Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 ON LINE
SPM: www.spm.uem.br/bspm

(3s.) **v. 2023 (41)** : 1–10.
ISSN-0037-8712 IN PRESS
doi:10.5269/bspm.62715

Relative Uniform Convergence of Double Sequence of Positive Functions Defined by Orlicz Function

Kshetrimayum Renubebeta Devi and Binod Chandra Tripathy

ABSTRACT: In this article, we introduce the notion of relative uniform convergence of double sequence of positive functions defined by using Orlicz function. We also introduce different classes of relative uniform convergent double sequence of functions and discuss their algebraic and topological properties.

Key Words: Double sequence, relative uniform convergence, scale function, solid, symmetric, monotone, convergence free, sequence algebra, completeness, Orlicz function.

Contents

1. Introduction

The notion of convergence for double sequences was given by Pringsheim [16]. Some initial works on double sequence spaces are found in Bromwich [2]. Later on the notion was investigated by Hardy [10], Moricz [13], Tripathy and Sarma [20,22,24], and many others.

A double sequence (x_{nk}) is said to converge in Pringsheim's sense if

$$\lim_{n,k\to\infty} x_{nk} = L, \text{ exists.}$$

The notion of regular convergence for double sequences was introduced by Hardy [10]. A double sequence (x_{nk}) is said to converge regularly if it converges in the pringsheim's sense and the following limits exist:

$$\lim_{k\to\infty} x_{nk} = L_n \text{ exists, for each } n \in N, \text{ and } \lim_{n\to\infty} x_{nk} = P_k \text{ exists, for each } k \in N.$$

When $L = L_n = P_k = \theta$, for all $n, k \in N$, we say that (x_{nk}) is regularly null.

The notion of uniform convergence of a sequence of functions relative to a scale function was introduced by E. H. Moore. Chittenden [3] gave the definition of relative uniform convergence of sequence of functions, which is defined as follows.

Definition 1.1. *(Chittenden [3]) A sequence (f_n) of real, single-valued functions f_n of a real variable x, ranging over a compact subset D of real numbers, converges relatively uniformly on D in case there exist functions g and σ, defined on D, and for every $\varepsilon > 0$, there exists an integer n_o (dependent on ε) such that for every $n \geq n_o$ the inequality*

$\mid g(x) - f_n(x) \mid < \varepsilon \mid \sigma(x) \mid$, *holds for every element x of D.*

2010 *Mathematics Subject Classification:* 40A30, 46A45, 46E30.
Submitted March 01, 2022. Published July 03, 2022

The function σ of the above definition is called a scale function.

The sequence (f_n) is said to converge uniformly relative to the scale function σ.

The notion was further studied by [4,5,6,7,8,12] and many others.

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$, which is continuous, non-decreasing and convex with $M(0) = 0, M(x) > 0$, for $x > 0$ and $M(x) \to \infty$, as $x \to \infty$.

An Orlicz function $M(x)$ satisfies the Δ_2-condition if for every $L > 1$, there exists a constant $K > 0$ and a positive number x such that $M(Lx) \leq KLM(x)$.

If the convexity of Orlicz function is replaced by $M(x + t) \leq M(x) + M(t)$, then this function is called modulus function.

Remark 1.2. *Let M be a convex function and $M(0) = 0$. Then, the inequality $M(\lambda x) \leq \lambda M(x)$ holds true, for all λ, with $0 < \lambda < 1$.*

Using Orlicz function $M(x)$, Lindenstrauss and Tzafriri [11] introduced the sequence space ℓ_M defined as following.

$$\ell_M = \sum_{n=1}^{\infty} M\left(\frac{|x_n|}{\rho}\right) < \infty, \text{ for some } \rho > 0.$$

The space ℓ_M is a Banach space with the following norm and is called as Orlicz sequence space.

$$||(x_n)|| = \inf\left\{\rho > 0 : \sum_{n=1}^{\infty} M\left(\frac{|x_n|}{\rho}\right) \leq 1\right\}.$$

Later on the notion was studied from different aspects by [1,9,14,15,17,18,19,21,23,25] and many others.

2. Definitions and preliminaries

In this section, we obtain the basic definitions which we shall use in establishing the main results of the article.

Definition 2.1. *A sequence space E is said to be* solid or normal *if $(x_{nk}) \in E$ implies $(\alpha_{nk}x_{nk}) \in E$, for all (α_{nk}), with $| \alpha_{nk} | \leq 1$, for all $n, k \in N$.*

Definition 2.2. *A sequence space E is said to be* monotone *if it contains the canonical pre-images of all its step spaces.*

Remark 2.3. *From the above notions, it follows that if a sequence space E is solid then, E is monotone.*

Definition 2.4. *A sequence space E is said to be* symmetric *if $(x_{nk}) \in E \Rightarrow (x_{\pi(n,k)}) \in E$, for all $n, k \in N \times N$, where π is a permutation of N, the set of natural numbers.*

Definition 2.5. *A sequence space E is said to be* convergence free *if $(x_{nk}) \in E$ and $x_{nk} = 0 \Rightarrow y_{nk} = 0$, together with $(y_{nk}) \in E$, for all $n, k \in N$.*

Definition 2.6. *A sequence space E is said to be a* sequence algebra *if $(x_{nk} * y_{nk}) \in E$ whenever (x_{nk}) and (y_{nk}) belongs to E, for all $n, k \in N$.*

Definition 2.7. *A sequence of functions $f_{nk} : D \to R$, of a real variable x, where $D \in R$, defined by an Orlicz function M is said to be relative uniform convergent on D, if there exist limiting function $f(x)$ and scale function $\sigma(x)$ defined on D, and for every $\varepsilon > 0$, there exists an integer $n_0 = n_0(\varepsilon)$ such that for all $n, k \geq n_0$,*

$$M\left(\frac{|f_{nk}(x) - f(x)|}{\rho|\sigma(x)|}\right) \leq \varepsilon, \textit{ for some } \rho > 0, \textit{ and for all } x \in D.$$

Remark 2.8. *When $f = \theta$, the zero function in the Definition 2.7, we get the definition of relative uniform null sequence of functions defined by Orlicz function M.*

Definition 2.9. *A sequence of functions $f_{nk} : D \to R$, of a real variable x, where $D \in R$, defined by an Orlicz function M is said to be relatively uniformly Cauchy on D if there exists scale function $\sigma(x)$ defined on D, and for every $\varepsilon > 0$, there exists an integer $n_0 = n_0(\varepsilon)$ such that for all $p \geq n \geq n_o$ and $q \geq k \geq n_0$,*

$$M\left(\frac{|f_{pq}(x) - f_{nk}(x)|}{\rho|\sigma(x)|}\right) \leq \varepsilon, \text{ for some } \rho > 0, \text{ and for all } x \in D.$$

Definition 2.10. *A sequence of functions $f_{nk} : D \to R$, of a real variable x, where $D \in R$, defined by an Orlicz function M is said to be regular relative uniform convergent on D if there exists function $f(x)$ and scale function $\sigma(x)$ defined on D, and for every $\varepsilon > 0$, there exists an integer $n_0 = n_0(\varepsilon)$ such that for all $x \in D$,*

$$M\left(\frac{|f_{nk}(x) - f(x)|}{\rho|\sigma(x)|}\right) \leq \varepsilon, \text{ for some } \rho > 0, \text{ and for all } n, k \geq n_0;$$

$$M\left(\frac{|f_{nk}(x) - f_n(x)|}{\rho|\xi_n(x)|}\right) \leq \varepsilon, \text{ for some } \rho > 0 \text{ ; for each } n \in N \text{ and for all } k \geq n_0;$$

$$M\left(\frac{|f_{nk}(x) - g_k(x)|}{\rho|\eta_k(x)|}\right) \leq \varepsilon, \text{ for some } \rho > 0 \text{ ; for each } k \in N \text{ and for all } n \geq n_0.$$

Remark 2.11. *When $f = f_n = g_k = \theta$, the zero function in the Definition 2.10, we get the definition of regular relative uniform null sequence of functions defined by Orlicz function M.*

Definition 2.12. *A double sequence of functions $(f_{nk}(x))$ defined on a compact domain $D \subseteq R$ is said to be relatively uniformly bounded if there exists a positive integer G such that*

$$|f_{nk}(x)| < G|\sigma(x)|, \text{ for all } x \in D, \text{ for all } n, k \in N.$$

Throughout we denote ${}_2\ell_\infty(M, ru)$, ${}_2c_0(M, ru)$, ${}_2c(M, ru)$, ${}_2c_0^R(M, ru)$, ${}_2c^R(M, ru)$ as the class of relatively uniformly bounded, relatively uniformly null, relatively uniformly convergent, regularly relatively uniformly null, regularly relatively uniformly convergent double sequence of functions respectively.

We define ${}_2c_0^B(M, ru) = {}_2c_0(M, ru) \cap {}_2\ell_\infty(M, ru)$; ${}_2c^B(M, ru) = {}_2c(M, ru) \cap {}_2\ell_\infty(M, ru)$.

3. Main results

Theorem 3.1. *The class of sequence of functions $Z(M, ru)$ is linear, for $Z = {}_2c_0, {}_2c, {}_2\ell_\infty, {}_2c_0^R, {}_2c^R, {}_2c_0^B, {}_2c^B$.*

Proof. Let $(f_{nk}(x)), (g_{nk}(x)) \in {}_2\ell_\infty(M, ru)$ and α, β be the scalars.

Since, $(f_{nk}(x)) \in {}_2\ell_\infty(M, ru)$ there exist $\rho_1 > 0$ and scale function $\sigma_1(x)$ on D such that for all $x \in D$,

$$\sup_{x \in D; n, k \geq 1} M\left(\frac{|f_{nk}(x)|}{\rho_1|\sigma_1(x)|}\right) < \infty. \tag{3.1}$$

Similarly, for all $x \in D$, there exist $\rho_2 > 0$ and scale function $\sigma_2(x)$ on D such that

$$\sup_{x \in D; n, k \geq 1} M\left(\frac{|g_{nk}(x)|}{\rho_2|\sigma_2(x)|}\right) < \infty. \tag{3.2}$$

Without loss of generality we can consider the same scale function $\sigma(x) = \max\{\sigma_1(x), \sigma_2(x)\}$ for $(f_{nk}(x))$ and $(g_{nk}(x))$.

Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Then, for all $x \in D$,

$$M\left(\frac{|(\alpha f_{nk}(x) + \beta g_{nk}(x))|}{\rho_3|\sigma(x)|}\right) \leq M\left(\frac{|\alpha|\ |f_{nk}(x)|}{\rho_3|\sigma(x)|}\right) + M\left(\frac{|\beta|\ |g_{nk}(x)|}{\rho_3|\sigma(x)|}\right)$$
$$\leq M\left(\frac{|f_{nk}(x)|}{\rho_1|\sigma_1(x)|}\right) + M\left(\frac{|g_{nk}(x)|}{\rho_2|\sigma_2(x)|}\right).$$

This implies, $\sup_{x\in D; n,k\geq 1} M\left(\frac{|(\alpha f_{nk}(x)+\beta g_{nk}(x))|}{\rho_3|\sigma(x)|}\right) \leq \sup_{x\in D; n,k\geq 1} M\left(\frac{|f_{nk}(x)|}{\rho_1|\sigma_1(x)|}\right) + \sup_{x\in D; n,k\geq 1} M\left(\frac{|g_{nk}(x)|}{\rho_2|\sigma_2(x)|}\right)$

$< \infty$.

This implies, $(\alpha f_{nk}(x) + \beta g_{nk}(x)) \in {}_2\ell_\infty(M, ru)$.

Hence, ${}_2\ell_\infty(M, ru)$ is a linear space.

Similarly, we can prove for the rest of the classes of sequence of functions. $\square$

Theorem 3.2. *The class of sequence of functions $Z(M, ru)$ is a normed space, for $Z = {}_2\ell_\infty$, ${}_2c_0^R$, ${}_2c^R$, ${}_2c_0^B$, ${}_2c^B$ normed by the following norm.*

For all $x \in D$,

$$||(f_{nk}(x))||_{(\sigma)} = \inf\left\{\rho > 0 : \sup_{x\neq\theta; ||x||\leq 1; n,k\geq 1} M\left(\frac{||f_{nk}(x)||}{\rho||x||\ ||\sigma(x)||}\right) \leq 1\right\}. \tag{3.3}$$

Proof. Let $(f_{nk}(x))$ and $(g_{nk}(x)) \in {}_2\ell_\infty(M, ru)$.

Evidently, $||(f_{nk}(x))||_{(\sigma)} = 0 \Rightarrow f_{nk}(x) = 0$.

Let $\rho = \rho_1 + \rho_2$ and ρ_1, ρ_2 are non-negative since, ρ is non-negative.

Without loss of generality, considering the same scale function $\sigma(x) = \max\{\sigma_1(x), \sigma_2(x)\}$ for $(f_{nk}(x))$ and $(g_{nk}(x))$, we have,

$$\sup_{x\in D; n,k\in N} M\left(\frac{|f_{nk}(x)|}{\rho|\sigma(x)|}\right) < \infty \text{ and } \sup_{x\in D; n,k\in N} M\left(\frac{|g_{nk}(x)|}{\rho|\sigma(x)|}\right) < \infty.$$

Then, we have, for all $x \in D$,

$$
\begin{aligned}
||(f_{nk}(x)+g_{nk}(x))||_{(\sigma)} &= \inf\left\{\rho>0:\sup_{x\neq\theta;||x||\leq 1;n,k\geq 1} M\left(\frac{||f_{nk}(x)+g_{nk}(x)||}{\rho||x||\;||\sigma(x)||}\right)\leq 1\right\}\\
&= \inf\left\{\rho_1,\rho_2>0:\sup_{x\neq\theta;||x||\leq 1;n,k\geq 1} M\left(\frac{||f_{nk}(x)+g_{nk}(x)||}{(\rho_1+\rho_2)||x||\;||\sigma(x)||}\right)\leq 1\right\}\\
&= \inf\left\{\rho_1,\rho_2>0:\sup_{x\neq\theta;||x||\leq 1;n,k\geq 1}\left\{\frac{\rho_1}{\rho_1+\rho_2}M\left(\frac{||f_{nk}(x)||}{\rho_1||x||\;||\sigma(x)||}\right)+\right.\right.\\
&\left.\left.\frac{\rho_2}{\rho_1+\rho_2}M\left(\frac{||g_{nk}(x)||}{\rho_2||x||\;||\sigma(x)||}\right)\right\}\leq 1\right\}\\
&\leq \inf\left\{\rho_1>0:\sup_{x\neq\theta;||x||\leq 1;n,k\geq 1} M\left(\frac{||f_{nk}(x)||}{\rho_1||x||\;||\sigma(x)||}\right)\leq 1\right\}+\\
&\inf\left\{\rho_2>0:\sup_{x\neq\theta;||x||\leq 1;n,k\geq 1} M\left(\frac{||g_{nk}(x)||}{\rho_2||x||\;||\sigma(x)||}\right)\leq 1\right\}\\
&\leq ||(f_{nk}(x))||_{(\sigma)}+||(g_{nk}(x))||_{(\sigma)}.
\end{aligned}
$$

For any scalar $\alpha>0$, $||(\alpha f_{nk}(x))||_{(\sigma)}=\inf\left\{\rho>0:\sup_{x\neq\theta;||x||\leq 1;n,k\geq 1} M\left(\frac{||\alpha f_{nk}(x)||}{\rho||x||\;||\sigma(x)||}\right)\leq 1\right\}$.

Let $r=\frac{\rho}{|\alpha|}$, then,

$||(\alpha f_{nk}(x))||_{(\sigma)}=\inf\left\{(|\alpha|r)>0:\sup_{x\neq\theta;||x||\leq 1;n,k\geq 1} M\left(\frac{||f_{nk}(x)||}{r||x||\;||\sigma(x)||}\right)\leq 1\right\}$.

$||(\alpha f_{nk}(x))||_{(\sigma)}=|\alpha|\inf\left\{r>0:\sup_{x\neq\theta;||x||\leq 1;n,k\geq 1} M\left(\frac{||f_{nk}(x)||}{r||x||\;||\sigma(x)||}\right)\leq 1\right\}$.

Hence proved. □

Theorem 3.3. *Let $(D,||.||)$ be a complete normed space. Then, the class of sequence of functions $Z(M,ru)$ is complete, for $Z={}_2\ell_\infty,\ {}_2c_0^R,\ {}_2c^R,\ {}_2c_0^B,\ {}_2c^B$.*

Proof. Let $(f^i(x))=(f^i_{nk}(x))$ be a relatively uniformly Cauchy double sequences of functions in ${}_2\ell_\infty(M,ru)$. Then, for a given $\varepsilon>0$, there exists $n_0\in N$ such that for all $x\in D$,

$$||f^i_{nk}(x)-f^j_{nk}(x)||_{(\sigma)}<\varepsilon,\ \text{for all } i,j\geq n_0.$$

We have, for all $x\in D$,

$$\inf\left\{\rho>0:\sup_{x\neq\theta;||x||\leq 1;n,k\geq 1} M\left(\frac{||f^i_{nk}(x)-f^j_{nk}(x))||}{\rho||x||\;||\sigma(x)||}\right)\leq 1\right\}\leq\varepsilon,\ \text{for all } i,j\geq n_0. \tag{3.4}$$

$\Rightarrow (f^i_{nk}(x))$ defined by Orlicz function M is relatively uniformly Cauchy on D for each $n,k\in N$.

$\Rightarrow (f^i_{nk}(x))$ defined by Orlicz function M is relatively uniformly convergent on D for each $n,k\in N$.

Therefore,

$$\lim_{i\to\infty} M\left(\frac{|f^i_{nk}(x)|}{\rho\;|\sigma(x)|}\right)=f_{nk}(x)$$

on D, for each $n, k \in N$.

From (3.4), for all $i \geq n_0$, taking limit when $j \to \infty$, we get,

$$\sup_{x \neq \theta; ||x|| \leq 1; n,k \geq 1} M\left(\frac{||f_{nk}^i(x) - f_{nk}(x))||}{\rho||x||\ ||\sigma(x)||}\right) \leq 1, \text{ for some } \rho > 0.$$

On taking infimum on ρ in the above expression by (3.4), we get, for all $i \geq n_0$,

$$\inf\left\{\rho > 0 : \sup_{x \neq \theta; ||x|| \leq 1; n,k \geq 1} M\left(\frac{||f_{nk}^i(x) - f_{nk}(x)||}{\rho||x||\ ||\sigma(x)||}\right) \leq 1\right\} \leq \varepsilon, \text{ for all } x \in D.$$

Hence, $(f_{nk}^i(x) - f_{nk}(x)) \in {}_2\ell_\infty(M, ru)$, for all $i \geq n_0$.

Since, ${}_2\ell_\infty(M, ru)$ is a linear space, for all $i \geq n_0$ and for all $x \in D$,

$$f_{nk}(x) = f_{nk}^i(x) - (f_{nk}^i(x) - f_{nk}(x)) \in {}_2\ell_\infty(M, ru).$$

Hence, ${}_2\ell_\infty(M, ru)$ is complete.

Similarly, we can prove for the rest of the cases. □

Result 3.1 *The class of sequence of functions $Z(M, ru)$ is not monotone and hence, not solid, for $Z = {}_2c_0,\ {}_2c,\ {}_2\ell_\infty,\ {}_2c_0^R,\ {}_2c^R,\ {}_2c_0^B,\ {}_2c^B$.*

The result follows from the example below.

Example 3.4. *Let $M(x) = x$, for all $x \in [0, \infty)$. Consider the double sequences of functions $(f_{nk}(x))$, $f_{nk} : [a, b] \to R, a \geq 0$ and $a < b; a, b \in R$ defined by*

$$f_{nk}(x) = \frac{nk}{nk + x}, \quad \textit{for all } n, k \in N.$$

Then, $(f_{nk}(x)) \in Z(M, ru)$, for $Z = {}_2c,\ {}_2\ell_\infty,\ {}_2c^R,\ {}_2c^B$.

Let $(g_{nk}(x))$ be the preimage of $(f_{nk}(x))$ and defined by

$$g_{nk}(x) = \begin{cases} f_{nk}(x), & \textit{for } n \textit{ is even;} \\ 0, & \textit{otherwise.} \end{cases}$$

This implies, $(g_{nk}(x)) \notin Z(M, ru)$, for $Z = {}_2c,\ {}_2\ell_\infty,\ {}_2c^R,\ {}_2c^B$.

Hence, $Z(M, ru)$ is not monotone and therefore, not solid, for $Z = {}_2c,\ {}_2\ell_\infty,\ {}_2c^R,\ {}_2c^B$.

Similarly, we can establish for the rest of the cases.

Result 3.2 *The class of sequence of functions $Z(M, ru)$ is not symmetric, for $Z = {}_2c_0,\ {}_2c,\ {}_2\ell_\infty,\ {}_2c_0^R,$ ${}_2c^R,\ {}_2c_0^B,\ {}_2c^B$.*
The result follows from the example below.

Example 3.5. *Let $M(x) = x$, for all $x \in [0, \infty)$. Consider the double sequences of functions $(f_{nk}(x))$, $f_{nk} : [a, b] \to R, a > 0$ and $a < b; a, b \in R$ defined by*

$$f_{nk}(x) = \frac{nkx}{nkx + 1}, \quad \textit{for all } n, k \in N.$$

This implies, $(f_{nk}(x)) \in Z(M, ru)$, *for* $Z = {}_2c,\ {}_2\ell_\infty,\ {}_2c^R,\ {}_2c^B$.

Let $(g_{nk}(x))$ *be the rearranged double sequences of functions of* $(f_{nk}(x))$ *defined by*

$$g_{nk}(x) = \begin{cases} \frac{nkx}{nkx+1}, & \text{for } k \text{ is even;} \\ \frac{n^2k^2x}{n^4k^4x+1}, & \text{otherwise.} \end{cases}$$

This implies, $(g_{nk}(x)) \notin Z(M, ru)$, *for* $Z = {}_2c,\ {}_2\ell_\infty,\ {}_2c^R,\ {}_2c^B$.

Hence, $Z(M, ru)$ *is not symmetric, for* $Z = {}_2c,\ {}_2\ell_\infty,\ {}_2c^R,\ {}_2c^B$.

Similarly, we can establish for the rest of the cases.

Result 3.3 *The class of sequence of functions* $Z(M, ru)$ *is not convergence free, for* $Z = {}_2c_0,\ {}_2c,\ {}_2\ell_\infty,\ {}_2c_0^R, {}_2c^R,\ {}_2c_0^B,\ {}_2c^B$.

The result follows from the following example.

Example 3.6. *Consider* $f_{nk} : [a, 1] \to R, g_{nk} : [a, 1] \to R, 0 < a < 1; a \in R$ *and* $M(x) = x^2$, *for all* $x \in [0, \infty)$ *defined by*

$f_{nk}(x) = \frac{1}{nkx}$, *and* $g_{nk}(x) = \frac{1}{nkx}$, *for* n *is even;*

$= \frac{nk+x}{nk}$, *otherwise.*

Then, $(f_{nk}(x)) \in Z(M, ru)$ *w.r.t the scale function* $\sigma(x) = \frac{1}{x^2}$, *for* $x \in [a, 1], 0 < a < 1$, *for which* $Z = {}_2c_0,\ {}_2c,\ {}_2\ell_\infty,\ {}_2c_0^R,\ {}_2c^R,\ {}_2c_0^B,\ {}_2c^B$ *but,* $(g_{nk}(x)) \notin Z(M, ru)$, *for* $Z = {}_2c_0,\ {}_2c,\ {}_2\ell_\infty,\ {}_2c_0^R,\ {}_2c^R,\ {}_2c_0^B,\ {}_2c^B$.
Hence, $Z(M, ru)$ *is not convergence free, for which* $Z = {}_2c_0,\ {}_2c, {}_2\ell_\infty,\ {}_2c_0^R,\ {}_2c^R,\ {}_2c_0^B,\ {}_2c^B$.

Theorem 3.7. *The class of sequence of functions* $Z(M, ru)$ *is sequence algebra, for* $Z = {}_2c_0,\ {}_2c,\ {}_2\ell_\infty,\ {}_2c_0^R,\ {}_2c^R,\ {}_2c_0^B,\ {}_2c^B$.

Proof. Let $(f_{nk}(x)), (g_{nk}(x)) \in {}_2c(M, ru)$.

Then, for every $\varepsilon > 0$ and for all $x \in D$, there exists an integer $n \geq n_0$ such that

$$M\left(\frac{|f_{nk}(x) - f(x)|}{\rho_1|\sigma_1(x)|}\right) \leq \varepsilon, \text{ for some } \rho_1 > 0.$$

Similarly, for all $x \in D$,

$$M\left(\frac{|g_{nk}(x) - g(x)|}{\rho_2|\sigma_2(x)|}\right) \leq \varepsilon, \text{ for some } \rho_2 > 0.$$

Without loss of generality, we can consider the same scale function $\sigma(x) = \max\{\sigma_1(x), \sigma_2(x)\}$, for $(f_{nk}(x))$ and $(g_{nk}(x))$.

Let $\rho_3 = \max\{\rho_1, \rho_2, \rho_1.\rho_2\}$.

By term wise addition and multiplication we can show that, for all $x \in D$,

$$M\left(\frac{|f_{nk}(x).g_{nk}(x) - h(x)|}{\rho_3|\sigma(x)|}\right) \leq \varepsilon, \text{ for some } \rho_3 > 0.$$

Hence, ${}_2c(M, ru)$ is sequence algebra.

Similarly, we can establish for the rest of the cases.

The proof of the following theorem is a routine verification and hence omitted. $\square$

Theorem 3.8. *The inclusion relation $Z(M,ru) \subset {}_2\ell_\infty(M,ru)$ strictly holds, for $Z = {}_2c_0^R,\ {}_2c^R,\ {}_2c_0^B,\ {}_2c^B$.*

The inclusions are strict follows from the example below.

Example 3.9. *Consider $f_{nk} : [a,1] \to R, 0 < a < 1; a \in R$ and $M(x) = x$, for all $x \in [0,\infty)$ defined by*

$f_{nk}(x) = \frac{1}{1+x^2}$, *for n, k both are odd and even;*

$= 0$, *otherwise.*

We get $(f_{nk}(x)) \in {}_2\ell_\infty(M,ru)$ w.r.t. the constant scale function $\sigma(x) = 1$, for all $x \in [a,1]$ but, $(f_{nk}(x)) \notin Z(M,ru)$, for $Z = {}_2c_0^R,\ {}_2c^R,\ {}_2c_0^B,\ {}_2c^B$.

In view of the Theorem 3.3 and Theorem 3.8, we formulate the following theorem without proof.

Theorem 3.10. *The classes of sequence of functions ${}_2c_0^B(M,ru),\ {}_2c^B(M,ru),\ {}_2c_0^R(M,ru),\ {}_2c^R(M,ru)$ are nowhere dense subsets of ${}_2\ell_\infty(M,ru)$.*

Theorem 3.11. *Let M, M_1 be two Orlicz functions that satisfy the Δ_2-condition. Then,*

1. *$Z(M_1,ru) \subseteq Z(M.M_1,ru)$,*

2. *$Z(M,ru) \cap Z(M_1,ru) \subseteq Z(M+M_1,ru)$, for $Z = {}_2c_0,\ {}_2c,\ {}_2\ell_\infty,\ {}_2c^R,\ {}_2c_0^R,\ {}_2c_0^B,\ {}_2c^B$.*

Proof. (1) Let $(f_{nk}(x)) \in {}_2\ell_\infty(M_1,ru)$. Then, there exist $\rho > 0$ and a scale function $\sigma(x)$ such that for all $x \in D$,

$$\sup_{x \in D; n,k \geq 1} M_1\left(\frac{|f_{nk}(x)|}{\rho|\sigma(x)|}\right) < \infty. \tag{3.5}$$

Let $0 < \varepsilon < 1$, and choose δ, with $0 < \delta < 1$ such that $M(x) < \varepsilon$, for $0 \leq x < \delta$.

Let, $g_{nk}(x) = M_1\left(\frac{|f_{nk}(x)|}{\rho|\sigma(x)|}\right)$ and consider

$$M(g_{nk}(x)) = M^{'}(g_{nk}(x)) + M^{''}(g_{nk}(x)),$$

where $M^{'}(g_{nk}(x))$ is $M(g_{nk}(x)),$ when $g_{nk}(x) \leq \delta,$ and $M^{''}(g_{nk}(x))$ is $M(g_{nk}(x)),$ when $g_{nk}(x) > \delta$.

By the Remark 1.2, for $g_{nk}(x) \leq \delta$,

$$M(g_{nk}(x)) \leq M(1)g_{nk}(x) \leq M(2)g_{nk}(x) \leq M(3)g_{nk}(x). \tag{3.6}$$

For $g_{nk}(x) > \delta$, we have

$$g_{nk}(x) < \frac{g_{nk}(x)}{\delta} \leq 1 + \frac{2g_{nk}(x)}{\delta}.$$

Since M is an Orlicz function, it follows that

$$M(g_{nk}(x)) < M\left(1 + 2\frac{g_{nk}(x)}{\delta}\right) < \frac{1}{3}M(3) + \frac{2}{3}M\left(\frac{3g_{nk}(x)}{\delta}\right).$$

Since M satisfies Δ_2- condition, we have

$$M(g_{nk}(x)) < \frac{1}{3}K\frac{g_{nk}(x)}{\delta}M(3) + \frac{2}{3}K\frac{g_{nk}(x)}{\delta}M(3) = K\frac{g_{nk}(x)}{\delta}M(3).$$

Hence

$$M^{''}(g_{nk}(x)) \leq \max(1, K\delta^{-1}M^{''}(3))g_{nk}(x). \tag{3.7}$$

From Equations (3.5), (3.6) and (3.7), we have,

$$(f_{nk}(x)) \in \ _2\ell_\infty(M.M_1, ru).$$

Thus, $_2\ell_\infty(M_1, ru) \subseteq \ _2\ell_\infty(M.M_1, ru)$.

Similarly, other cases can be established.

(2) Suppose $(f_{nk}(x)) \in \ _2\ell_\infty(M, ru) \cap \ _2\ell_\infty(M_1, ru)$. Then, there exist $\rho > 0$ and scale function $\sigma(x)$ such that for all $x \in D$,

$$\sup_{x \in D; n,k \geq 1} M\left(\frac{|f_{nk}(x)|}{\rho|\sigma(x)|}\right) < \infty, \text{ and } \sup_{x \in D; n,k \geq 1} M_1\left(\frac{|f_{nk}(x)|}{\rho|\sigma(x)|}\right) < \infty.$$

Then,

$$\sup_{x \in D; n,k \geq 1}(M + M_1)\left(\frac{|f_{nk}(x)|}{\rho|\sigma(x)|}\right) = \sup_{x \in D; n,k \geq 1} M\left(\frac{|f_{nk}(x)|}{\rho|\sigma(x)|}\right) + \sup_{x \in D; n,k \geq 1} M_1\left(\frac{|f_{nk}(x)|}{\rho|\sigma(x)|}\right)$$

$< \infty$.

This implies, $(f_{nk}(x)) \in \ _2\ell_\infty(M + M_1, ru)$.

Hence, $_2\ell_\infty(M, ru) \cap \ _2\ell_\infty(M_1, ru) \subseteq \ _2\ell_\infty(M + M_1, ru)$.

Similarly, the rest of the cases can be established.

Remark 3.12. *In Theorem 3.11(1) Δ_2- condition is necessary because we cannot consider the inequality $g_{nk}(x) < \delta$ without the Δ_2- condition and hence, we cannot obtain the result.*

On taking $M_1(x) = x$ in Theorem 3.11(1), we get the following result. □

Corollary 3.13. *Let M be an Orlicz function that satisfy the Δ_2-condition. Then,*

$$Z(ru) \subseteq Z(M, ru), \text{ for } Z = \ _2c_0, \ _2c, \ _2\ell_\infty, \ _2c^R, \ _2c_0^R, \ _2c_0^B, \ _2c^B.$$

4. Conclusions

In this article, we studied about the notion of relative uniform convergence of double sequences of functions defined by using Orlicz function M w.r.t. a scale function $\sigma(x)$. We defined the classes of double sequences of functions $_2c_0(M, ru)$, $_2c(M, ru)$, $_2\ell_\infty(M, ru)$, $_2c_0^B(M, ru)$, $_2c^B(M, ru)$, $_2c_0^R(M, ru)$, $_2c^R(M, ru)$ and studied their properties like solid, monotone, symmetric, convergence free, denseness, sequence algebra. We have also investigated the properties of the above sequence spaces when Δ_2- condition is satisfied.

References

1. Y. Altin, M. Et and B. C. Tripathy, *The sequence space* $|\bar{N}_p|(M, r, q, s)$ *on seminormed spaces.* Applied Math. Comput. 154, 423-430, (2004).
2. T. J. IA. Bromwich, *An Introduction to the Theory of Infinite Series.* Macmillan & Co. Ltd., Newyork, (1965).
3. E. W. Chittenden, *Relatively uniform convergence of sequences of functions.* Trans. Amer. Math. Soc. 15, 197-201, (1914).
4. K. Demirci, A. Boccuto, S. Yıldız and F. Dirik, *Relative uniform convergence of a sequence of functions at a point and Korovkin-type approximation theorems.* Positivity 24(10), DOI: 10.1007/s11117-019-00656-6.
5. K. Demirci and S. Orhan, *Statistically relatively uniform convergence of positive linear operators.* Results Math. 69, 359-367, (2016).
6. K. R. Devi and B. C. Tripathy, *Relative uniform convergence of difference double sequence of positive linear functions*, Ric. Mat. (2021) DOI:10.1007/s11587-021-00613-0(online published).
7. K. R. Devi and B. C. Tripathy, *Relative uniform convergence of difference sequence of positive linear functions.* Trans. A. Razmadze Math. Inst. 176(1), 37-43, (2022).
8. K. R. Devi and B. C. Tripathy, *On relative uniform convergence of double sequences of functions.* Proc. Natl. Acad. Sci., India, Sect. A Phys. Sci (2022), DOI: 10.1007/s40010-022-00768-x (online published).
9. M. Et, Y. Altin, B. Choudhary and B. C. Tripathy, *On some classes of sequences defined by sequence of Orlicz functions.* Math. Ineq. Appl. 9(2), 335-342, (2006).
10. G. H. Hardy, *On the convergence of certain multiple series.* Proc. Camb. Philos. Soc. 19, 86-95, (1917).
11. J. Lindenstrauss and L. Tzafriri, *On Orlicz sequence spaces.* Israel J. Math. 10, 379-390, (1971).
12. S. A. Mohiuddine, B. Hazarika and Mohammed A. Alghamdi, *Ideal relatively uniform convergence with Korovkin and Voronoskaya type approximation theorems.* Filomat 33(14), 4549-4560, (2019).
13. F. Moričz, *Extension of spaces* c *and* c_o *from single to double sequences.* Acta Math. Hungar. 57(1-2), 129-136, (1991).
14. P. K. Nath and B. C. Tripathy, *Convergent complex uncertain sequences defined by Orlicz function.* Ann. Univ. Craiova Math. Comput. Sci. Ser. 46(1), 139-149, (2019).
15. P. K. Nath and B. C. Tripathy, *Statistical convergence of complex uncertain sequences defined by Orlicz function.* Proyecciones J. Math. 39(2), 301-315, (2020).
16. A. Pringsheim, *Zur Ttheorie der zweifach unendlichen Zahlenfolgen.* Math. Ann. 53, 289-321, (1900).
17. B. C. Tripathy and S. Mahanta, *On a class of sequences related to the* ℓ^p *space defined by Orlicz functions.* Soochow J. Math. 29(4), 379-391, (2003).
18. B. C. Tripathy and S. Mahanta, *On a class of generalized lacunary difference sequence spaces defined by Orlicz function.* Acta Math. Appl. Sin. Engl. Ser. 20(2), 231-238, (2004).
19. B. C. Tripathy and S. Mahanta, *On a class of difference sequences related to the* ℓ^p *space defined by Orlicz functions.* Math. Slovaca, 57(2), 171-178, (2007).
20. B. C. Tripathy and B. Sarma, *Vector valued paranormed statistically convergent double sequence spaces*, Math. Slovaca 57(2), 179-188, (2007).
21. B. C. Tripathy, Y. Altin and M. Et, *Generalized difference sequences spaces on seminormed spaces defined by Orlicz functions.* Math. Slovaca 58(3), 315-324, (2008).
22. B. C. Tripathy and B. Sarma, *Vector valued double sequence spaces defined by Orlicz function.* Math. Slovaca 59(6), 767-776, (2009).
23. B. C. Tripathy and H. Dutta, *On some new paranormed difference sequence spaces defined by Orlicz functions.*Kyungpook J. Math. 50(1), 59-69, (2010).
24. B. C. Tripathy and B. Sarma, *Double sequence spaces of fuzzy numbers defined by Orlicz function*, Acta Math. Sci. 31B(1), 134-140, (2011).
25. B. C. Tripathy and H. Dutta, *On some lacunary difference sequence spaces defined by a sequence of Orlicz functions and q-lacunary* Δ_n^m *-statistical convergence.* An. Stiintt. Univ. “Ovidius” Constanta Ser. Mat. 20(1), 417-430, (2012).

Kshetrimayum Renubebeta Devi,
Department of Mathematics,
Tripura University, Agartala - 799022; Tripura, INDIA.
E-mail address: `renu.ksh11@gmail.com`

and

Binod Chandra Tripathy,
Department of Mathematics,
Tripura University, Agartala - 799022; Tripura, INDIA.
E-mail address: `tripathybc@yahoo.com and binodtripathy@tripurauniv.ac.in`