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Weak Solution to p(x)-Kirchoff Type Problems Under no-flux Boundary Condition by Topological Degree

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ABSTRACT: This paper is concerned with the existence of weak solutions of $p(x)$-Kirchhoff type problems with no-flux boundary condition. Our technical approach is based on topological degree methods of Berkovits.

Key Words: Generalized $p(x)$-Kirchhoff-type problems, Sobolev spaces with variable exponent, Berkovits topological degree, Sobolev space with variable exponents.

Contents

1. Introduction

In this paper, we study the existence of weak solution of the following Kirchhoff type problem,

$$\begin{cases} -\mathcal{M}\Big(\displaystyle\int_\Omega (A(x,\nabla u) + \frac{1}{p(x)}|\nabla u|^{p(x)})dx\Big)\Big[\operatorname{div} a(x,\nabla u) - |\nabla u|^{p(x)-2}\nabla u\Big] = \lambda f(x,u,\nabla u) & \text{in} \quad \Omega, \\ u = \text{constant} & \text{on} \quad \partial\Omega \\ \displaystyle\int_{\partial\Omega} a(x,\nabla u)\nu d\Gamma = 0. \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N (N \geqslant 2)$ with a Lipschitz boundary denoted by $\partial\Omega$, $p \in C_+(\overline{\Omega})$, λ is a real parameter, $-\operatorname{div} a(x,\nabla u)$ is a Leray-Lions operator. In the statement of problem (1.1). $a, f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are two Carathéodoryâ€™s functions and $\mathcal{M} : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous function.

The study of various mathematical problems with variable exponent growth condition has been received considerable attention in recent years, we can for example refer to [11,7,4]. This great interest may be justified by their various physical applications.
As it is well known, problem (1.1) is related to the stationary version of the Kirchhoff equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \Big(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \Big|\frac{\partial u}{\partial x}\Big|^2 dx\Big)\frac{\partial^2 u}{\partial x^2} = 0 \tag{1.2}$$

presented by Kirchhoff in 1883 [16], where E, ρ_0, ρ, h, L are constants. This equation is an extension of the classical d'Alembert's wave equation by considering the effects of the changes in the length of the strings during the vibrations. Kirchhoff-type boundary value problems model several physical and biological systems where u describes a process which depend on the average of itself, as for example, the population density. Recently, Kirchhoff-type problems have been studied in many papers, we refer to see

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[1,7,8,9,14,12,20,19]. In the present work, by using the topological degree methods of Berkovits, which is frequently utilised in the study of nonlinear equations, particularly elliptic equations, Brouwer created the first topological degree in 1912 for continuous mappings in finite dimensional Euclidean spaces [6], Leray and Schauder generalized it in 1934 for compact operators in Banach spaces of infinite dimension [18], later, the theory was constructed by Berkovits [3,5], the existence of weak solutions for the problem (1.1) is established.

The remaining part of the paper is the following: In Section 2, we introduce some notations and functional spaces. In Section 3, we show some basic assumptions and we define the notion of weak solution. We end in Section 4 by proving the existence of weak solution for problem(1.1).

2. Mathematical background

2.1. Lebesgue-Sobolev spaces with variable exponent

In this subsection we give some definitions and results about Lebesgue-Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$.

Let Ω be a smooth bounded domain in $\mathbb{R}^N(N \geq 2)$, with a Lipschitz boundary denoted by $\partial\Omega$. Set

$$C_+(\overline{\Omega}) = \Big\{p : p \in C(\overline{\Omega}) \text{ such that } p(x) > 1 \text{ for any } x \in \overline{\Omega}\Big\}.$$

For each $p \in C_+(\overline{\Omega})$, we define

$$p^+ \ := \max\Big\{p(x),\ x \in \overline{\Omega}\Big\} \ \text{ and } \ p^- := \min\Big\{p(x),\ x \in \overline{\Omega}\Big\}.$$

For every $p \in C_+(\overline{\Omega})$, we define

$$L^{p(x)}(\Omega) = \Big\{u : \Omega \to \mathbb{R} \ \text{ is measurable such that } \ \int_\Omega |u(x)|^{p(x)}dx < +\infty\Big\},$$

equipped with the Luxemburg norm

$$|u|_{p(x)} = \inf\{\lambda > 0 : \rho_{p(x)}\left(\frac{u}{\lambda}\right) \leq 1\},$$

where

$$\rho_{p(x)}(u) = \int_\Omega |u(x)|^{p(x)}dx,\ \forall\ u \in L^{p(x)}(\Omega).$$

Proposition 2.1. *[13] Let (u_n) and $u \in L^{p(\cdot)}(\Omega)$, then*

$$|u|_{p(x)} < 1(resp. = 1; > 1) \ \Leftrightarrow \ \rho_{p(x)}(u) < 1(resp. = 1; > 1) \tag{2.1}$$

$$|u|_{p(x)} > 1 \ \Rightarrow \ |u|^{p^-}_{p(x)} \leq \rho_{p(x)}(u) \leq |u|^{p^+}_{p(x)}, \tag{2.2}$$

$$|u|_{p(x)} < 1 \ \Rightarrow \ |u|^{p^+}_{p(x)} \leq \rho_{p(x)}(u) \leq |u|^{p^-}_{p(x)}, \tag{2.3}$$

$$\lim_{n\to\infty} |u_n - u|_{p(x)} = 0 \ \Leftrightarrow \ \lim_{n\to\infty} \rho_{p(x)}(u_n - u) = 0. \tag{2.4}$$

Remark 2.2. *According to (2.2) and (2.3), we have*

$$|u|_{p(x)} \leq \rho_{p(x)}(u) + 1, \tag{2.5}$$

$$\rho_{p(x)}(u) \leq |u|^{p^-}_{p(x)} + |u|^{p^+}_{p(x)}. \tag{2.6}$$

Proposition 2.3. *[17] The spaces $L^{p(x)}(\Omega)$ is a separable and reflexive Banach spaces.*

Proposition 2.4. *[17] The conjugate space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$ where $\frac{1}{p(x)}+\frac{1}{p'(x)}=1$ for all $x\in\Omega$. For any $u\in L^{p(x)}(\Omega)$ and $v\in L^{p'(x)}(\Omega)$, we have the following Hölder-type inequality*

$$|\int_\Omega uv\ dx|\leq\left(\frac{1}{p-}+\frac{1}{p'-}\right)|u|_{p(x)}|v|_{p'(x)}\leq 2|u|_{p(x)}|v|_{p'(x)}. \tag{2.7}$$

Remark 2.5. *If $r_1,\, r_2\in C_+(\overline{\Omega})$ with $r_1(x)\leq r_2(x)$ for any $x\in\overline{\Omega}$, then there exists the continuous embedding $L^{r_2(x)}(\Omega)\hookrightarrow L^{r_1(x)}(\Omega)$.*

Now, let $p\in C_+(\overline{\Omega})$ and we define $W^{1,p(x)}(\Omega)$ as

$$W^{1,p(x)}(\Omega)=\Big\{u\in L^{p(x)}(\Omega)\text{ such that }|\nabla u|\in L^{p(x)}(\Omega)\Big\},$$

equipped with the norm

$$||u||=|u|_{p(x)}+|\nabla u|_{p(x)}.$$

Proposition 2.6. *[13,17] The space $\Big(W^{1,p(x)}(\Omega),|\cdot|_{1,p(x)}\Big)$ is separable and reflexive Banach space.*

In this paper we will try to find weak solution for problem (1.1) in the following space

$$\mathcal{X}:=\Big\{u\in W^{1,p(x)}(\Omega):\quad u|_{\partial\Omega}=constant\Big\}. \tag{2.8}$$

The space $\mathcal{X}$ is a closed subspace of the separable and reflexive Banach space $W^{1,p(x)}(\Omega)$ (See [4]) , so $\mathcal{X}$ is also separable and reflexive Banach space with the norm $\|\cdot\|$.

2.2. Review on some classes of mappings and topological degree theory

Now, we give some results and properties from the Berkovits degree theory for demicontinuous operators of generalized (S_+) type in real reflexive Banach. We start by defining some classes of mappings. In what follows, let X be a real separable reflexive Banach space with dual X^* and with continuous dual pairing $\langle\cdot\,,\,\cdot\rangle$ and given a nonempty subset Ω of X , and $\rightharpoonup$ represents the weak convergence.

Let Y be another real Banach space.

Definition 2.7. *The operator $F:\Omega\subset X\to Y$ is said to be bounded, if it takes any bounded set into a bounded set.*

Definition 2.8. *The operator $F:\Omega\subset X\to Y$ is said to be demicontinuous, if for any $(u_n)\subset\Omega$, $u_n\to u$ implies $F(u_n)\rightharpoonup F(u)$.*

Definition 2.9. *The operator $F:\Omega\subset X\to Y$ is said to be compact, if it is continuous and the image of any bounded set is relatively compact.*

Definition 2.10. *A mapping $F:\Omega\subset X\to X^*$ is said to be of type (S_+), if for any $(u_n)\subset\Omega$ with $u_n\rightharpoonup u$ and $\limsup\limits_{n\to\infty}\langle Fu_n,u_n-u\rangle\leq 0$, we have $u_n\to u$.*

Definition 2.11. *The operator $F:\Omega\subset X\to X^*$ is said to be quasimonotone, if $u_n\rightharpoonup u$ implies $\limsup\limits_{n\to\infty}\langle Fu_n,\ u_n-u\rangle\geq 0$.*

Definition 2.12. *Let $T:\Omega_1\subset X\to X^*$ be a bounded operator such that $\Omega\subset\Omega_1$. For any operator F $:\Omega\subset X\to X$, we say that F satisfies condition $(S_+)_T$, if for any $(u_n)\subset\Omega$ with $u_n\rightharpoonup u$, $y_n:=Tu_n\rightharpoonup y$ and $\limsup\limits_{n\to\infty}\langle Fu_n,\ y_n-y\rangle\leq 0$, we have $u_n\to u$.*

Remark 2.13. *(see [22])*

1. *If a mapping is compact in a set, then it is quasi-monotone in that set.*

2. *If the mapping is demi-continuous and satisfies the condition (S_+) in a set, then it is quasimonotone in that set.*

In the following, we consider the following classes of operators :

$$\mathcal{F}_1(\Omega) := \{F : \Omega \to X^* \mid F \text{ is bounded, demicontinuous and satifies condition}(S_+)\},$$
$$\mathcal{F}_T(\Omega) := \{F : \Omega \to X \mid F \text{ is demicontinuous and satifies condition}(S_+)_T\},$$

Lemma 2.14. *[5] Lets $T \in \mathcal{F}_1(\overline{G})$ be continuous and $S : D_S \subset X^* \to X$ be demicontinuous such that $T(\overline{G}) \subset D_s$, where G is a bounded open set in a real reflexive Banach space X. Then the following statements are true :*

1. If *S is quasimonotone, then $I + SoT \in \mathcal{F}_T(\overline{G})$, where I denotes the identity operator.*
2. *If S is of class (S_+), then $SoT \in \mathcal{F}_T(\overline{G})$.*

Definition 2.15. *Let G is to be a bounded open subset of a real reflexive Banach space X, $T \in \mathcal{F}_1(\overline{G})$ be continuous and let $F, S \in \mathcal{F}_T(\overline{G})$. We define an affine homotopy $\Lambda : [0,1] \times \overline{G} \to X$ by*

$$\Lambda(t,u) := (1-t)Fu + tSu \qquad \text{for} \qquad (t,u) \in [0,1] \times \overline{G}$$

is called an admissible affine homotopy with the common continuous essential inner map T.

Remark 2.16. *[5]* The above affine homotopy satisfies condition $(S_+)_T$.

Let $\mathcal{O}$ be the collection of all bounded open set in X. we give the Berkovits topological degree for a class of demicontinuous operator satisfying condition $(S_+)_T$ for more details see [5].

Theorem 2.17. *Let $M = \big\{(F, G, h) \mid G \in \mathcal{O},\ T \in \mathcal{F}_1(\overline{G}),\ F \in \mathcal{F}_T(\overline{G}),\ h \notin F(\partial E)\big\}$. There exists a unique degree function $d : M \longrightarrow \mathbb{Z}$ which satisfies the following properties :*

1. *(* Normalization*) For any $h \in G$, we have $d(I, E, h) = 1$.*
2. *(* Additivity*) Let $F \in \mathcal{F}_T(\overline{G})$. If G_1 and G_2 are two disjoint open subsets of G such that $h \notin F(\overline{G} \backslash (G_1 \cup G_2))$ then we have*
$$d(F, G, h) = d(F, G_1, h) + d(F, G_2, h)\ .$$
3. *(* Homotopy invariance*) If $\Lambda : [0,1] \times \overline{G} \to X$ is a bounded admissible affine homotopy with a common continuous essential inner map and h: $[0,1] \to X$ is a continuous path in X such that $h(t) \notin \Lambda(t, \partial G)$ for all $t \in [0,1]$, then the value of $d(\Lambda(t,\cdot), G, h(t))$ is constant for all $t \in [0,1]$.*
4. *(* Existence*) if $d(F, G, h) \neq 0$, then the equation $Fu = h$ has a solution in G.*

3. Basic assumptions and technical Lemmas

Definition 3.1. *We call that $u \in \mathcal{X}$ is a weak solution of problem (1.1) if*

$$\mathcal{M}\Big(\int_\Omega (A(x,\nabla u) + \frac{1}{p(x)}|\nabla u|^{p(x)})\,dx\Big)\Big[\int_\Omega a(x,\nabla u)\nabla v + \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v\Big] = \lambda \int_\Omega f(x,\ u, \nabla u) v\,dx,$$

for all $v \in \mathcal{X}$.

In this paper, we assume that $a(x,\xi) : \Omega \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a Carathéodory and a continuous derivative with respect to ξ of the continuous mapping $A(x,\xi) : \Omega \times \mathbb{R}^N \longrightarrow \mathbb{R}$. $A = A(x,\xi)$, $a(x,\xi) = \nabla_\xi A(x,\xi)$, and for a. e. x in Ω and all $\xi, \xi' \in \mathbb{R}^N$, $(\xi \neq \xi')$.

(A_1) $\qquad A(x,0) = 0,$

(A_2) $\qquad \alpha|\xi|^{p(x)} \leq a(x,\xi)\cdot\xi \leq p(x)A(x,\xi),$

(A_3) $\qquad |a(x,\xi)| \leq \eta\big(k(x) + |\xi|^{p(x)-1}\big)$

(A_4) $\qquad [a(x,\xi) - a(x,\xi')]\cdot(\xi-\xi') > 0,$

where $\alpha,\ \eta$ are some positive constants and $k(x)$ is a positive function in $L^{p'(x)}(\Omega)$, ($p'(x)$ is the conjugate exponent of $p(x)$).

The Carathéodory's function f is defined from $\Omega\times\mathbb{R}\times\mathbb{R}^N$ into $\mathbb{R}$ and it is satisfies only the growth condition, for all $t\in\mathbb{R}^N$, $s\in\mathbb{R}$ and a.e. $x\in\Omega$.

(f_1) $\qquad |f(x,s,\xi)| \leq \varrho(e(x) + |s|^{p(x)-1} + |\xi|^{p(x)-1}),$

where ϱ is a positive constant, $e(x)$ is a positive function in $L^{q'(x)}(\Omega)$.

(M_1) $\mathcal{M}:\ \mathbb{R}^+\to\mathbb{R}^+$ is continuous and non-decreasing function, for which there exist two positive constant and m_1 such that,

$$t^{r(x)-1} \leq \mathcal{M}(t) \leq m_1 t^{r(x)-1}$$

where $r(x)\in C_+(\overline{\Omega})$ and $1\leq r^-\leq r(x)\leq r^+\leq p^-\leq p(x)\leq p^+$, for all $t\in[0,+\infty[$.

Lemma 3.2. *([2]) Let $g\in L^{r(x)}(\Omega)$ and $g_n\subset L^{r(x)}(\Omega)$ such that $\|g_n\|_{r(x)}\leq C$, If $g_n(x)\to g(x)$ a.e. in Ω then $g_n\rightharpoonup g$ weakly in $L^{r(x)}(\Omega)$.*

Lemma 3.3. *([2]) Assume that (A_2)-(A_4) hold, let $(u_n)_n$ be a sequence in $W^{1,p(x)}(\Omega)$ such that $u_n\rightharpoonup u$ weakly in $W^{1,p(x)}(\Omega)$ and*

$$\int_\Omega \big[a(x,\nabla u_n) - a(x,\nabla u)\big]\nabla(u_n-u)dx \longrightarrow 0, \tag{3.1}$$

then $u_n\longrightarrow u$ strongly in $\mathcal{X}$.

Let us consider the following functional

$$\mathcal{J}(u) = \widehat{\mathcal{M}}\Big(\int_\Omega \big(A(x,\nabla u) + \frac{1}{p(x)}|\nabla u|^{p(x)}\big)dx\Big), \qquad \forall u\in\mathcal{X}$$

where $\widehat{M}:\ [0,+\infty[\longrightarrow[0,+\infty[$ be the primitive of the function M, defned by

$$\widehat{\mathcal{M}}(t) = \int_0^t \mathcal{M}(\xi)d\xi.$$

It is well known that $\mathcal{J}$ is well defined and continuously Gâteaux differentiable whose Gâteaux derivatives at point $u\in\mathcal{X}$ is the functional $\mathcal{J}'(u)\in\mathcal{X}^*$ setting by

$$\langle\mathcal{J}'(u)\,,\,v\rangle = \mathcal{M}\Big(\int_\Omega (A(x,\nabla u) + \frac{1}{p(x)}|\nabla u|^{p(x)})\,dx\Big)\,\Big[\int_\Omega a(x,\nabla u)\nabla v + \int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla v\Big]$$

for all $u,v\in\mathcal{X}$.
On other hand, we consider the functional $\mathcal{L}:\mathcal{X}\to\mathbb{R}$ defined by:

$$\mathcal{L}(u) = \int_\Omega (A(x,\nabla u) + \frac{1}{p(x)}|\nabla u|^{p(x)})\,dx \quad \text{For all} \quad v\in\mathcal{X}$$

then $\mathcal{L}\in C^1(\mathcal{X},\mathbb{R})$ and,

$$\langle\mathcal{L}'(u)\,,\,v\rangle = \int_\Omega a(x,\nabla u)\nabla v + \int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla v.$$

Let us consider the operator $\mathcal{T}$ acting from $\mathcal{X}$ to its dual $\mathcal{X}^*$ is defined by

$$\begin{aligned}\langle\mathcal{T}u\,,\,v\rangle &= \langle\mathcal{J}'(u)\,,\,v\rangle\\ &= \mathcal{M}\Big(\int_\Omega (A(x,\nabla u) + \frac{1}{p(x)}|\nabla u|^{p(x)})\,dx\Big)\,\Big[\int_\Omega a(x,\nabla u)\nabla v + \int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla v\Big]\end{aligned} \tag{3.2}$$

for all $u,v\in\mathcal{X}$.

Proposition 3.4. *Suppose that* (M_1), $(A_1)-(A_4)$ *hold, then*

(i) $\mathcal{T}$ *is bounded, coercive, continuous operator.*

(ii) $\mathcal{T}$ *is of type* (S_+).

Proof.

$i)$ It is clear that $\mathcal{T}$ is continuous, because $\mathcal{T}$ is the Fréchet derivative of $\mathcal{J}$.

Now, we prove that the operator $\mathcal{T}$ is bounded.

Let $u, v \in \mathcal{X}$, by the Hölder's inequality and (M_1), we obtain

$$
\begin{aligned}
|< \mathcal{T}u, v >| &= \Big|\mathcal{M}\Big(\int_\Omega (A(x,\nabla u) + \frac{1}{p(x)}|\nabla u|^{p(x)})\,dx\Big)\Big[\int_\Omega a(x,\nabla u)\nabla v + \int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla v dx\Big]\Big| \\
&\leq m_1\Big(\int_\Omega (A(x,\nabla u) + \frac{1}{p(x)}|\nabla u|^{p(x)})\,dx\Big)^{r(x)-1}\Big[\int_\Omega |a(x,\nabla u)\ \nabla v|\ dx + \int_\Omega |\nabla u|^{p(x)-1}\|\nabla v| dx\Big] \\
&\leq \text{const}\Big(\Big(\int_\Omega A(x,\nabla u)dx\Big)^{r(x)-1} + \Big(\int_\Omega |\nabla u|^{p(x)}dx\Big)^{r(x)-1}\Big)\times \\
&\Big[|a(x,\nabla u)\,|_{p'(x)}|\nabla v\ |_{p(x)} + \Big(\int_\Omega |\nabla u^{p(x)-1}|^{p'(x)}dx\Big)^{1/\theta}|\nabla v|_{p(x)}\Big] \\
&\leq \text{const}\Big(\Big(\int_\Omega A(x,\nabla u)dx\Big)^{r(x)-1} + \|u\|^{\gamma(r(x)-1)}\Big)\times\Big[|a(x,\nabla u)|_{p'(x)}\|v\| + \|u\|^{\gamma/\theta}\|v\|\Big],
\end{aligned}
$$

wehere

$$
\gamma = \begin{cases} p^- & \text{if} \quad \|u\| \leq 1 \\ p^+ & \text{if} \quad \|u\| \geq 1, \end{cases}
$$

and

$$
\theta = \begin{cases} p'^- & \text{if} \quad |\nabla u^{p(x)-1}|_{p'(x)} \leq 1 \\ p'^+ & \text{if} \quad |\nabla u^{p(x)-1}|_{p'(x)} \geq 1. \end{cases}
$$

By (A_1) we have for any $x \in \Omega$ and $\xi \in \mathbb{R}^n$

$$
A(x,\xi) = \int_0^1 \frac{d}{ds}A(x,s\xi)ds = \int_0^1 a(x,s\xi)\xi ds
$$

and by combining (A_3), Fubini's theorem and Young's inequality we have

$$
\begin{aligned}
\int_\Omega A(x,\nabla u)dx &= \int_\Omega\int_0^1 a(x,s\nabla u)\nabla u ds\, dx \\
&= \int_0^1 \Big[\int_\Omega a(x,s\nabla u)\nabla u dx\Big]ds \\
&\leq \int_0^1 \Big[C_{p'}\int_\Omega \big|a(x,s\nabla u)\big|^{p'(x)}dx + C_p\int_\Omega |\nabla u|^{p(x)}dx\Big]ds \\
&\leq C_1 + C'\int_0^1\int_\Omega |s\,\nabla u|^{p(x)}\,dx\,ds + C_p\|u\|^\gamma \\
&\leq C_1 + C_2\int_\Omega |\nabla u|^{p(x)}\,dx + C_p\|u\|^\gamma \\
&\leq C_m(\|u\|^\gamma + 1), \quad \big(\gamma \quad \text{is defined above}\big).
\end{aligned}
\tag{3.3}
$$

From (A_3), we can easily show that $|a(x,\nabla u)|_{p'(x)}$ is bounded for all u in $\mathcal{X}$. Therefore

$$
|\langle \mathcal{T}u, v\rangle| \leq const\,\|v\|\,,
$$

as a result the operator $\mathcal{T}$ is bounded.

Next, we prove that the operator $\mathcal{T}$ is coercive.

Let $u \in \mathcal{X}$, from (A_2) and (M_1), we obtain

$$\begin{aligned}
\frac{\langle \mathcal{T}u, u\rangle}{\|u\|} &= \frac{\mathcal{M}\Big(\displaystyle\int_\Omega (A(x,\nabla u) + \frac{1}{p(x)}|\nabla u|^{p(x)})\,dx\Big)\Big[\displaystyle\int_\Omega a(x,\nabla u)\nabla u + \int_\Omega |\nabla u|^{p(x)}dx\Big]}{\|u\|} \\
&\geq \frac{\Big(\displaystyle\int_\Omega (A(x,\nabla u) + \frac{1}{p(x)}|\nabla u|^{p(x)})\,dx\Big)^{r(x)-1}\Big[\displaystyle\int_\Omega a(x,\nabla u)\nabla u + \int_\Omega |\nabla u|^{p(x)}dx\Big]}{\|u\|} \\
&\geq (\frac{\Big(\displaystyle\frac{\alpha}{p^+}\int_\Omega |\nabla u|^{p(x)}dx + \frac{1}{p^+}\int_\Omega |\nabla u|^{p(x)})\,dx\Big)^{r(x)-1}\Big[\alpha\displaystyle\int_\Omega |\nabla u|^{p(x)} + \int_\Omega |\nabla u|^{p(x)}dx\Big]}{\|u\|} \\
&\geq C_1\frac{\Big(\displaystyle\int_\Omega |\nabla u|^{p(x)} + |u|^{p(x)}dx\Big)^{r(x)}}{\|u\|} - C_2\frac{\Big(\displaystyle\int_\Omega |u|^{p(x)}\Big)^{r(x)}}{\|u\|} \\
&\geq C_1\ \frac{\|u\|^{\gamma r(x)}}{\|u\|} - C_2\frac{|u|^{\beta r(x)}}{\|u\|} \\
&\geq C\|u\|^{\gamma r(x)-1} - C'.
\end{aligned}$$

Which means that $\dfrac{\langle \mathcal{T}u, u\rangle}{\|u\|} \to \infty$ as $\|u\| \to \infty$.

Where γ is defined above, and

$$\beta = \begin{cases} p^- & \text{if} \quad |u|_{p(x)} \leq 1 \\ p^+ & \text{if} \quad |u|_{p(x)} \geq 1. \end{cases}$$

Now, we prove that $\mathcal{T}$ is strictly monotone operator. The monotonicity of $\mathcal{L}$ follows easily from the following inequalities By using (A_4), and taking into the inequality (see [15]), For all $\xi, \eta \in \mathbb{R}^N$,

$$\begin{aligned}
(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)(\xi-\eta).(|\xi|^p + |\eta|^p)^{\frac{2-p}{p}} &\geq (p-1)|\xi-\eta|^p \quad \text{if} \quad 1<p<2, \\
(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)(\xi-\eta) &\geq (\frac{1}{2})^p|\xi-\eta|^p \quad \text{if} \qquad p \geq 2.
\end{aligned} \tag{3.4}$$

We obtain for all u, $v \in \mathcal{X}$ with $u \neq v$,

$$\langle \mathcal{L}'(u) - \mathcal{L}'(v), u-v\rangle > 0$$

which implies that $\mathcal{L}'$ is strictly monotone. Thus, by Prop. 25.10 in [22], $\mathcal{L}$ is strictly convex. Furthermore, as M is nondecreasing, then $\widehat{M}$ is convex in $[0,+\infty[$. So, for any $u, v \in \mathcal{X}$ with $u \neq v$, and every $s, t \in (0,1)$ with $s+t=1$, we have

$$\widehat{\mathcal{M}}(\mathcal{L}(su+tv)) < \widehat{\mathcal{M}}(s\mathcal{L}(u) + t\mathcal{L}(v)) \leq s\widehat{M}(\mathcal{L}(u)) + t\widehat{\mathcal{M}}(\mathcal{L}(v)).$$

This proves that $\mathcal{J}$ is strictly convex, since $\mathcal{J}'(u) = \mathcal{T}(u)$ in $\mathcal{X}^*$ we infer that $\mathcal{T}$ is strictly monotone in $\mathcal{X}$.

$ii)-$ We verify that the operator $\mathcal{T}$ is of type (S_+).

Assume that $(u_n)_n \subset \mathcal{X}$ and

$$\begin{cases} u_n \rightharpoonup u & \text{in } \mathcal{X} \\ \limsup\limits_{n\to\infty}\langle \mathcal{T}u_n, u_n - u\rangle \leq 0. \end{cases} \tag{3.5}$$

We will show that $u_n \to u$ in $\mathcal{X}$.

On the one hand, in fact $u_n \rightharpoonup u$ in $\mathcal{X}$, so $(u_n)_n$ is a bounded sequence in $\mathcal{X}$, then there exist a subsequence still denoted by $(u_n)_n$ such that $u_n \rightharpoonup u$ in $\mathcal{X}$, under the strict monotonicity of $\mathcal{T}$ we get

$$0 = \limsup_{n\to\infty} \langle \mathcal{T}u_n - \mathcal{T}u,\ u_n - u\rangle = \lim_{n\to\infty} \langle \mathcal{T}u_n - \mathcal{T}u,\ u_n - u\rangle. \tag{3.6}$$

Then

$$\lim_{n\to\infty} \langle \mathcal{T}u_n,\ u_n - u\rangle = 0,$$

which means

$$\lim_{n\to\infty} \mathcal{M}(\mathcal{L}(u_n)) \Big[\int_\Omega a(x, \nabla u_n)\nabla(u_n - u) + \int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n(u_n - u)dx \Big] = 0. \tag{3.7}$$

Since $\displaystyle\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\, dx$ is bounded and by (3.3), we infer that $\mathcal{L}(u_n)$ is bounded.

As M is continuous, up to a subsequence there is $k \geq 0$ such that

$$\mathcal{M}(\mathcal{L}(u_n)) \longrightarrow \mathcal{M}(k) \geq k^{r(x)-1} \qquad \text{as} \quad n \to \infty. \tag{3.8}$$

From (3.7) and (3.8), we get

$$\Big[\lim_{n\to\infty} \int_\Omega a(x, \nabla u_n)\nabla(u_n - u)dx + \int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n(\nabla u_n - \nabla u)dx \Big] = 0.$$

Using the continuous embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$, we have $\displaystyle\lim_{n\to\infty} \int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n(\nabla u_n - \nabla u)dx = 0$. Then

$$\lim_{n\to\infty} \int_\Omega a(x, \nabla u_n)\nabla(u_n - u)dx = 0$$

In light of Lemma 3.3, we obtain

$$u_n \longrightarrow u \qquad \text{strongly in } \ \mathcal{X},$$

which implies that $\mathcal{T}$ is of type (S_+).

Lemma 3.5. *If the condition (f_1) hold. Then the operator $S : \mathcal{X}^* \to \mathcal{X}^*$ defined by*

$$\langle \mathcal{S}\, u,\ v\rangle = -\lambda \int_\Omega f(x, u, \nabla u)\, v\, dx, \qquad \forall u,\ v \in \mathcal{X}$$

is compact.

Proof.

In ordre to prove this lemma, we proceed in tow steps.

Step 1:

Let us consider the operator $\psi : \mathcal{X} \to L^{p'(x)}(\Omega)$ defined by

$$\psi u(x) = -\lambda f(x, u, \nabla u)$$

Now we show that the operator ψ is bounded and continuous. For this, let $u, v \in \mathcal{X}$, using the growth condition (f_1) we have

$$
\begin{aligned}
|\psi\, u|^{\theta}_{p'(x)} &\leq \int_\Omega |\lambda f(x,u,\nabla u)|^{p'(x)}dx \\
&\leq \int_\Omega |\lambda|^{p'(x)}\Big|\varrho\Big(e(x)+|u|^{p(x)-1}+|\nabla u|^{p(x)-1}\Big)\Big|^{p'(x)}dx \\
&\leq const(|\lambda|^{p'^-}+|\lambda|^{p'^+})\int_\Omega \big(|e(x)|^{p'(x)}+|u|^{(p(x)-1)p'(x)}+|\nabla u|^{(p(x)-1)p'(x)}\big)dx \\
&\leq const(|\lambda|^{p'^-}+|\lambda|^{p'^+})\int_\Omega \big(|e(x)|^{p'(x)}+|u|^{p(x)}+|\nabla u|^{p(x)}\big)dx \\
&\leq const\int_\Omega |e(x)|^{p'(x)}dx + const\int_\Omega |u|^{p(x)}+|\nabla u|^{p(x)}\,dx \\
&\leq const|e|^{\theta_1}_{p'(x)} + const\|u\|^{\gamma} \\
&\leq C_{max}(\|u\|^{\gamma}+1),
\end{aligned}
\tag{3.9}
$$

where $C_{max}=\max\big(const\|e\|^{\theta_1}_{p'(x)}, const\big)$, and

$$\theta=\begin{cases} p'^+ & \text{if} \quad |\psi\, u|_{p'(x)}\leq 1\\ p'^- & \text{if} \quad |\psi\, u|_{p'(x)}\geq 1.\end{cases}$$

$$\theta_1=\begin{cases} p'^+ & \text{if} \quad |e|_{p'(x)}\leq 1\\ p'^- & \text{if} \quad |e|_{p'(x)}\geq 1\end{cases} \quad and, \quad \gamma=\begin{cases} p^+ & \text{if} \quad \|u\|\leq 1\\ p^- & \text{if} \quad \|u\|\geq 1.\end{cases}$$

Therefore ψ is bounded on $\mathfrak{X}$.
Next, we show that ψ is continuous, Not that if $u_n\to u$ in $\mathfrak{X}$, then $u_n\to u$ in $L^{p(x)}(\Omega)$ and $\nabla u_n\to\nabla u$ in $\big(L^{p(x)}(\Omega)\big)^N$. Thus there exist a subsequence still denoted by (u_n) and measurable functions φ in $L^{p(x)}(\Omega)$ and σ in $\big(L^{p(x)}(\Omega)\big)^N$ such that

$$u_n(x)\to u(x) \quad \text{and} \quad \nabla u_n(x)\to\nabla u(x),$$

$$|u_n(x)|\leq\varphi(x) \quad \text{and} \quad |\nabla u_n(x)|\leq|\sigma(x)|,$$

for a.e. $x\in\Omega$ and all $n\in\mathbb{N}$. Since satisfies the Carathéodory condition, we obtain

$$f(x,u_n(x),\nabla u_n(x))\to f(x,u(x),\nabla u(x)) \quad \text{a.e. } x\in\Omega. \tag{3.10}$$

Thanks to (f_1) we obtain

$$|f(x,u_n(x),\nabla u_n(x))|\leq\varrho\big(e(x)+|\varphi(x)|^{p(x)-1}+|\sigma(x)|^{p(x)-1}\big)$$

for a.e. $x\in\Omega$ and for all $k\in\mathbb{N}$.
Since

$$e(x)+|\varphi(x)|^{p(x)-1}+|\sigma(x)|^{p(x)-1}\in L^{p'(x)}(\Omega),$$

and from (3.10), we get

$$\int_\Omega |f(x,u_k(x),\nabla u_k(x))-f(x,u(x),\nabla u(x))|^{p'(x)}dx\longrightarrow 0,$$

by using the dominated convergence theorem we have

$$\psi u_k\to\psi u \quad \text{in} \quad L^{p'(x)}(\Omega).$$

Thus the entire sequence (ψu_n) converges to ψu in $L^{p'(x)}(\Omega)$ and then ψ is continuous.

The canonical linear embedding $I:\mathfrak{X}\to L^{p(x)}(\Omega)$ is compact by Rellichâ€™s embedding theorem, so the adjoint operator, then $I^*: L^{p'(x)}(\Omega)\to\mathfrak{X}^*$ is compact. Hence the compositions $I^*\circ\psi$ is compact. that means $\mathcal{S}=I^*\circ\psi$ is compact.

4. Main results

Theorem 4.1. *Suppose that the hypotheses $(A_1)-(A_4)$, (f_1) and (M_1) hold true. Then there exists at least one weak solution u in $\mathcal{X}$ of the problem (1.1).*

Proof.
Let $u \in \mathcal{X}$ be a weak solutions of the problem (1.1) if and only if

$$\mathcal{T}u = -\mathcal{S}u, \tag{4.1}$$

where $\mathcal{T}$, $\mathcal{S}$ be two operators as defined in (3.2) and Lemma 3.5 respectively.

On the one hand, from Proposition 3.4 the operator $\mathcal{T}$ given in (3.2) is strictly monotone, bounded, continuous, coercive and satisfies condition (S_+). Then, by using the Minty-Browder Theorem (see [22], Theorem 26 A), the inverse operator $\mathcal{G} = \mathcal{T}^{-1} : \mathcal{X}^* \to \mathcal{X}$ exists and is bounded. Moreover, it is continuous and satisfies condition (S_+).

On the other hand, notice by Lemma 3.5 the operator $\mathcal{S}$ is bounded, quasimonotone and continuous. Hence, equation (4.1) is equivalent to the abstract Hammerstein equation

$$u = \mathcal{G}v \quad \text{and} \quad v + \mathcal{S} \circ \mathcal{G}v = 0. \tag{4.2}$$

To solve the equations (4.2), we will employ the Berkovits topological degree seen in section above. For this, let us consider the set

$$\mathcal{B} := \Big\{ \ v \ \in \ (\mathcal{X}^* \setminus \quad v + t\mathcal{S} \circ \mathcal{G}v = 0 \quad \text{for some} \quad t \in [0,1] \Big\}.$$

First, we show that the set B is bounded in $\mathcal{X}^*$.

Let $\ v \in \mathcal{B}$ and take $\ u := \mathcal{G}v$. According to (A_2), (f_1) and by the Young's inequality, we obtain

$$\begin{aligned}
\|\mathcal{G}v\|^{\gamma} &= \int_\Omega (|u|^{p(x)} + |\nabla u|^{p(x)})dx \\
&\leq \int_\Omega |u|^{p(x)}dx + \frac{1}{\alpha}\mathcal{M}\Big(\int_\Omega (A(x,\nabla u) + \frac{1}{p(x)}|u|^{p(x)})\,dx\Big)\Big[\int_\Omega a(x,\nabla u)\nabla u + \int_\Omega |\nabla u|^{p(x)}\Big] \\
&= \int_\Omega |u|^{p(x)}dx + \frac{1}{\alpha}\langle \mathcal{T}u, u\rangle \\
&= \int_\Omega |u|^{p(x)}dx + \frac{1}{\alpha}\langle v, \mathcal{G}v\rangle \ \leq \ \int_\Omega |u|^{p(x)}dx + \frac{t}{\alpha}|\langle S\circ\mathcal{G}v, \mathcal{G}v\rangle| \\
&\leq \int_\Omega |u|^{p(x)}dx + \frac{t}{\alpha}\int_\Omega |\lambda f(x,u,\nabla u)u|dx \\
&\leq \int_\Omega |u|^{p(x)}dx + C_{p'}\int_\Omega |f(x,u,\nabla u)|^{p'(x)}dx + C_p\int_\Omega |u|^{p(x)}dx \\
&\leq (1 + C_{2,p})\int_\Omega |u|^{p(x)}dx + C_{1,p}\int_\Omega |f(x,u,\nabla u)|^{p'(x)}dx \\
&\leq C_1\|u\|^{\gamma} + C_2(\|u\|^{\gamma} + 1). \\
&\leq (C_1 + C_2)\|u\|^{\gamma} + C_2 \\
&\leq C_T\big(1 + \|u\|^{\gamma}\big).
\end{aligned}$$

Therefore $\big\{\mathcal{G}v \setminus v \in \mathcal{B}\big\}$ is bounded.

As the operator S is bounded and from (4.2), it follows that the set $\mathcal{B}$ is bounded in $\mathcal{X}^*$. Then there exists a positive constant R such that

$$\|v\|_{\mathcal{X}^*} < R \quad \text{for all} \quad v \in B.$$

Thus

$$v + t\mathcal{S}\circ\mathcal{G}v \neq 0 \quad \text{for all} \quad v \in \partial\mathcal{B}_R(0) \quad \text{and all} \quad t \in [0,1].$$

By Lemma 2.14, we get

$$I + \mathcal{S} \circ \mathcal{G} \in \mathcal{F}_T(\overline{\mathcal{B}_R(0)}) \quad \text{and} \quad I = \mathcal{T} \circ \mathcal{G} \in \mathcal{F}_T(\overline{\mathcal{B}_R(0)}).$$

Let us define an affine homotopy Λ from $[0,1] \times \overline{\mathcal{B}_R(0)}$ into $\mathcal{X}^*$ by

$$\Lambda(t,v) := v + \mathcal{S} \circ \mathcal{G}v \quad \text{for} \quad (t,v) \in [0,1] \times \overline{\mathcal{B}_R(0)}.$$

Using the homotopy invariance and normalization property of the degree d stated in Theorem 2.17, we have

$$d(I + \mathcal{S} \circ \mathcal{G}, \mathcal{B}_R(0), 0) = d(I, \mathcal{B}_R(0), 0) = 1,$$

consequently, we can find a point $v \in \mathcal{B}_R(0)$ such that

$$v + \mathcal{S} \circ \mathcal{G}v = 0.$$

it follows that $u = \mathcal{G}v$ is a weak solution of (1.1). This ends the proof.

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