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The Infimum Eigenvalue for Degenerate $p(x)$-biharmonic Operator with the Hardy Potential

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ABSTRACT: This article aims to study the existence of at least one unbounded nondecreasing sequence of nonnegative eigenvalues $(\lambda_k)_{k\geq 1}$ for a class of elliptic Navier boundary value problems involving the degenerate $p(x)$-biharmonic operator with q(x)-Hardy inequality by using the variational technique based on the Ljusternik-Schnirelmann theory on C^1-manifolds and the theory of the variable exponent Lebesgue spaces. We also obtain the positivity of the infimum eigenvalue for the problem.

Key Words: Degenerate $p(x)$-biharmonic operator, Ljusternik Schnirelmann theory, $q(x)$-Hardy inequality, variational methods.

Contents

1. Introduction

In recent years, the study of nonlinear elliptic differential equations and variational problems with variable exponent has been very successful. It has many applications in several scientific fields, including physics, nonlinear electrorheological fluids, and elastic mechanics. In that situation, we refer to Ruzicka [18] and Zhikov [23], the reference in that respect, and see also [8,9,10,11].

Fourth order elliptic equations arise in many applications such as: Micro Electro Mechanical systems, thin film theory, surface diffusion on solids, interface dynamics, flow in Hele-Shaw cells, and phase field models of multiphase systems (see [12]). Additionally, this type of equation can describe the static from a beam change or the sport of a rigid body.

In this work, we consider the problem

$$\begin{cases}\Delta\left((a(|\Delta u|^{p(x)})\;|\Delta u|^{p(x)-2}\Delta u\right)=\lambda\left(\omega(x)|u|^{p(x)-2}u-\mu\frac{|u|^{q(x)-2}u}{\delta(x)^{2q(x)}}\right) & \text{in } \Omega,\\ u=\Delta u=0 \quad \text{on } \partial\Omega, \end{cases}\tag{1.1}$$

where $\Omega\subset\mathbb{R}^N$ $(N\geq 3)$ is a bounded domain with smooth boundary, $\delta(x)=dist(x,\partial\Omega)$ denotes the distance from the boundary $\partial\Omega$. λ and μ are two real parameters, ω is a positive function such that $\omega\in L^\infty(\Omega)$, p,q are continuous functions on $\overline{\Omega}$ verifying:

$$1<p^-=\inf_{x\in\overline{\Omega}}p(x)\leq p^+=\max_{x\in\overline{\Omega}}p(x)<q^-=\inf_{x\in\overline{\Omega}}q(x)\leq q^+=\max_{x\in\overline{\Omega}}q(x)<\frac{N}{2}\quad \forall\, x\in\overline{\Omega},\quad \textit{and}\tag{1.2}$$

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($\mathbf{A}_1$) $a : \mathbb{R}^+ \to \mathbb{R}$ is a real continuous function, and the mapping $\Theta : \mathbb{R}^N \to \mathbb{R}$, given by $\Theta(\xi) = A(|\xi|^{p(x)})$ is strictly convex, where A is the primitive of a that is

$$A(t) = \int_0^t a(s)ds.$$

($\mathbf{A}_2$) There exist two constants L and K $0 < L < K$ such that $\forall (t \geq 0) \quad L \leq a(t) \leq K$.

We indicate that elliptic equations requiring the $p(\cdot)$-biharmonic equations are not trivial generalizations of identical problems investigated in the stable case since the $p(x)$-biharmonic operator is not homogeneous, so few techniques utilized in the situation of the p-biharmonic operators will lose out in that new case, for example, the Lagrange Multiplier Theorem.

The group of $p(x)$-biharmonic equations deemed by several authors in later years. Several scientists have studied gleam equations under diverse boundary conditions and by different approaches.

In [4] Ayoujil and El Amrouss are studied a class of $p(\cdot)$-biharmonic of the form

$$\begin{cases} \Delta(|\Delta u|^{p(x)-2}\Delta u) = \lambda |u|^{q(x)-2}u & \text{in } \Omega, \\ u = \Delta u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $N \geq 1$, $\lambda \geq 0$; by applying variational methods based on the Mountain Pass Lemma and Ekeland Variational Principle, they established several existence criteria for eigenvalues.

By using mainly adequate variational techniques, Ayoujil in [2] and El Allali et al. in [20] studied some problems with an indefinite weight under Neumann boundary conditions and Navier boundary conditions.

In the case where $p(x) = q(x)$, Ayoujil and El Amrouss in [5] proved the existence of infinitely many eigenvalue sequences regarding the problem (1.3) by using the Ljusternik-Schnirelmann theory on C^1-manifolds.

In [16], Lin Li et al. considered the above problem and using variational methods, problem assuming that the function f is Carathéodory, using the mountain pass theorem, fountain theorem, local linking theorem, and symmetric mountain pass theorem, have establish the existence of at least one solution and infinitely many solutions of this problem, respectively.

In [3], Tsouli et al. have been considered the fourth-order quasi-linear elliptic equation involving the $(p_1(\cdot), p_2(\cdot))$-biharmonic operator, and applying variational methods, by the assumptions on the Carathéodory function f, they establish the existence of at least one solution and infinitely many solutions to this problem, respectively

$$\begin{cases} \Delta(|\Delta u|^{p_1(x)-2}\Delta u) + \Delta(|\Delta u|^{p_2(x)-2}\Delta u) = f(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

Motivated by the above papers inspired by some techniques from Belaouidel et al. in [6], and based on Ljusternik-Schnirelmann theory, this article aims to study the existence of at least one unbounded nondecreasing sequence of nonnegative eigenvalues $(\lambda_k)_{k\geq 1}$ for a class of elliptic Navier boundary value issues involving the degenerate $p(x)$-biharmonic operator with q(x)-Hardy inequality. Specifically, it is a generalization of the work of El Khalil et al. [13].

Here, Problem (1.1) has been stated in the framework of the generalized Sobolev space

$$X := W^{2,p(\cdot)}(\Omega) \cap W_0^{1,p(\cdot)}(\Omega),$$

The remainder of this paper has been organized as follows: In Section 2, we present the main results. In Section 3, we introduce some basic preliminary results for the variable exponent Lebesgue spaces, $L^{p(x)}(\Omega)$, and Sobolev spaces, $W^{k,p(x)}(\Omega)$. In Section 4, we give $q(x)$-Hardy Inequality. In Section 5, we provide some results and proofs of the functionals. In Section 6, we demonstrate the existence of at least one non-decreasing sequence of nonnegative eigenvalues to Problem (1.1) by applying Ljusternik–Schnirelman theory. In Section 7, we provide proof of the positivity of the infimum of all eigenvalues.

2. Main results

Definition 2.1. *We say that a function $u \in X$ is a weak solution of (1.1) if*

$$\int_\Omega \left(|\Delta((a(|\Delta u|^{p(x)})\ |\Delta u|^{p(x)-2}\Delta u\Delta v\right) dx = \lambda\left(\int_\Omega \omega(x)|u|^{p(x)-2}uvdx - \mu\int_\Omega \frac{|u|^{q(x)-2}u}{\delta(x)^{2q(x)}}vdx\right), \tag{2.1}$$

for all $v \in X$.

If u is not equivalently zero, we say that λ is the eigenvalue of (1.1) corresponding to the eigenfunction u of degenerate $p(x)$ biharmonic operator associated to the eigenvalue of λ.
Set

$$\Lambda = \left\{\lambda : \lambda \textit{ is an eigenvalue of } (1.1)\right\}.$$

Define the following functionals on X. For $u \in X$, define

$$I_\lambda(u) = \phi(u) - \lambda\varphi(u).$$

Where $\varphi, \phi : X \to \mathbb{R}$

$$\varphi(u) = \int_\Omega \frac{1}{p(x)}\omega(x)|u|^{p(x)}dx - \mu\int_\Omega \frac{|u|^{q(x)}u}{\delta(x)^{2q(x)}}dx;$$
$$\phi(u) = \int_\Omega \frac{1}{p(x)}A(|\Delta u|^{p(x)})dx.$$

We also define the following sets:

$$\Sigma_j = \{K \subset \mathcal{H} \mid K \text{ is symmetric, compact, and } \gamma(K) \geq j\},$$

where $\gamma(K) = j$ is the Krasnoselskii genus of the set K, i.e., the smallest integer j such that there exists an odd continuous map from K to $\mathbb{R}^j\backslash\{0\}$. For more details, the reader may refer to [19]. And set

$$\mathcal{H} = \{u \in X \mid \varphi(u) = 1\}.$$

The main results of this work are the following.

Theorem 2.2. *: For any integer $j \in \mathbb{N}^*$,*

$$\lambda_j = \inf_{k\in\Sigma_j}\max_{u\in K}\phi(u),$$

is a critical value of ϕ restricted on $\mathcal{H}$. More precisely, there exists $u_j \in K$ such that :

$$\lambda_j = \phi(u_j) = \sup_{u\in K}\phi(u),$$

and u_j is a solution of Problem (1.1) associated to the positive eigenvalue λ_j. Moreover,

$$\lambda_j \to \infty \textit{ as } j \to \infty.$$

Theorem 2.3. *: If there exists an open subset $U \subset \Omega$ and a point $x_0 \in \Omega$ such that*

$$p(x_0) < \ (or >)\, p(x) \ for\, all\, x \in\ \partial U,\ then,\ \lambda_1 = 0.$$

We set:

$$\lambda_1 = \inf_{u\in X}\left\{\int_\Omega \frac{1}{p(x)}A(|\Delta u|^{p(x)})dx \ \Big|\ \int_\Omega \frac{1}{p(x)}\omega(x)|u|^{p(x)}dx - \mu\int_\Omega \frac{|u|^{q(x)}u}{\delta(x)^{2q(x)}}dx = 1\right\}. \tag{2.2}$$

The value defined in (2.2) can be as the Rayleigth quotient:

$$\lambda_1 = \inf_{u\in X}\frac{\int_\Omega \frac{1}{p(x)}A(|\Delta u|^{p(x)})dx}{\int_\Omega \frac{1}{p(x)}\omega(x)|u|^{p(x)}dx - \mu\int_\Omega \frac{1}{q(x)}\frac{|u|^{q(x)}}{\delta(x)^{2q(x)}}},$$

where the infimum is taken over $X \setminus \{0\}$.

Corollary 2.4. *: The following statements hold true*

(i) $\lambda_1 = \inf_{u\in X}\left\{\int_\Omega \frac{1}{p(x)}A(|\Delta u|^{p(x)})dx \ \mid u\in X \text{ and } \int_\Omega \frac{\omega(x)}{p(x)}|u|^{p(x)}dx - \mu\int_\Omega \frac{|u|^{q(x)}u}{\delta(x)^{2q(x)}}dx = 1\right\};$

(ii) $0<\lambda_1\le\lambda_2\le\ldots\le\lambda_n\to+\infty.$

3. Preliminaries

To guarantee the completeness of this paper, we need some results on the variable exponent spaces $L^{p(\cdot)}(\Omega)$ and $W^{k,p(\cdot)}(\Omega)$, and some properties, which we use later. Let Ω be a bounded domain of $\mathbb{R}^N$, denote

$$C_+(\overline{\Omega}) = \left\{h(x): \quad h(x)\in C(\overline{\Omega}), \quad h(x)>1, \quad \forall x\in\overline{\Omega}\right\}.$$

For any $h\in C_+(\overline{\Omega})$, we define

$$h^+ = \max\left\{h(x):\ x\in\overline{\Omega}\right\}, \quad h^- = \min\left\{h(x):\ x\in\overline{\Omega}\right\};$$

For any $p\in C_+(\overline{\Omega})$, we define the *variable exponent Lebesgue space*

$$L^{p(\cdot)}(\Omega) = \left\{u:\Omega\to\mathbb{R} \text{ measurable and } \int_\Omega |u(x)|^{p(x)}dx<\infty\right\},$$

equipped with the so-called *Luxemburg norm*:

$$|u|_{p(\cdot)} = \inf\left\{\mu>0:\ \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(\cdot)}dx\le 1\right\}.$$

Then, $(L^{p(\cdot)}(\Omega), |\cdot|_{p(\cdot)})$ becomes a Banach space.

Proposition 3.1 ([21]). *The space $(L^{p(\cdot)}(\Omega), |\cdot|_{p(\cdot)})$ is separable, uniformly convex, reflexive and its conjugate space is $L^{q(\cdot)}(\Omega)$ where $q(\cdot)$ is the conjugate function of $p(\cdot)$, i.e.,*

$$\frac{1}{p(\cdot)}+\frac{1}{q(\cdot)}=1.$$

For $u\in L^{p(\cdot)}(\Omega)$ and $v\in L^{q(\cdot)}(\Omega)$, we have

$$\left|\int_\Omega uv dx\right| \le \left(\frac{1}{p^-}+\frac{1}{q^-}\right)|u|_{p(\cdot))}|v|_{q(\cdot)} \le 2|u|_{p(\cdot)}|v|_{q(\cdot)}.$$

The Sobolev space with variable exponent $W^{k,p(\cdot)}(\Omega)$ is defined as

$$W^{k,p(\cdot)}(\Omega) = \left\{u\in L^{p(\cdot)}(\Omega): D^\alpha u\in L^{p(\cdot)}(\Omega), |\alpha|\le k\right\},$$

where $D^\alpha u = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\ldots\partial x_N^{\alpha_N}}u$, with $\alpha=(\alpha_1,\ldots,\alpha_N)$ is a multi-index and $|\alpha| = \sum_{i=1}^N \alpha_i$. The space $W^{k,p(\cdot)}(\Omega)$ equipped with the norm:

$$\|u\|_{k,p(\cdot)} = \sum_{|\alpha|\le k}|D^\alpha u|_{p(\cdot)}.$$

It also becomes a separable and reflexive Banach space. For more details, we refer the reader to [21]. Denote

$$p_k^*(\cdot) = \begin{cases} \frac{Np(\cdot)}{N-kp(\cdot)} & \text{if } kp(\cdot) < N, \\ +\infty & \text{if } kp(\cdot) \geq N, \end{cases}$$

for any $k \geq 1$.

Proposition 3.2 ([5]). *For $p, r \in C_+(\overline{\Omega})$ such that $r(\cdot) \leq p_k^*(\cdot)$, there is a continuous embedding*

$$W^{k,p(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega).$$

If we replace $\leq$ with $<$, the embedding is compact.

We denote by $W_0^{k,p(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{k,p(\cdot)}(\Omega)$.
Let us choose on X the norm $\|.\|$ define by:

$$\|u\| = \inf\left\{\mu > 0 : \int_\Omega \left(\left|\frac{\Delta u(x)}{\mu}\right|^{p(x)}\right) dx \leq 1\right\}.$$

We note that space X, equipped with the above norm, is a separable, reflexive, and Banach space.

Remark 3.1. We can know that for any $u \in X$, $\|u\| = \|u\|_{1,p(x)} + \|u\|_{2,p(x)}$ where

$$\|u\|_{1,p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)};$$

$$\|u\|_{2,p(x)} = \sum_{|\alpha|=2} |D^\alpha u|_{p(x)}.$$

According to [[21] Theorem 4.4], the norm $\|\cdot\|_{2,p(x)}$ is equivalent to the norm $|\cdot|_{p(\cdot)}$ in the space X. Consequently, the norms $\|\cdot\|_{2,p(\cdot)}$, $\|\cdot\|$ and $|\cdot|_{p(\cdot)}$ are equivalent.

Similar to Proposition 3.2, we have:

Proposition 3.3 ([1]). *Let $I(u) = \int_\Omega A\left(|\Delta u(x)|^{p(x)}\right) dx$. Then, for $u, u_n \in X$, we have*

(1) $\|u\| < 1$ (respectively=1; > 1) $\Longleftrightarrow$ $I(u) < 1$ (respectively $= 1$; > 1);

(2) $\|u\| \leq 1 \Rightarrow \|u\|^{p^+} \leq I(u) \leq \|u\|^{p^-}$;

(3) $\|u\| \geq 1 \Rightarrow \|u\|^{p^-} \leq I(u) \leq \|u\|^{p^+}$;

(4) $\|u_n\| \to 0$ (respectively $\to \infty$) $\Longleftrightarrow$ $I(u_n) \to 0$ (respectively $\to \infty$).

4. On the $q(x)$-Hardy inequality

Lemma 4.1. *Let Ω be a regular bounded domain in $\mathbb{R}^N$ and $1 < p^- \leq p^+ < q^- \leq q^+ < \frac{N}{2}$ $\forall x \in \Omega$. Then, there exists a positive constant C such as the $q(x)$-Hardy Inequality*

$$\int_\Omega \frac{1}{p(x)} A(|\Delta u|^{p(x)}) dx \geq C \int_\Omega \frac{|u|^{q(x)}}{\delta(x)^{2q(x)}} dx, \tag{4.1}$$

holds for all $u \in X$ in the following two cases:

(i) $|u| \leq (\delta(x))^2$ and $|A(|\Delta u|^{p(x)})| \geq 1$,

(ii) $|u| \geq (\delta(x))^2$ and $|A(|\Delta u|^{p(x)})| \leq 1$.

Proof. We have

$$\phi(u) = \int_\Omega \frac{1}{p(x)} A(|\Delta u|^{p(x)}) dx \geq K \int_\Omega \frac{|\Delta u|^{p(x)}}{p(x)} dx.$$

(i) Since A is bounded, under (i), we have $|u| \leq (\delta(x))^2$ and $|A(|\Delta u|^{p(x)})| \geq 1$, then,

$$\int_\Omega \frac{p^+}{p(x)} A\left(|\Delta u|^{p(x)}\right) dx \geq K \int_\Omega \frac{p^+}{p(x)} |\Delta u|^{p(x)} dx \geq K \int_\Omega |\Delta u|^{p^-} dx. \tag{4.2}$$

On the other hand, the subsequent inequality:

$$\int_\Omega |\Delta u|^p dx \geq \left(\frac{N(p-1)(N-2p)}{p^2}\right)^p \int_\Omega \frac{|u|^p}{\delta(x)^{2p}} dx \quad \textit{whenever} \quad u \in C_c^\infty(\Omega). \tag{4.3}$$

We obtain:

$$\int_\Omega |\Delta u|^{p^-} dx \geq KC^- \int_\Omega \frac{|u|^{p^-}}{(\delta(x))^{2p^-}} dx,$$

where

$$C^- = \left(\frac{N(N-2p(x))(p^- - 1)}{(p^-)^2}\right)^{p^-}.$$

Thus, we deduce from (4.2) that

$$\int_\Omega \frac{p^+}{p(x)} A\left(|\Delta u|^{p(x)}\right) dx \geq p^- C^- K \int_\Omega \frac{1}{p^-} \frac{|u|^{p^-}}{\delta^{2p^-(x)}(x)} dx.$$

Now, since $|u| \leq (\delta(x))^2$, we obtain :

$$\left(\frac{|u|}{(\delta(x))^2}\right)^{p^-} \geq \left(\frac{|u|}{(\delta(x))^2}\right)^{p(x)}.$$

Thus,

$$p^- C^- \int_\Omega \frac{1}{p^-} \left(\frac{|u|}{(\delta(x))^2}\right)^{p^-} dx \geq p^- C^- \int_\Omega \frac{1}{p(x)} \left(\frac{|u|}{(\delta(x))^2}\right)^{p(x)} dx.$$

Therefore,

$$\int_\Omega \frac{p^+}{p(x)} A\left(|\Delta u|^{p(x)}\right) dx \geq p^- C^- K \int_\Omega \frac{p^+}{p(x)} \left(\frac{|u|}{\delta^2(x)}\right)^{p(x)} dx,$$

then,

$$\int_\Omega \frac{1}{p(x)} A\left(|\Delta u|^{p(x)}\right) dx \geq \frac{p^-}{p^+} C^- K \int_\Omega \left(\frac{1}{p(x)} \frac{|u|^{p(x)}}{\delta(x)^{2p(x)}}\right) dx.$$

Since $p(x) < q(x)$ in Ω, then, there exists the continuous embedding

$$L^{q(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega),$$

i.e., there exists a non negative constant M such that:

$$|u|_{L^{q(x)}(\Omega)} \leq M |u|_{L^{p(x)}(\Omega)}, \tag{4.4}$$

so,

$$\begin{aligned} \int_\Omega \frac{1}{p(x)} A\left(|\Delta u|^{p(x)}\right) dx &\geq \frac{p^-}{p^+} C^- K \int_\Omega \left(\frac{1}{p(x)} \frac{|u|^{p(x)}}{\delta(x)^{2p(x)}}\right) dx, \\ &\geq \frac{q^-}{q^+} C^- \frac{K}{M} \int_\Omega \left(\frac{1}{q(x)} \frac{|u|^{q(x)}}{\delta(x)^{2q(x)}}\right) dx. \end{aligned}$$

Hence, we deduce that

$$\int_\Omega \frac{1}{p(x)} A\left(|\Delta u|^{p(x)}\right) dx \geq C \int_\Omega \frac{1}{q(x)} \frac{|u|^{q(x)}}{\delta(x)^{2q(x)}} dx,$$

where

$$C = \frac{q^-}{q^+} C^- \frac{K}{M}.$$

(ii) We use a similar process to (i), we have $|u| \geq (\delta(x))^2$ and $|A(|\Delta u|^{p(x)})| \leq 1$, thus,

$$\int_\Omega \frac{p^+}{p(x)} A\left(|\Delta u|^{p(x)}\right) dx \geq K \int_\Omega |\Delta u|^{p^+} dx. \tag{4.5}$$

On the other hand, by (3.2) in [13], we have:

$$\int_\Omega A\left(|\Delta u|^{p^+}\right) dx \geq K C^+ \int_\Omega \frac{|u|^{p^+}}{\delta(x)^{2p^+}} dx, \tag{4.6}$$

where

$$C^+ = \frac{(N(N-2p(x))(p^+-1))^{p^+}}{(p^+)^2}.$$

Thus, we deduce from (4.6) that :

$$\int_\Omega \frac{p^+}{p(x)} A\left(|\Delta u|^{p(x)}\right) dx \geq C^+ p^- K \int_\Omega \frac{1}{p^-} \frac{|u|^{p^+}}{\delta(x)^{2p^+}} dx.$$

Since $|u| \geq (\delta(x))^2$, we get

$$\left(\frac{|u|}{(\delta(x))^2}\right)^{p^+} \geq \left(\frac{|u|}{(\delta(x))^2}\right)^{p(x)}.$$

Thus,

$$p^- C^+ K \int_\Omega \frac{1}{p^-} \left(\frac{|u|}{\delta^2(x)}\right)^{p^+} dx \geq p^- C^+ K \int_\Omega \frac{1}{p(x)} \left(\frac{|u|}{\delta^2(x)}\right)^{p(x)} dx.$$

Therefore,

$$\int_\Omega \frac{p^+}{p(x)} A\left(|\Delta u|^{p(x)}\right) dx \geq p^- C^+ K \int_\Omega \frac{1}{p(x)} \left(\frac{|u|}{\delta^2(x)}\right)^{p(x)} dx.$$

Then,

$$\int_\Omega \frac{1}{p(x)} A\left(|\Delta u|^{p(x)}\right) dx \geq \frac{p^-}{p^+} C^+ K \int_\Omega \frac{1}{p(x)} \left(\frac{|u|}{\delta^2(x)}\right)^{p(x)} dx.$$

Hence, we deduce that

$$\int_\Omega \frac{1}{p(x)} A\left(|\Delta u|^{p(x)}\right) dx \geq \frac{p^-}{p^+} C^+ K \int_\Omega \frac{1}{p(x)} \frac{|u|^{p(x)}}{(\delta(x))^{2p(x)}} dx.$$

By using the continuous embedding $L^{q(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$ and equation (4.4), we have:

$$|u|_{L^{q(x)}(\Omega)} \leq M |u|_{L^{p(x)}(\Omega)}.$$

Then,

$$\int_\Omega \frac{1}{p(x)} A\left(|\Delta u|^{p(x)}\right) dx \geq C \int_\Omega \frac{1}{q(x)} \frac{|u|^{q(x)}}{\delta(x)^{2q(x)}} dx.$$

□

Remark: It is assumed that the function φ verifies the hypothesis throughout the article: There exists the constants $D_1 > 0$ and $\eta > 0$ such that $\eta > p^+$ and

$$0 < \int_\Omega \frac{1}{p(x)}\omega(x)|u|^{p(x)}dx - \mu\int_\Omega \frac{|u|^{q(x)}}{\delta(x)^{2q(x)}}dx \leq \int_\Omega \frac{\omega(x)|u|^{p(x)}}{\eta}dx - \mu\int_\Omega \frac{1}{\eta}|u|^{q(x)}dx,$$

for all $|u| > D_1$.

Using the properties found in [14] and in Chapter 3, Section 3.2 of [17], we can say that $\varphi(u)$ is well defined on X,

5. Functional framework

Throughout this paper, we note X^* as the dual of X. To show the main results of this work, we need to know some properties of the functionals ϕ and φ, which have already been defined in Section 2.

Lemma 5.1. *The following statements hold :*

(i) φ and ϕ are even and of class C^1 on X,

(ii) $\mathcal{H}$ is a closed C^1-manifold.

Proof. Similar to the argument to [Lemma 3.3 in [13]] that φ and ϕ are even and of class C^1 on X and $\mathcal{H} = \varphi^{-1}(1)$. In view of Lemma 4.1 $\mathcal{H}$ is closed.

The derivative operator $\varphi^{'}$ satisfies $\varphi^{'}(u) \neq 0$ for all $u \in \mathcal{H}$ (i.e., : $\varphi^{'}(u)$ is into for all $u \in \mathcal{H}$).

Hence, φ is a submersion, which proves that it is a C^1-manifold.

The operator $T := \phi^{'} : X \longrightarrow X^*$ defined as

$$\langle T(u), v\rangle = \int_\Omega a(|\Delta u|^{p(x)})|\Delta u|^{p(x)-2}\Delta u\Delta v dx,$$

for any $u, v \in X$ satisfies the assertions of the following lemma, which is the key to establishing our main results in the next section. □

Lemma 5.2. *: The following statements hold:*

(i) T is continuous, bounded, and strictly monotone,

(ii) T is of (S_+) type,

(iii) T is a homeomorphism.

Proof. (i) $T = \phi^{'}$ is the Frèchet derivative of ϕ, it follows that T is continuous and bounded, and hence, for all $u, v \in X$, we deduce that $\langle T(u) - T(v), u - v\rangle > 0$. It means that T is strictly monotone.

(ii) Let $(u_n)_n$ be a sequence of X and $\limsup_{n\to+\infty}\langle T(u_n), u_n - u\rangle \leq 0$, under the monotony of T, we obtain $\langle T(u_n) - T(u), u_n - u\rangle \geq 0$ and since $u_n \rightharpoonup u$ in X, we have $\limsup\limits_{n\to+\infty}\langle T(u_n), u_n - u\rangle = 0$.

Put:

$$\mathcal{U}_p = \{x \in \Omega : p(x) \geq 2\}, \quad \mathcal{V}_p = \{x \in \Omega : 1 < p(x) < 2\}.$$

Using the following elementary inequalities:

$$|x - y|^p \leq 2^p(|x|^{p-2}x - |y|^{p-2}y)\cdot(x - y) \quad \text{if} \quad p \geq 2 \quad \textit{and}$$

$$|x - y|^2 \leq \frac{1}{(p-1)}(|x| + |y|)^{2-p}(|x|^{p-2}x - |y|^{p-2}y)\cdot(x - y) \quad \text{if} \quad 1 < p < 2,$$

for all $(x,y)\in(\mathbb{R}^N)^2$, where $x.y$ denotes the usual inner product in $\mathbb{R}^N$, we obtain,

$$\begin{aligned}\langle T(u_n)-T(u),u_n-u\rangle &= \int_\Omega \Big(a(|\Delta u_n|^{p(x)})|\Delta u_n|^{p(x)-2}\Delta u_n\Delta(u_n-u)\Big)\,dx \\ &- \int_\Omega \Big(a(|\Delta u|^{p(x)})|\Delta u|^{p(x)-2}|\Delta u|\Delta(u_n-u)\Big)\,dx.\end{aligned} \tag{5.1}$$

Since a is bounded,

$$\begin{aligned}\langle T(u_n)-T(u),u_n-u\rangle &\geq L\int_\Omega \Big(|\Delta u_n|^{p(x)-2}\Delta u_n\Delta(u_n-u)\Big)\,dx \\ &\quad -K\int_\Omega |\Delta u|^{p(x)-2}\Delta u\,(\Delta(u_n-u))\,dx, \\ &\geq \max(L,K)\left[\int_\Omega \Big(|\Delta u_n|^{p(x)-2}|\Delta u_n|-|\Delta u|^{p(x)-2}|\Delta u|\Big)\,\Delta(u_n-u)\right]dx \\ &\geq K\int_\Omega \Big(|\Delta u_n|^{p(x)-2}|\Delta u_n|-|\Delta u|^{p(x)-2}|\Delta u|\Big)\,\Delta(u_n-u)dx.\end{aligned}$$

Therefore, we have: $\int_{\mathcal{U}_p}|\Delta u_n-\Delta u|^{p(x)}\leq 2^{(p^--2)}\int_\Omega A(u_n,u)dx$,

$$\int_{\mathcal{V}_p}|\Delta u_n-\Delta u|^{p(x)}\,dx\leq (p^+-1)\int_\Omega (A(u_n,u))^{\frac{p(x)}{2}}\,(B(u_n,u))^{\frac{(2-p(x))p(x)}{2}}\,dx,$$

where

$$A(u_n,u)=\Big(|\Delta u_n|^{p(x)-2}\Delta u_n-|\Delta u|^{p(x)-2}\Delta u\Big)\,(\Delta u_n-\Delta u),$$

and

$$B(u_n,u)=(|\Delta u_n|+|\Delta u|)^{2-p(x)}\,.$$

Using $\langle T(u_n)-T(u),u_n-u\rangle\geq 0$ and

$$\int_\Omega A(u_n,u)dx=\langle T(u_n)-T(u),u_n-u\rangle$$

we have $0\leq\int_\Omega A(u_n,u)dx\leq 1$.

Now, we distinguish two cases:

(1) If $\int_\Omega A(u_n,u)dx=0$ then, since $A(u_n,u)\geq 0$ so $A(u_n,u)=0$.

(2) If $0<\int_\Omega A(u_n,u)dx<1$ then, $t^{p(x)}:=\Big(\int_{\mathcal{V}_p}A(u_n,u)dx\Big)^{-1}$ is positive.

By applying Young′s inequality, we deduce that:

$$\int_{\mathcal{V}_p} t\,(A(u_n,u))^{\frac{p(x)}{2}}\,(B(u_n,u))^{(2-p(x))\frac{p(x)}{2}}\,dx \quad\leq\quad \int_{\mathcal{V}_p}\Big[A(u_n,u)(t)^{\frac{2}{p(x)}}+(B(u_n,u))^{p(x)}\Big]\,dx.$$

Now, by the fact that $\frac{2}{p(x)}<2$, we have

$$\begin{aligned}\int_{\mathcal{V}_p}\Big[A(u_n,u)(t)^{\frac{2}{p(x)}}+(B(u_n,u))^{p(x)}\Big]\,dx &\leq \int_{\mathcal{V}_p}\Big[A(u_n,u)(t)^2+(B(u_n,u))^{p(x)}\Big]\,dx, \\ &\leq 1+\int_{\mathcal{V}_p}(B(u_n,u))^{p(x)}\,dx.\end{aligned}$$

Hence,

$$\int_{\mathcal{V}_p}|\Delta u_n-\Delta u|^{p(x)}dx \quad\leq\quad \left(\int_{\mathcal{V}_p}A(u_n,u)dx\right)^{\frac{1}{2}}\left(\int_{\mathcal{V}_p}(B(u_n,u))^{p(x)}dx\right).$$

Since, $\int_\Omega (B(u_n,u))^{p(x)}\,dx$ is bounded, we have $\int_{\mathcal{V}_p} |\Delta u_n - \Delta u| \to 0$ as $n \to +\infty$.

(iii) Now, we prove that T is a homeomorphism. First, by the strict monotonicity, T is an injection. Furthermore, for any $u \in X$ with $\|u\| > 1$, we have

$$\frac{\langle T(u), u\rangle}{\|u\|} = \frac{\phi^{'}(u)}{\|u\|} \geq \|u\|^{p^- -1} \to +\infty \; as \; \|u\| \to +\infty$$

i.e., T is coercive. Thus, T is surjection in view of the Minty-Browder Theorem (see Theorem 26 A(d)in [22]). Hence, T has an inverse mapping $T^{-1} : X^* \longrightarrow X$. Therefore, the continuity of T^{-1} is sufficient to ensure T to be a homeomorphism. Indeed, let $(f_n)_n$ be a sequence of X^* so that $f_n \to f$ in X. Let u_n and u such that $T^{-1}(f_n) = u_n$ and $T^{-1}(f) = u$. Through the coercivity of T, we conclude that the sequence $(u_n)_n$ is bounded in the reflexive space X. For a subsequence, if necessary, we have $u_n \rightharpoonup \hat{u}$ in X for some $\hat{u}$, then,

$$\lim_{n\to+\infty} \langle T(u_n) - T(\hat{u}), u_n - \hat{u}\rangle = \lim_{n\to+\infty} \langle f_n - f, u_n - \hat{u}\rangle = 0.$$

By assertion (ii) and the continuity of T, it follows that $u_n \to \hat{u}$ in X and $T(u_n) \to T(\hat{u}) = T(u)$ in X^*. In addition, since T is an into the operator, we conclude that $u = \hat{u}$. The following lemma plays the central role in proving our fundamental result related to existence.

□

Lemma 5.3. *The following statements hold:*

(i) $\varphi^{'}$ is completely continuous,

(ii) ϕ satisfies the Palais-Smale condition on $\mathcal{H}$, i.e., for $u_n \subset \mathcal{H}$, if $\{\phi(u_n)\}_n$ is bounded and

$$\alpha_n = \phi^{'}(u_n) - \beta_n \varphi^{'}(u_n) \to 0 \; as \; n \to +\infty,$$

where $\beta_n = \frac{\langle \phi^{'}(u_n), u_n\rangle}{\langle \varphi^{'}(u_n), u_n\rangle}$ then, $\{u_n\}_{n\geq 1}$ has a convergent subsequence in X.

Proof. (i) Let us prove that $\varphi^{'}$ is completely continuous. Let $u, v \in X$. We have:

$$\langle \varphi^{'}(u), v\rangle = \int_\Omega \omega(x)|u|^{p(x)-2}uv dx - \mu \int_\Omega \frac{|u|^{q(x)-2}u}{\delta(x)^{2q(x)}} v dx.$$

Indeed, let $(u_n)_n \subset X$, $u_n \rightharpoonup u$ in X implies $\varphi^{'}(u_n) \to \varphi^{'}(u)$ in $X^{'}$.

For any $v \in X$, by Hölder's inequality in X and continuous embedding of X into $L^{p(x)}(\Omega)$, it follows that

$$\begin{aligned} \langle \varphi^{'}(u_n) - \varphi^{'}(u), v\rangle &= \int_\Omega \omega(x)(|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u) v dx \\ &- \mu \int_\Omega \frac{1}{\delta(x)^{2q(x)}} (|u_n|^{q(x)-2}u_n - |u|^{q(x)-2}u) v dx. \end{aligned}$$

On the one hand, we will show that

$$\begin{aligned} \int_\Omega (\omega(x)|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u) v dx \leq& \|\omega\|_\infty \left| \int_\Omega (|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u) v dx \right|, \\ \leq& d_1 \|\omega\|_\infty \||u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u\|_{r(x)} \|v\|_{p(x)}, \; d_1 > 0, \\ \leq& d_2 \|\omega\|_\infty \||u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u\|_{r(x)} \|v\|, \; d_2 > 0, \end{aligned}$$

where $r(x)=\frac{p(x)}{p(x)-1}$.
Since $X\hookrightarrow L^{p(x)}(\Omega)$, we have $u_n\to u$ in $L^{p(x)}(\Omega)$. We deduce that

$$|u_n|^{p(x)-2}u_n\to|u|^{p(x)-2}u\quad as\quad n\to+\infty\quad in\ L^{r(x)}(\Omega).$$

On the other hand,

$$B(x)=\int_\Omega\frac{|u|^{q(x)-2}}{\delta(x)^{2q(x)}}vdx\le\int_{\{x\in\Omega/\delta(x)>1\}}\frac{|u|^{q(x)-2}}{\delta(x)^{2q(x)}}dx+\int_{\{x\in\Omega/\delta(x)\le1\}}\frac{|u|^{q(x)-2}}{\delta(x)^{2q(x)}}dx.$$

By applying Hölder's inequality, we obtain

$$|B(x)|\le2|u|_{q(x)}^{q(x)-2}|v|_{q(x)}+2|\frac{u}{\delta(x)^2}|_{q(x)}^{q(x)-2}|\frac{v}{\delta(x)^2}|_{q(x)}.$$

This and the $q(.)$-Hardy inequality (4.1) yield

$$|B(x)|\le2|u|_{q(x)}^{q(x)-2}|v|_{q(x)}+\frac{2}{C^2}|\Delta u|^{q(x)-2}|\Delta v|_{q(x)}.$$

Then,

$$|B(x)|\le2K_1|u|^{q(x)-2}\|v\|+\frac{2K_2}{C^2}\|u\|^{q(x)-2}\|v\|,$$

where K_1 is a constant given by the embedding of X in $L^{q(\cdot)}(\Omega)$ and K_2 is given by the equivalence of norms $|\Delta\cdot|_{q(\cdot)}$ and $\|.\|$.

$$\|B(x)\|_*\le(2K_1+\frac{2K_2}{C^2})\|u\|^{q(x)-2},$$

where $\|.\|_*$ is the dual norm associated with $\|.\|$. Since $g\in L^{q(x)-1}(\Omega)$, it follows from the dominated convergence theorem that

$$|u_n|^{q(x)-2}u_n\to|u|^{q(x)-2}u\quad in\quad L^{q^{'}(x)}(\Omega).$$

Recall that the embedding

$$L^{q^{'}(x)}(\Omega)\hookrightarrow X^*,$$

and

$$L^{r(x)}(\Omega)\hookrightarrow X^*,$$

That is $\varphi^{'}(u_n)\to\varphi^{'}(u)$ in X^*. this proves assertion (i).

(ii) The functional ϕ satisfies the Palais-Smale condition on $\mathcal{H}$, i.e., $u_n\subset\mathcal{H}$, if $\{\phi(u_n)\}_n$ is bounded and $\alpha_n=\phi^{'}(u_n)-\beta_n\varphi^{'}(u_n)\to0$ as $n\to+\infty$, where $\beta_n=\frac{\langle\phi^{'}(u_n),u_n\rangle}{\langle\varphi^{'}(u_n),u_n\rangle}$ then, $\{u_n\}_{n\ge1}$, has a convergent subsequence in X. Indeed, we show that $\{u_n\}_{n\ge1}$ has a convergent subsequence that converges strongly. First, we show ϕ is coercive.
We have

$$\phi(u)=\int_\Omega\frac{1}{p(x)}A(|\Delta u|^{p(x)})dx\ge\int_\Omega\frac{1}{p^+}A(|\Delta u|)^{p(x)}dx\ge\frac{L}{p^+}\|u\|^{p^-},$$

for all $u\in X$ such that $\|u\|>1$, and so ϕ is coercive, then, $(u_n)_n$ is bounded, we have $u_n\rightharpoonup u$ in X and u_n converges strongly to u in $L^{p(x)}(\Omega)$. Throughout this, we easily deduce the boundedness of $\{u_n\}$. By the definition of ϕ , we have that $\rho(\Delta u_n)$ is bounded in $\mathbb{R}$; thus, without loss of generality, we assume that u_n converge weakly in X for some functions $u\in X$, and $\rho(\Delta u_n)\to l$, for the rest we distinguish two cases:

(1) If $l = 0$, then, u_n converges to 0 in X.

(2) If $l \neq 0$, since $T = \phi^{'}$ is of (S^+) type then, it suffices to show that

$$\limsup_{n\to+\infty}\langle T(u_n), u_n - u\rangle \leq 0.$$

Indeed, notice that

$$\langle T(u_n), u_n - u\rangle = \rho(\Delta u_n) - \langle T(u_n), u\rangle.$$

Set

$$\beta_n = \frac{\langle \phi^{'}(u_n), u_n\rangle}{\langle \varphi^{'}(u_n), u_n\rangle},$$

then,

$$\alpha_n = \phi^{'}(u_n) - \beta_n\varphi^{'}(u_n) \to_{n\to+\infty} 0.$$

Therefore,

$$\langle T(u_n), u_n - u\rangle = \rho(\Delta u_n) - \alpha_n - \frac{\langle \phi^{'}(u_n), u_n\rangle}{\langle \varphi^{'}(u_n), u_n\rangle}\langle \varphi^{'}(u_n), u\rangle.$$

That is,

$$\langle T(u_n), u_n - u\rangle = \frac{\rho(\Delta u_n)}{\langle \varphi^{'}(u_n), u_n\rangle}\langle \varphi^{'}(u_n), u_n\rangle - \langle \varphi^{'}(u_n), u\rangle - \alpha_n.$$

On the other hand, from Lemma 5.3, $\varphi^{'}$ is completely continuous, thus,

$$\varphi^{'}(u_n) \to \varphi^{'}(u),$$

and

$$\langle \varphi^{'}(u_n), u_n\rangle \to \langle \varphi^{'}(u), u\rangle.$$

Then,

$$|\langle \varphi^{'}(u_n), u_n\rangle - \langle \varphi^{'}(u_n), u\rangle| \leq |\langle \varphi^{'}(u_n), u_n\rangle - \langle \varphi^{'}(u), u\rangle| + |\langle \varphi^{'}(u_n), u\rangle - \langle \varphi^{'}(u), u\rangle|.$$

It follows that

$$|\langle \varphi^{'}(u_n), u_n\rangle - \langle \varphi^{'}(u_n), u\rangle| \leq |\langle \varphi^{'}(u_n), u_n\rangle - \langle \varphi^{'}(u), u\rangle| + \|\varphi^{'}(u_n) - \varphi^{'}(u))\|_*\|u\|.$$

It implies that

$$\langle \varphi^{'}(u_n), u_n\rangle - \langle \varphi^{'}(u_n), u\rangle \to 0 \; as \; n \to +\infty.$$

Combining with the above equalities, we obtain

$$\limsup_{n\to+\infty}\langle T(u_n), u_n - u\rangle \leq \frac{l}{\langle \varphi^{'}(u), u\rangle}\limsup_{n\to+\infty}\left(\langle \varphi^{'}(u_n), u_n\rangle - \langle \varphi^{'}(u_n), u\rangle\right).$$

We deduce

$$\limsup_{n\to+\infty}\langle T(u_n), u_n - u\rangle \leq 0. \tag{5.2}$$

On the other hand,

$$\langle \phi^{'}(u_n), u_n - u\rangle = \langle T(u_n), u_n - u\rangle,$$

according to (5.2), we conclude that

$$\limsup_{n\to+\infty}\langle \phi^{'}(u_n), u_n - u\rangle \leq 0.$$

In view of Lemma 5.2 $u_n \to u$ strongly in X.

□

As is well known, to solve the eigenvalue problem (1.1), the constrained variational method can be applied. We will use the following Ljusternik-Schnirelmann principle on C^1-manifolds [19].

6. Existence results

Recall that the main result of this work is to show that problem (1.1) has at least one unbounded nondecreasing sequence of nonnegative eigenvalues $(\lambda_k)_{k\geq 1}$. We use a variational technique based on the Ljusternik-Schnirelmann theory on C^1-manifolds [19] to attain this objective. We give a direct characterization of (λ_k) involving a mini-max argument over sets of genus greater than k.

Proposition 6.1. *: Suppose that $\mathcal{H}$ is a closed symmetric C^1-manifold of real Banach space X and $0 \notin \mathcal{H}$, also suppose that $\phi \in C^1(\mathcal{H}, \mathbb{R})$ is even and bounded below.*
Define

$$\lambda_j = \inf_{k\in\Sigma_j} \sup_{u\in K} \phi(i), \quad j = 1, 2, \ldots,$$

where $\Sigma_j = \{K \subset \mathcal{H} : \text{compact, symmetric, and } \gamma(K) \geq j\}$. If $\Sigma_k \neq \emptyset$ for some $k \geq 1$ and if ϕ satisfies the Palais-Smale condition (PS) for all $\lambda = \lambda_j$, $j = 1, 2, \ldots, k$, then, ϕ has at least k distinct pairs of critical points.

Lemma 6.2. *: For any $j \in \mathbb{N}^*$ $\Sigma_j \neq \emptyset$.*

Proof. : Since X is separable, for any $j \in \mathbb{N}^*$, there exists $(e_i)_{i\geq 1}$ lineary dense in X such that

$$\begin{cases} supp(e_i) \cap supp(e_j) = \emptyset & \text{if } i \neq j, \\ meas(supp(e_i)) > 0 & \text{for } i \in \{1, 2, \cdots, j\}. \end{cases}$$

Let

$$F_j = Span\{e_1, e_2, \ldots, e_j\}.$$

F_j be the vector subspace of X generated by j vectors $\{e_1, e_2, \ldots, e_j\}$ then, $dimF_j = j$. We may assume that $e_i \in \mathcal{H}$ (if not, we take $e_i^{'} = \frac{e_i}{[p(x)\varphi(e_i)]^{\frac{1}{p(x)}}}$). Let $j \in \mathbb{N}^*$ clearly, F_j is vector subspace with $dimF_j = j$.

If $v \in F_j$ then, there exist $\alpha_1, \alpha_2, .., \alpha_j$ in $\mathbb{R}$, such that $v = \displaystyle\sum_{i=1}^{j} \alpha_i e_i$.

Thus,

$$\varphi(v) = \sum_{i=1}^{j} |\alpha_i|^{p(.)} \varphi(e_i) = \sum_{i=1}^{j} |\alpha_i|^{p(.)}.$$

It follows that the map $v \mapsto (\varphi(v))^{\frac{1}{p(.)}} = |||v|||$ defines a norm on F_j. Consequently, there is constant $c > 0$ such that $\|v\| \leq |||v||| \leq \frac{1}{c}\|v\|$. It implies that the set $\mathcal{V}_j = F_j \cap \{v \in X/\varphi(v) \leq 1\}$ is bounded because $\mathcal{V}_k \subset B(0, \frac{1}{c})$ where $B(0, \frac{1}{c}) = \{u \in X \quad / \quad \|u\| \leq \frac{1}{c}\}$.
Thus, $\mathcal{V}_j$ is a symetric bounded neighborhood of $0 \in F_j$. Moreover, $F_j \cap \mathcal{H}$ is a compact set. By Proposition 2.3 (f) [19], we conclude that $\gamma(F_j \cap \mathcal{H}) = j$ and then, we finally obtain that $\Sigma_j \neq \emptyset$. □

Lemma 6.3. $\lambda_j \to \infty$ *as* $j \to \infty$.

Proof. Let $(e_k, e_n^*)_{k,n}$ be a bi-orthogonal system satisfying: $(e_k)_k$ are linearly dense in X, $(e_{n^*})_n$ are total for X^* , $\langle e_n^*, e_k\rangle = 0$ if $n \neq k$, $\langle e_n^*, e_k\rangle = 1 \quad \forall n \in \mathbb{N}^*$.

Set for every $k \in \mathbb{N}^*$ $F_k = Span\{e_1, e_2, \ldots, e_k\}$ and $F_k^\perp = Span\{e_{k+1}, e_{k+2}, \ldots\}$. From (g) in Proposition 2.3 [19], we have $K \cap F_{k-1}^\perp = \emptyset$ for any $K \in \Sigma_k$.

Let's prove that $t_k := \displaystyle\inf_{K\in\Sigma_k} \sup_{u\in K\cap F_{k-1}^\perp} \phi(u) \to \infty$ as $k \to \infty$.

Arguing by contradiction, assume that for k is large enough, there exists $u_k \in F_{k-1}^\perp$ with

$$\int_\Omega \frac{\omega(x)|u|^{p(x)-2}}{p(x)} dx - \mu \int_\Omega \frac{|u|^{q(x)}}{q(x)\delta^{2q(x)}} dx = 1.$$

Such that : $t_k \leq \phi(u_k) \leq M$. For some $M > 0$ independent of K. Thus, $|\Delta u_k|_{p(.)} \leq M$.
It implies that $(u_k)_k$ is bounded in X. We may assume that $u_k \rightharpoonup u$ in X and $u_k \to u$ in $L^{p(\cdot)}(\Omega)$. By choice of $F_{k-1}^{\perp}$. We observe that $u_k \rightharpoonup u$ in X because $\langle e_n^*, e_k \rangle = 0$ for any $k > n$. It contradicts the fact:

$$1 = \int_\Omega \frac{\omega(x)|u_k|^{p(x)-2}}{p(x)} dx - \mu \int_\Omega \frac{|u_k|^{q(x)}}{q(x)\delta^{2q(x)}} dx \to 0,$$

for every k. Hence, $\lambda_k \geq t_k$ for every $k \geq 1$. □

Proof of Theorem 2.2 : Applying Lemma 6.2, 6.3, and Ljusternik-Schnirelmann theory to the Problem (1.1), we have for each $j \in \mathbb{N}^*$. λ_j is a critical value of (1.1) on C^1-manifold $\mathcal{H}$ such that $\lambda_j \to \infty$ as $j \to \infty$.

7. On the positivity of inf Λ

Now, we give the following lemma, which will be used in Theorem 2.3, which is the second main result of this paper.

We introduce the following Rayleigh quotients

$$R(u) = R_{p(.),q(.)}(u) = \inf_{u \in X, u \neq 0} \frac{\int_\Omega A(|\Delta u|^{p(x)}) dx}{\int_\Omega \omega(x)|u|^{p(x)} dx - \mu \int_\Omega \frac{1}{\delta(x)^{2q(x)}} |u|^{q(x)}}.$$

Lemma 7.1. *: $\lambda_1 = 0 \Leftrightarrow R(u) = 0$.*

Proof. **of Lemma 7.1**: Similar the argument to [Lemma 5.1, [13]], we have the bounds

$$\frac{p^-}{p^+} R(u) \leq \lambda_1 \leq \frac{p^+}{p^-} R(u),$$

for which it follows that $\lambda_1 = 0 \Leftrightarrow R(u) = 0$. □

Proof. **of Theorem 2.3** : We try to show the case where $p(x_0) < p(x)$. For the case $p(x_0) > p(x)$ it's similar. For $O \subset \overline{\Omega}$ and $\delta > 0$.
Set $B(O, \delta) = \{x \in \mathbb{R}^N / dist(x, O) < \delta\}$. Without loss of generality, we may assume that $\overline{U} \subset \Omega$.
Then, there exists $\exists\, \varepsilon_0 > 0 \quad p(x_0) < p(x) - 4\varepsilon_0 \quad \forall x \in \partial U$

$$\exists\, \varepsilon_1 > 0 : \quad p(x_0) < p(x) - 2\epsilon_0 \quad \forall x \in B(\partial U, \varepsilon_1), \tag{7.1}$$

where

$$B(\partial U, \varepsilon_1) = \{x \mid \exists\, y \in \partial U : |x - y| < \varepsilon_1\} \subset \Omega,$$

and $\exists\, \varepsilon_2 > 0 : B(x_0, \varepsilon_2) \subset U \backslash B(\partial U, \varepsilon_1)$ and

$$|p(x_0) - p(x)| < \varepsilon_0 \quad \forall x \in B(x_0, \varepsilon_2). \tag{7.2}$$

Take $u_0 \in C_0^\infty(\Omega)$ such that $0 \leq u_0 \leq 1$, $|\Delta(u_0(x))| \leq C$, and

$$u_0(x) = \begin{cases} 1 & \text{if } x \in U \backslash B(\partial\Omega, \varepsilon_1), \\ 0 & \text{if } x \notin U \cup B(\partial\Omega, \varepsilon_1). \end{cases}$$

Then, for sufficiently small $t > 0$, we obtain

$$\begin{aligned} R(tu_0) &= \frac{\int_\Omega A\left(|\Delta(tu_0(x))|^{p(x)}\right) dx}{\int_\Omega \omega(x)|tu_0(x)|^{p(x)} dx - \mu \int_\Omega \frac{1}{\delta(x)^{2q(x)}} |tu_0|^{q(x)} dx} \\ &\leq \frac{\int_{B(\partial U, \varepsilon_1)} A\left(|\Delta(tu_0(x))|^{p(x)}\right) dx}{\int_{B(x_0,\varepsilon_2)} \omega(x)|tu_0|^{p(x)} dx - \mu \int_{B(x_0,\varepsilon_2)} \frac{1}{\delta(x)^{2q(x)}} |tu_0|^{q(x)}} dx \\ &\leq \frac{c_1}{c_2} t^{p(\xi_1) - p(\xi_2)}, \end{aligned}$$

where $c_1 = \int_{B(\partial U,\varepsilon_1)} A\left(|\Delta u_0(x)|^{p(x)}\right) dx$ and

$$c_2 = \int_{B(x_0,\varepsilon_2)} \omega(x)|u_0(x)|^{p(x)}dx \;-\; \mu \int_{B(x_0,\varepsilon_2)} \frac{1}{\delta(x)^{2q(x)}}|u_0(x)|^{q(x)}dx.$$

Note that c_1 and c_2 are positive constants independent of t, with $\xi_1 \in \overline{B(\partial U,\varepsilon_1)}$ and $\xi_2 \in \overline{B(x_0,\varepsilon_2)}$. Using (7.1) and (7.2) we get $|p(\xi_1) - p(\xi_2)| > \varepsilon_0$.

Therefore,

$$R(tu_0) \leq \frac{c_1}{c_2} t^{\varepsilon_0} \quad \textit{for all } t \in (0;1),$$

when $t \to 0^+$, we obtain $R(u) = 0$. In view of Lemma 7.1, we deduce $\lambda_1 = 0$. This complete the proof. □

Proof. **of Corralary 2.4**

(i) For $u \in \mathcal{H}$, set $K_1 = \{u, -u\}$. It is clear that $\gamma(K_1) = 1$.
ϕ is even and $\phi(u) = \max\limits_{K_1} \phi \geq \inf\limits_{K\in\Sigma_1} \max\limits_{u\in K} \phi(u) = \lambda_1$.
On the other hand, for all $K \in \Sigma_1$ and $u \in K$, we have $\sup\limits_{u\in K} \phi \geq \phi(u) \geq \inf\limits_{u\in\mathcal{H}} \phi(u)$. It follows that $\inf\limits_{k\in\Sigma_1} \max\limits_{K} \phi = \lambda_1 \geq \inf\limits_{u\in\mathcal{H}} \phi(u)$.
Then,

$$\lambda_1 = \inf\left\{\int_\Omega \frac{1}{p(x)} A(|\Delta u|)^{p(x)}dx \;\mid\; u \in X \text{ and } \int_\Omega \frac{\omega(x)}{p(x)}|u|^{p(x)}dx - \mu\int_\Omega \frac{1}{q(x)}\frac{|u|^{q(x)-2}u}{\delta(x)^{2q(x)}}dx = 1\right\}.$$

(ii) For all $i \geq j$, we have $\Sigma_i \subset \Sigma_j$, and in view of λ_i, $i \in \mathbb{N}^*$, we get $\lambda_i \geq \lambda_j$. As regards $\lambda_n \to \infty$ it has been proved in Theorem 2.2.

□

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