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Weak Structures on Texture Spaces and Weak Semiopen Sets

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ABSTRACT: The purpose of this paper is to introduce and study weak structure on texture spaces. In this context, the notion of weak semiopen sets and weak semibicontinuity are defined in weak distructure texture spaces, and is presented some characterization.

Key Words: Texture, fuzzy set, weak structure, weak semiopen, weak semibicontinuity.

Contents

1. Introduction

In mathematics, weaker structures than topological spaces offer more fruitfull environments for application. In this context, weak structures that are without the union axiom in the topological space have been the field of interest of mathematicians in recent years. Recall that [1] a collection $\mathcal{B}$ of subsets of a set X is said to be a base on X iff $\emptyset, X \in \mathcal{B}$ and for all $A, B \in \mathcal{B}$ we have $A \cap B \in \mathcal{B}$. Then $(X, \mathcal{B})$ is called a base space. Further, the concept base $\mathcal{B}$ was defined as a weak structure in [17], and $(X, \mathcal{B})$ was called weak space.

Texture spaces were introduced by L. M. Brown as a point-set setting for the study of fuzzy sets [3]. Ditopologies [2] on textures unify the fuzzy topologies and topologies and in a non-complemented setting by means of duality in the textural concepts [5]. The theory of texture spaces continues its development through various fields such as generalized sets, rough set theory and selection principles [6,7,8,9,10,11,12,13,14,15,16]. Recently, the notion of base space leads to the analogous concept for texture spaces, and di-base texture space was introduced in [6]. Further, the relationship between base spaces and di-base spaces were given in the categorical context.

In this study, we shall use the terminology *diweak texture space* instead of di-base texture space, and we shall study generalized weak open sets in diweak texture spaces.

In the next section, we shall briefly the basic motivation and its study for texture spaces. For more details, we refer to [2,3,5].

2. Texture spaces and ditopology

Definition 2.1. *Let U be a set. A* texturing $\mathcal{U}$ *of U is a subset of $\mathcal{P}(U)$ which is a point-separating, complete, completely distributive lattice containing U and $\emptyset$, and for which meet coincides with intersection and finite joins with union. The pair $(U, \mathcal{U})$ is then called a* texture space*, or shortly* texture.

For $u \in U$, the *p-sets* and the *q-sets* are defined by

$$P_u = \bigcap\{A \in \mathcal{U} \mid u \in A\}, \quad Q_u = \bigvee\{A \in \mathcal{U} \mid u \notin A\}, \quad \text{respectively.}$$

In general a texturing of U need not be closed under set complementation, but it may be that there

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exists a mapping $\sigma : \mathcal{U} \to \mathcal{U}$ satisfying $\sigma(\sigma(A)) = A, \ \forall A \in \mathcal{U}$ and $A \subseteq B \implies \sigma(B) \subseteq \sigma(A), \forall A, B \in \mathcal{U}$. In this case σ is called a *complementation* on $(U, \mathcal{U})$, and $(U, \mathcal{U}, \sigma)$ is said to be a *complemented texture.*

Example 2.2. *(1) For any set U, $(U, \mathcal{P}(U), c_U)$ is the complemented discrete texture representing the usual set structure of X. Here the complementation $c_U(A) = U \setminus A$, $A \subseteq U$, is the usual set complement. Clearly, $P_u = \{u\}$ and $Q_u = U \setminus \{u\}$ for all $u \in U$.*

(2) Let $L = (0, 1]$, $\mathcal{L} = \{(0, r] \mid r \in [0, 1]\}$ and $\lambda((0, r]) = (0, 1 - r]$, $r \in [0, 1]$. Then $(L, \mathcal{L}, \lambda)$ is complemented texture space. Here $P_r = Q_r = (0, r]$ for all $r \in L$. $(L, \mathcal{L})$ is said to be Hutton texture.

(3) For $\mathbb{I} = [0, 1]$ define $\mathcal{I} = \{[0, t] \mid t \in [0, 1]\} \cup \{[0, t) \mid t \in [0, 1]\}$, $\iota([0, t]) = [0, 1-t)$ and $\iota([0, t)) = [0, 1-t]$, $t \in [0, 1]$. Again $(\mathbb{I}, \mathcal{I}, \iota)$ is a complemented texture, which is called unit interval texture. Here $P_t = [0, t]$ and $Q_t = [0, t)$ for all $t \in I$.

Definition 2.3. *A* ditopology *on a texture $(U, \mathcal{U})$ is a pair (τ, κ) of subsets of $\mathcal{U}$ where the set of open sets τ and the set of closed sets κ satisfy*

$$\begin{aligned} U, \emptyset \in \tau, &\qquad U, \emptyset \in \kappa \\ G_1, G_2 \in \tau \implies G_1 \cap G_2 \in \tau, &\qquad K_1, K_2 \in \kappa \implies K_1 \cup K_2 \in \kappa \\ G_i \in \tau, i \in I \implies \bigvee_{i \in I} G_i \in \tau, &\qquad K_i \in \kappa, i \in I \implies \bigcap_{i \in I} K_i \in \kappa. \end{aligned}$$

Hence a ditopology is essentially a "topology" for which there is no *a priori* relation between the open and closed sets. If (τ, κ) is a ditopology on $(U, \mathcal{U})$ then $(S, \mathcal{S}, \tau, \kappa)$ is called ditopological texture space or shortly, ditopological space.

Difunctions arise often in the study of textures and ditopological texture spaces. A difunction is a direlation [5] (f, F) satisfying certain additional conditions.

Difunctions: Let (f, F) be a direlation from $(U, \mathcal{U})$ to $(V, \mathcal{V})$. Then (f, F) is called a *difunction from* $(U, \mathcal{U})$ *to* $(V, \mathcal{V})$ if it satisfies the following two conditions.

DF1 For $u, u' \in U$, $P_u \not\subseteq Q_{u'} \implies \exists v \in V$ with $f \not\subseteq \overline{Q}_{(u,v)}$ and $\overline{P}_{(u',v)} \not\subseteq F$.

DF2 For $v, v' \in T$ and $u \in U$, $f \not\subseteq \overline{Q}_{(u,v)}$ and $\overline{P}_{(u,v')} \not\subseteq F \implies P_{v'} \not\subseteq Q_v$.

Image and Inverse Image: Let $(f, F) : (U, \mathcal{U}) \to (V, \mathcal{V})$ be a difunction.

1. For $A \in \mathcal{U}$, the *image* $f^{\to}A$ and the *co-image* $F^{\to}A$ are defined by

$$\begin{aligned} f^{\to}A &= \bigcap \{Q_v \mid \forall u, \ f \not\subseteq \overline{Q}_{(u,v)} \implies A \subseteq Q_u\}, \\ F^{\to}A &= \bigvee \{P_v \mid \forall u, \ \overline{P}_{(u,v)} \not\subseteq F \implies P_u \subseteq A\}. \end{aligned}$$

2. For $B \in \mathcal{V}$, the *inverse image* $f^{\leftarrow}B$ and the *inverse co-image* $F^{\leftarrow}B$ are defined by

$$\begin{aligned} f^{\leftarrow}B &= \bigvee \{P_u \mid \forall v, \ f \not\subseteq \overline{Q}_{(u,v)} \implies P_v \subseteq B\}, \\ F^{\leftarrow}B &= \bigcap \{Q_u \mid \forall v, \ \overline{P}_{(u,v)} \not\subseteq F \implies B \subseteq Q_v\}. \end{aligned}$$

For a difunction, the inverse image and the inverse co-image are equal, but the image and co-image are usually not.

3. Revisited weak structures in texture spaces

First, let's recall the concept of weak structure in texture spaces, previously called base and co-base in [6].

Definition 3.1. *A weak-distructure on a texture $(U, \mathcal{U})$ is a pair $(\mathcal{W}, c\mathcal{W})$ of subsets of $\mathcal{U}$, where the set of weak-open, or shortly w-open, sets $\mathcal{W}$ satisfies*

(1) $U,\, \emptyset \in \mathcal{W}$,

(2) $G_1,\, G_2 \in \mathcal{W} \implies G_1 \cap G_2 \in \mathcal{W}$.

and the set of weak-closed, or shortly w-closed, sets $c\mathcal{W}$ satisfies

(1) $U,\, \emptyset \in c\mathcal{W}$,

(2) $K_1,\, K_2 \in c\mathcal{W} \implies K_1 \cup K_2 \in c\mathcal{W}$.

Hence a weak-distructure is essentially a "weak structure [17]" for which there is no a priori relation between the weak-open and weak-closed sets. If $(\mathcal{W}, c\mathcal{W})$ is a weak-distructure on $(U, \mathcal{U})$ then $(U, \mathcal{U}, \mathcal{W}, c\mathcal{W})$ is called diweak texture space or shortly, diw-texture space.

We denote by $\mathcal{W}O(U, \mathcal{U}, \mathcal{W}, c\mathcal{W})$, or when there can be no confusion by $\mathcal{W}O(U)$, the set of w-open sets in $\mathcal{U}$. Likewise, $\mathcal{W}C(U, \mathcal{U}, \mathcal{W}, c\mathcal{W})$ or $\mathcal{W}C(U)$ will denote the set of w-closed sets.

Definition 3.2. *Let $(U, \mathcal{U}, \mathcal{W}, c\mathcal{W})$ be a diw-texture space and $A \in \mathcal{U}$. Then the weak-closure of A and the weak-interior of A are defined as follows:*

(a) $cl_w\,(A) = \bigcap\{K \in c\mathcal{W} \mid A \subseteq K\}$

(b) $int_w\,(A) = \bigvee\{G \in \mathcal{W} \mid G \subseteq A\}$.

Clearly, $A \subseteq cl_w\,(A)$ and $int_w\,(A) \subseteq A$ by above definition.

If $(\mathcal{W}, c\mathcal{W})$ is a weak-distructure on a complemented texture $(U, \mathcal{U}, \sigma)$ we say $(\mathcal{W}, c\mathcal{W})$ is complemented if $c\mathcal{W} = \sigma(\mathcal{W})$. In this case we have $\sigma(cl_w\,(A)) = int_w(\sigma(A))$ and $\sigma(int_w(A)) = cl_w(\sigma(A))$.

Example 3.3. *(1) For any texture $(U, \mathcal{U})$ a weak-distructure $(\mathcal{W}, c\mathcal{W})$ with $\mathcal{W} = \mathcal{U}$ is called discrete, and one with $c\mathcal{W} = \mathcal{U}$ is called codiscrete.*

(2) For any texture $(U, \mathcal{U})$ a weak-distructure $(\mathcal{W}, c\mathcal{W})$ with $\mathcal{W} = \{\emptyset, U\}$ is called indiscrete, and one with $c\mathcal{W} = \{\emptyset, U\}$ is called co-indiscrete.

(3) For any weak structure w on X (in the sense [17]), (w, w^c), $w^c = \{X \setminus G \mid G \in w\}$, is a complemented weak-distructure on the discrete texture $(X, powX, c_X)$.

(4) Consider unit interval texture $(\mathbb{I}, \mathcal{I}, \iota)$. Then $\mathcal{W}_{\mathbb{I}} = \{[0, r) \mid 0 \leq r \leq 1\} \cup \{\mathbb{I}\}$, $c\mathcal{W}_{\mathbb{I}} = \{[0, r] \mid 0 \leq r \leq 1\} \cup \{\emptyset\}$ defines a complemented weak-distructure, called the natural weak-distructure on $(\mathbb{I}, \mathcal{I}, \iota)$.

(5) Let (τ, κ) be a ditopology on the texture $(U, \mathcal{U})$. Then, automatically, the pair (τ, κ) is a weak-distructure on $(U, \mathcal{U})$.

Recall that [17] a function $f : (X, w_X) \to (Y, w_Y)$ between weak spaces is called weak continuous iff for all $B \in w_Y$ we have $f^{-1}(B) \in w_X$. This leads to the following analogous concepts for texture spaces [6].

Definition 3.4. *Let $(U_j, \mathcal{U}_j, \mathcal{W}_j, c\mathcal{W}_j)$, $j = 1, 2$, be a diw-texture spaces. Then the difunction $(f, F) : (U_1, \mathcal{U}_1) \to (U_2, \mathcal{U}_2)$ is called*

1. *w-continuous if $F^{\leftarrow}(G) \in \mathcal{W}O(U_1)$ for every $G \in \mathcal{W}O(U_2)$.*
2. *w-cocontinuous if $f^{\leftarrow}(K) \in \mathcal{W}C(U_1)$ for every $K \in \mathcal{W}C(U_2)$.*
3. *w-bicontinuous if it is w-continuous and w-cocontinuous.*

4. Semiopen sets in diweak texture spaces

Recall that [18] that a subset A of a weak space (X, w_X) is called semi-open if $A \subseteq cl_w(int_w((A))$. This leads to the following analogous concepts in a diweak texture spaces.

Definition 4.1. *Let $(U, \mathcal{U}, \mathcal{W}, c\mathcal{W})$ be a diw-texture space. A set $A \in \mathcal{U}$ is called*

1. *weak semiopen, or shortly w-semiopen, if $A \subseteq cl_w\, int_w\,(A)$, and*
2. *weak semiclosed, or shortly w-semiclosed, if $int_w\, cl_w\,(A) \subseteq A$.*

We denote by $\mathcal{WSO}(U, \mathcal{U}, \mathcal{W}, c\mathcal{W})$, or when there can be no confusion by $\mathcal{WSO}(U)$, the set of w-semiopen sets in $\mathcal{U}$. Likewise, $\mathcal{WSC}(U, \mathcal{U}, \mathcal{W}, c\mathcal{W})$, or $\mathcal{WSC}(U)$ will denote the set of w-semiclosed sets.

Proposition 4.2. *Let $(U, \mathcal{U}, \mathcal{W}, c\mathcal{W})$ be a diw-texture space.*

(i) $A \in \mathcal{U}$ is w-semiopen if and only if there exists a set $G \in \mathcal{WO}(U)$ such that $G \subseteq A \subseteq cl_w\,(G)$.

(ii) $B \in \mathcal{U}$ is w-semiclosed if and only if there exists a set $K \in \mathcal{WC}(U)$ such that $int_w\,(K) \subseteq B \subseteq K$.

Proof: The proofs are elementary and are omitted. □

Lemma 4.3. *For a given diw-texture space $(U, \mathcal{U}, \mathcal{W}, c\mathcal{W})$:*

(i) $\mathcal{WO}(U) \subseteq \mathcal{WSO}(U)$ *and* $\mathcal{WC}(U) \subseteq \mathcal{WSC}(U)$.

(ii) *Arbitrary join of w-semiopen sets is w-semiopen.*

(iii) *Arbitrary intersection of weak semiclosed sets is w-semiclosed.*

Proof: (i) Let $G \in \mathcal{WO}(U)$. Then $G = int_w(G)$, and so $cl_w(G) = cl_w(int_w(G))$. Since $G \subseteq cl_w(G)$, we have $G \subseteq cl_w(int_w(G))$. That is, $G \in \mathcal{WSO}(U)$. Likewise, it is proved that $\mathcal{WC}(U) \subseteq \mathcal{WSC}(U)$.

(ii) Let $\{G_j\}$, $j \in J$, be a family of w-semiopen sets. Then $G_j \subseteq cl_w(int_w(G_j)) \subseteq cl_w(int_w(\bigvee G_j))$. Thus, $\bigvee G_j \subseteq cl_w(int_w(\bigvee G_j))$, and so $\bigvee G_j$ is w-semiopen.

(iii) Suppose that Let $\{K_j\}$, $j \in J$ be a family of w-semiclosed sets. Then

$$int_w(cl_w(\bigcap K_j)) \subseteq int_w(cl_w(K_j)) \subseteq K_j$$

. Then $int_w(cl_w(\bigcap K_j)) \subseteq \bigcap K_j$, and so $\bigcap K_j$ is w-semiclosed. □

The next results are obvious for a diweak texture space $(U, \mathcal{U}, \mathcal{W}, c\mathcal{W})$. Let $A, B \in \mathcal{U}$:

$$A \in \mathcal{WSO}(U),\ A \subseteq B \subseteq cl_w(A) \implies B \in \mathcal{WSO}(U),$$

$$A \in \mathcal{WSC}(U),\ int_w(A)\ \subseteq B \subseteq A \implies B \in \mathcal{WSC}(U).$$

In general, there is no relation between the w-semiopen and w-semiclosed sets, but for a complemented diw-texture space we have:

Proposition 4.4. *For a complemented diw-texture space $(U, \mathcal{U}, \sigma, \mathcal{W}, c\mathcal{W})$:*

$$A \in \mathcal{U} \quad \text{is weak semiopen if and only if} \quad \sigma(A) \text{ is weak semiclosed.}$$

Proof: For $A \in \mathcal{U}$, since $\sigma(int_w\,(A)) = cl_w(\sigma(A))$ and $\sigma(cl_w\,(A)) = int_w(\sigma(A))$, the proof is trivial. □

Example 4.5. *(1) If (X, w) is a weak space then $(X, \mathcal{P}(X), c_X, w, w^c)$ is a complemented diw-texture space (see Examples 3.3 (3)) Clearly, the weak semiopen and weak semiclosed sets in (X, w) correspond precisely to the w-semiopen and w-semiclosed respectively, in $(X, \mathcal{P}(X), c_X, w, w^c)$.*

(2) For the unit interval complemented diw-texture space $(\mathbb{I}, \mathcal{I}, \iota, \mathcal{W}_{\mathbb{I}}, c\mathcal{W}_{\mathbb{I}})$ of Examples 3.3 (4), we have $\mathcal{W}SO(\mathbb{I}) = \mathcal{W}SC(\mathbb{I}) = \mathcal{I}$.

Definition 4.6. *Let $(U, \mathcal{U}, \mathcal{W}, c\mathcal{W})$ be a diw-texture space and $A \in \mathcal{U}$. We define:*

1. *The weak semiclosure $scl_w(A)$ of A under $(\mathcal{W}, c\mathcal{W})$ by the equality*

$$scl_w(A) = \bigcap\{B \mid B \in \mathcal{W}SC(U) \text{ and } A \subseteq B\}$$

and

2. *The weak semi-interior $sint_w(A)$ of A under $(\mathcal{W}, c\mathcal{W})$ by the equality*

$$sint_w(A) = \bigvee\{B \mid B \in \mathcal{W}SO(U) \text{ and } B \subseteq A\}.$$

Note that, by Lemma 4.3, we have $scl_w(A) \in \mathcal{W}SC(U)$ and $sint_w(A) \in \mathcal{W}SO(U)$, while

$$A \in \mathcal{W}SC(U) \iff A = scl_w(A)$$

and

$$A \in \mathcal{W}SO(U) \iff A = sint_w(A).$$

Obviously, $scl_w(A)$ is w-semiclosed set which contains A and $sint_w(A)$ is the greatest w-semiopen set which is contained in A, and we have $A \subseteq scl_w(A) \subseteq cl_w(A)$ and $int_w(A) \subseteq sint_w(A) \subseteq A$.

Proposition 4.7. *Let $(U, \mathcal{U}, \mathcal{W}, c\mathcal{W})$ be a diw-texture space and $A \in \mathcal{U}$.*

1. *$scl_w(A) = A \cup int_w\, cl_w(A)$.*
2. *$sint_w(A) = A \cap cl_w\, int_w(A)$.*

Proof: We prove (2), leaving the essentially dual proof of (1) to the interested reader.
Since $sint_w(A)$ is w-semiopen, we have $sint_w(A) \subseteq cl_w\, int_w(sint_w(A))$. Therefore, $sint_w(A) \subseteq cl_w\, int_w(A)$, and so $sint_w(A) \subseteq (A \cap cl_w\, int_w(A))$. To obtain the opposite inclusion we observe that $int_w(A) \subseteq (A \cap cl_w\, \mathrm{int}_w(A))$ and $int_w(A) \subseteq cl_w\, int_w(A \cap cl_w\, int_w(A))$ and $(A \cap cl_w\, int_w(A)) \subseteq cl_w\, int_w(A) \subseteq cl_w\, int_w(A \cap cl_w\, int_w(A))$. Hence $A \cap cl_w\, int_w(A)$ is w-semiopen, and so $(A \cap cl_w\, int_w(A)) \subseteq sint_w(A)$. □

Now, we recall that a function between weak spaces is called weak semicontinuous [18] if the inverse image of each weak open set is weak semiopen. This leads to the following concepts for a difunction between diweak texture spaces.

Definition 4.8. *Let $(U_j, \mathcal{U}_j, \mathcal{W}_j, c\mathcal{W}_j)$, $j = 1, 2$, be a diw-texture spaces. Then the difunction $(f, F) : (U_1, \mathcal{U}_1) \to (U_2, \mathcal{U}_2)$ is called*

1. *w-semicontinuous if $F^{\leftarrow}(G) \in \mathcal{W}SO(U_1)$ for every $G \in \mathcal{W}O(U_2)$.*
2. *w-semicocontinuous if $F^{\leftarrow}(K) \in \mathcal{W}SC(U_1)$ for every $K \in \mathcal{W}C(U_2)$.*
3. *w-semibicontinuous if it is w-semicontinuous and w-semicocontinuous.*

Since every w-open (w-closed) set is w-semiopen (w-semiclosed) set, every w-bicontinuous difunction is w-semibicontinuous.

Remark 4.9. *Recall that [5] if* $f : X \to Y$ *is a point function then* (f, f') *is a difunction from* $(X, \mathcal{P}(X))$ *to* $(Y, \mathcal{P}(Y))$ *where* $f' = (X \times Y) \setminus f$*. Conversely, if* (f, F) *is a difunction from* $(X, \mathcal{P}(X))$ *to* $(Y, \mathcal{P}(Y))$ *then* $F = (X \times Y) \setminus f$ *and* $F^{\leftarrow} = f^{-1}$*.*

f *is weak semicontinuous point function from* (X, w_X) *to* (Y, w_Y) *if and only if* (f, f') *is* w*-semibicontinuous difunction from* $(X, \mathcal{P}(X), w_X, {w_X}^c)$ *to* $(Y, \mathcal{P}(Y), w_Y, {w_Y}^c)$*.*

Proposition 4.10. *Let* $(U_j, \mathcal{U}_j, \mathcal{W}_j, c\mathcal{W}_j)$*,* $j = 1, 2$*, be a diweak texture spaces and Then* $(f, F) : (U_1, \mathcal{U}_1) \to (U_2, \mathcal{U}_2)$ *be a difunction.*

1. *The following are equivalent:*

   (a) (f, F) *is* w*-semicontinuous.*

   (b) $int_w(F^{\to}(A)) \subseteq F^{\to}(sint_w(A))$, $\forall\, A \in \mathcal{U}_1$.

   (c) $f^{\leftarrow}(int_w(B)) \subseteq sint_w(f^{\leftarrow}(B))$, $\forall\, B \in \mathcal{U}_2$.

2. *The following are equivalent:*

   (a) (f, F) *is* w*-semicocontinuous.*

   (b) $f^{\to}(scl_w(A)) \subseteq cl_w(f^{\to}(A))$, $\forall\, A \in \mathcal{U}_1$.

   (c) $scl_w(F^{\leftarrow}(B)) \subseteq F^{\leftarrow}(cl_w(B))$, $\forall\, B \in \mathcal{U}_1$.

Proof: We prove (1), leaving the dual proof of (2) to the interested reader.

(a)$\Longrightarrow$(b) Let $A \in \mathcal{U}_1$. From [5, Theorem 2.24 (2 a)] and the definition of weak-interior,

$$f^{\leftarrow}(int_w(F^{\to}(A))) \subseteq f^{\leftarrow}(F^{\to}(A)) \subseteq A.$$

Since inverse image and co-image under a difunction is equal, $f^{\leftarrow}(int_w(F^{\to}(A))) = F^{\leftarrow}(int_w(F^{\to}(A)))$. Thus, $f^{\leftarrow}(int_w(F^{\to}(A))) \in SO(U_1)$, by w-semicontinuity. Hence $f^{\leftarrow}(int_w(F^{\to}(A))) \subseteq sint_w(A)$ and applying [5, Theorem 2.4 (2 b)] gives

$$int_w(F^{\to}(A)) \subseteq F^{\to}\big(f^{\leftarrow}(int_w(F^{\to}(A)))\big) \subseteq F^{\to}(sint_w(A)),$$

which is the required inclusion.

(b)$\Longrightarrow$(c) Take $B \in \mathcal{U}_2$. Applying inclusion (b) to $A = f^{\leftarrow}(B)$ and using [5, Theorem 2.4 (2 b)] gives

$$int_w(B) \subseteq int(F^{\to}(f^{\leftarrow}(B))) \subseteq F^{\to}(sint_w(f^{\leftarrow}(B))).$$

Hence, we have $f^{\leftarrow}(int_w(B)) \subseteq f^{\leftarrow}\big(F^{\to}\, sint_w(f^{\leftarrow}(B))\big) \subseteq\ sint_w(f^{\leftarrow}B)$ by [5, Theorem 2.24 (2 a)].

(c)$\Longrightarrow$(a) Applying (c) for $B \in \mathcal{W}O(U_2)$ gives

$$f^{\leftarrow}(B) = f^{\leftarrow}(int_w(B)) \subseteq\ sint_w(f^{\leftarrow}(B)),$$

so $F^{\leftarrow}(B) = f^{\leftarrow}(B) =\ sint_w(f^{\leftarrow}(B)) \in SO(U_1)$. Hence, (f, F) is w-semicontinuous. □

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References

1. H. L. Bentley and R. G. Ori, *Topology without the union axiom(revisited)*, Quaestiones Mathematicae, **37**, (2014), 265–277.

2. L. M. Brown and M. Diker, *Ditopological texture spaces and intuitionistic sets*, Fuzzy sets and systems **98**, (1998), 217–224.

3. L. M. Brown and R. Ertürk, *Fuzzy Sets as Texture Spaces, I. Representation Theorems*, Fuzzy Sets and Systems **110** (2) (2000), 227–236.

4. L. M. Brown and R. Ertürk, *Fuzzy sets as texture spaces, II. Subtextures and quotient textures*, Fuzzy Sets and Systems **110** (2) (2000), 237–245.

5. L. M. Brown, R. Ertürk and Ş. Dost, *Ditopological texture spaces and fuzzy topology, I. Basic Concepts*, Fuzzy Sets and Systems **147** (2) (2004), 171–199.

6. Ş. Dost, *Textures and Base spaces*, South Asian Journal of Mathematics, **9** (1) (2019), 12–18.

7. Ş. Dost, *Precise Sets in Approximation Spaces and Textures*, Fundamenta Informaticae, **158** (4) (2018), 353–368.

8. Ş. Dost, *Metric spaces and textures*, Applied General Topology, **18** (1) (2017), 203–217.

9. Ş. Dost, *Quotient structure of interior-closure texture spaces*, Filomat, **29** (5) (2015), 947–962.

10. Ş. Dost, *A textural view of soft fuzzy rough sets*, Journal of Intelligent & Fuzzy Systems, **28** (6) (2016), 2519–2535.

11. Ş. Dost, *Serial relation and textural rough set*, Mathematica Moravica, **25** (2) (2021), 69–79.

12. Ş. Dost, *Minimal and maximal open sets in ditopological texture space*, Scientia Magna, **16** (1) (2021), 72–78.

13. Ş. Dost, L. M.Brown, *Intuitionistic Textures Revisited*, Hacettepe Journal of Mathematics and Statistics, **34** (2) (2005), 115–130.

14. R. Ekmekçi, *Graded Diextremities*, Filomat, 32 (1) (2018), 149–164.

15. R. Ekmekçi, *A Tychonoff Theorem For Graded Ditopological Texture Spaces*, Communications Faculty Of Science University Of Ankara Series A1mathematics and Statistics (69)1, 193–212.

16. M. Kule and Ş. Dost, *A textural view of soft fuzzy rough sets*, Annals of Fuzzy Mathematics and Informatics, **12** (1) (2016), 47–62.

17. W. K. Min and Y. K. Kim, *On Weak Structures and w- spaces*, Far East Journal of Mathematical Sciences, 97(5) (2015), 549-561. http://dx.doi.org/10.17654/fjmsjul2015 549–561.

18. W. K. Min and Y. K. Kim, *w-Semi open sets and w-semicontinuity in weak spaces*, Int. Journal of Pure and Applied Mathematics, **110** (1) (2016), 49–56.

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