} P_j(A), \tag{2.4}$$

where $P = \chi_A$,

$$B_{1j} = (A-\alpha_1 I)^j \prod_{l=2}^{s} (A-\alpha_l I) \sum_{i=0}^{m_1-j-1} \sum_{l=2}^{s} \frac{(-1)^i a_l}{(\alpha_1-\alpha_l)^{i+1}} (A-\alpha_1 I)^i$$

and

$$a_l = \begin{cases} 1/\prod_{p=2, p\neq l}^{s} (\alpha_l-\alpha_p) & \text{if} \quad s \geq 3, \\ 1 & \text{if} \quad s = 2. \end{cases}$$

Proof. In this case, Formula (2.2) becomes

$$\exp(tA) = \sum_{j=0}^{m_1-1} \frac{t^j}{j!} \mathrm{e}^{\alpha_1 t} B_{1j} + \mathrm{e}^{\alpha_2 t} B_{20} + \cdots + \mathrm{e}^{\alpha_s t} B_{s0}$$

where, for $j = 2, \ldots, s$,

$$B_{j0} = L_{j0}(A)[P] = P_j(A) g_j(\alpha_j) = \frac{1}{P_j(\alpha_j)} P_j(A).$$

On the other hand, for $j = 0, \ldots, m_1 - 1$, we have $B_{1j} = L_{1j}(A)[P]$, where

$$L_{1j}(x)[P] = P_1(x)(x-\alpha_1)^j \sum_{i=0}^{m_1-1-j} \frac{1}{i!} g_1^{(i)}(\alpha_1)(x-\alpha_1)^i,$$

$$P_1(x) = \prod_{l=2}^{s}(x-\alpha_l),$$

$$g_1(x) = 1/\prod_{l=2}^{s}(x-\alpha_l).$$

But since

$$g_1(x) = \sum_{l=2}^{s} \frac{a_l}{x-\alpha_l},$$

we have

$$g_1^{(i)}(x) = \sum_{l=2}^{s} \frac{(-1)^i i! a_l}{(x-\alpha_l)^{i+1}}.$$

Then

$$L_{1j}(x)[P] = (x-\alpha_1)^j \prod_{l=2}^{s}(x-\alpha_l) \sum_{i=0}^{m_1-1-j} \sum_{l=2}^{s} \frac{(-1)^i a_l}{(\alpha_1-\alpha_l)^{i+1}} (x-\alpha_1)^i.$$

Therefore

$$B_{1j} = (A-\alpha_1 I)^j \prod_{l=2}^{s}(A-\alpha_l I) \sum_{i=0}^{m_1-j-1} \sum_{l=2}^{s} \frac{(-1)^i a_l}{(\alpha_1-\alpha_l)^{i+1}} (A-\alpha_1 I)^i.$$

Thus, the proof is completed. □

To illustrate Corollary 2.13, consider the following example.

Example 2.14. *Let*

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 3 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}.$$

The characteristic polynomial of A is $\chi_A(x) = x^2(x+2)(x-2)$. Let us choose, for example, $\alpha_1 = 0, \alpha_2 = -2$ and $\alpha_3 = 2$. Then, in light of Formula (2.4), the exponential matrix of A is given by

$$\exp(tA) = B_{10} + tB_{11} + \mathrm{e}^{-2t} B_{20} + \mathrm{e}^{2t} B_{30}, \tag{2.5}$$

where

$$\begin{cases} B_{10} = (A+2I)(A-2I)\Big[\big(\frac{a_2}{\alpha_1-\alpha_2} + \frac{a_3}{\alpha_1-\alpha_3}\big)I - \big(\frac{a_2}{(\alpha_1-\alpha_2)^2} + \frac{a_3}{(\alpha_1-\alpha_3)^2}\big)A\Big], \\ B_{11} = A(A+2I)(A-2I)\Big[\big(\frac{a_2}{\alpha_1-\alpha_2} + \frac{a_3}{\alpha_1-\alpha_3}\big)\Big] \end{cases}$$

and

$$\begin{cases} B_{20} &=& P_2(A)/P_2(\alpha_2) = (-1/16)A^2(A-2I) = (-1/16)A^3 + (1/8)A^2, \\ B_{30} &=& P_3(A)/P_3(\alpha_3) = (1/16)A^2(A+2I) = (1/16)A^3 + (1/8)A^2. \end{cases}$$

In this case $a_2 = 1/(\alpha_2 - \alpha_3) = -1/4$ *and* $a_3 = 1/(\alpha_3 - \alpha_2) = 1/4$. *Consequently, we have*

$$\begin{cases} B_{10} &=& (-1/4)A^2 + I, \\ B_{11} &=& (-1/4)A^3 + A, \\ B_{20} &=& (-1/16)A^3 + (1/8)A^2, \\ B_{30} &=& (1/16)A^3 + (1/8)A^2. \end{cases}$$

Hence Formula (2.5) *becomes*

$$\exp(tA) = \frac{e^{2t} + e^{-2t} - 4t}{16}A^3 + \frac{e^{2t} + e^{-2t} - 2}{8}A^2 + tA + I.$$

Since

$$A^2 = \begin{pmatrix} 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 4 & 3 & 2 & -2 \\ 3 & 4 & -2 & 2 \end{pmatrix}$$

and

$$A^3 = \begin{pmatrix} 4 & 4 & 0 & 0 \\ 4 & 4 & 0 & 0 \\ 9 & 11 & -4 & 4 \\ 5 & 3 & 4 & -4 \end{pmatrix},$$

we obtain

$$\exp(tA) = \begin{pmatrix} \frac{e^{2t}+1}{2} & \frac{e^{2t}-1}{2} & 0 & 0 \\ \frac{e^{2t}-1}{2} & \frac{e^{2t}+1}{2} & 0 & 0 \\ \frac{17e^{2t}-e^{-2t}-4t-16}{16} & \frac{17e^{2t}-5e^{-2t}+4t-12}{16} & \frac{e^{-2t}+1}{2} & \frac{-e^{-2t}+1}{2} \\ \frac{11e^{2t}+e^{-2t}-4t-12}{16} & \frac{11e^{2t}+5e^{-2t}+4t-16}{16} & \frac{-e^{-2t}+1}{2} & \frac{e^{-2t}+1}{2} \end{pmatrix}.$$

The following result is due to W. A. Harris et al. [5]

Corollary 2.15. *If* $\chi_A(x) = (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_k)$ *has distinct roots* $\alpha_1, \alpha_2, \ldots, \alpha_k$*, then*

$$\exp(tA) = e^{\alpha_1 t}B_1 + e^{\alpha_2 t}B_2 + \cdots + e^{\alpha_k t}B_k,$$

where $B_i = \prod_{j=1, j\neq i}^{k} \dfrac{1}{\alpha_i - \alpha_j}(A - \alpha_j I_k)$.

Proof. Is a particular case of Theorem 2.9. □

When A is a matrix with only one eigenvalue, we have the following known result shown by Apostol [2]

Corollary 2.16. *If* $\chi_A(x) = (x-\alpha)^k$ *has a single root* α*, then*

$$\exp(tA) = e^{\alpha t}B_1 + te^{\alpha t}B_2 + \cdots + \frac{t^{k-1}}{(k-1)!}e^{\alpha t}B_k,$$

where $B_i = (A - \alpha I)^{i-1}, 1 \leq i \leq k$.

Proof. Follows immediately from Theorem 2.9. □

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