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Solving Fuzzy Fractional Atangana-Baleanu Differential Equation Using Adams-Bashforth-Moulton Method

Said Melliani, Fouziya Zamtain, M'hamed Elomari and Lalla Saadia Chadli

ABSTRACT: This work is concerned with the numerical study of the fuzzy fractional equation involving the Atangana-Baleanu derivative in the sense of Caputo. We are going to apply the Adams Bashforth Moulton method to the equation concerned, which is an interconnection between the Lagrange approximation and the trapezoidal rule. We achieved this work by giving examples that illustrate this method.

Key Words: Generalization Hukuhara difference, Lagrange Interpolation, Fuzzy Atangana-Baleanu fractional derivatives, Modified Trapezoidal rule, Adams-Bashforth-Moulton method.

Contents

1. Introduction

In recent years, the mathematical study of fuzzy fractional equations has gained a lot of attention in the literature. The motivation for this arises from several aspects: From the realms of physics and engineering [4]. But the theory of fractional calculus, which deals with the investigation and applications of derivatives and integrals of arbitrary order, has a long history. The theory of fractional calculus was developed mainly as a purely theoretical field of mathematics.
In 2016, Atangana and Baleanau introduced a new operator for more proprieties, which we refer to as [5]. That was called the Atangana-Baleanau derivative with fractional order. This new operator characterized by the derivative has nonlocal and nonsingular kernel which is built by the generalized Mittag–Leffler function and finds applications in many problems in the field of groundwater and thermal science. The importance of fractional derivatives with non-singular kernels comes also from the fact that certain models of dissipative phenomena cannot be adequately described by the classical fractional operators. Atangana with Goufo enhanced the version based upon the Riemann–Liouville approach. But the latter has a bit of complexity. This is the reason for creating the Atangana Baleanu derivative in the sense of Caputo. The detailed definitions of ABC fractional derivative and AB fractional integral will be introduced in section 3.
The authors studied in [1] a fuzzy fractional integral equation. The fractional derivative is considered in the sense of Riemann-Liouville. T.Allahviranloo et al. presented in [3] the fuzzy Caputo fractional differential equation FCFDE under the Generalized Hukuhara differentiability. In addition, the existence and uniqueness of solutions for a class of fuzzy Caputo fractional differential equations with an initial value are investigated. As we know, the concept of the fuzzy set was introduced by L. Zadeh [8]. The

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fuzzy systems with the interval approach use an infinite valued parameter in the range of $[0,1]$ as a condensed degree of belief. This parameter increases complicity while also increasing the fuzzy solution in fuzzy systems. In the solving process of the model, the Atangana Baleanu derivative in the fractional case of differential equations has a memory to use all the previous information. This takes place in [2]. In recent decades, the predictor-corrector method has been used by many researchers to solve problems in stochastic processes, estimation theory, signal processing, and hybrid wavelets. Currently, some researchers have expanded the application of the Predictor-Corrector method (PCM for short) for finding the solution to fractional problems. From these studies, it is necessary to focus on the application of this method in solving fuzzy differential equations of fractional order. Therefore, in this study, we propose a new procedure for solving fuzzy differential equation of fractional order based on PC. We will try to study the fuzzy fractional equation using the predictor-corrector method, based on the idea of paper [2] The following paper is organized as follows: In Section 2 we will recall some basic definitions and concepts are brought as a necessary step to the next section. The fuzzy Atangana-Baleanu takes place in Section 3. Section 4 preserves the main result and we will end this paper with a conclusion.

2. Preliminaries

In this section some basic definitions and concepts are brought as a necessary step to the next sections.

Definition 2.1. *A fuzzy number u is a map $u:\mathbb{R}\to[0,1]$ which satisfies*
- *u is upper semi-continuous.*
- *u is fuzzy convex, i.e. $u(\lambda x+(1-\lambda)y)\geq\min\{u(x),u(y)\}$ for all $x,y\in\mathbb{R},\lambda\in[0,1]$.*
- *u is normal, i.e. $\exists x_0\in\mathbb{R}$ such that $u(x_0)=1$.*
- *The closure of $\operatorname{Supp}(u)=\{x\in\mathbb{R}\mid u(x)>0\}$ is compact.*

Definition 2.2. *The parametric interval form of a fuzzy number u is shown as, $u[r]=[\underline{u}(r),\overline{u}(r)]$, $0\leq r\leq 1$. And have the following properties,*
- *$\underline{u}(r)$ is a left continuous and non-decreasing function with respect to r.*
- *$\overline{u}(r)$ is a left continuous and non-increasing function with respect to r.*
- *For each $r\in[0,1]$, we have $\underline{u}(r)\leq\overline{u}(r)$.*

Definition 2.3. *For two fuzzy numbers u,v in the parametric interval form the arithmetic operations are.*

$$(u+v)[r]=[\underline{u}(r)+\underline{v}(r),\overline{u}(r)+\overline{v}(r)]$$

$$(\lambda\odot u)[r]=\begin{cases}[\lambda\underline{u}(r),\lambda\overline{u}(r)], & \lambda\geq 0\\ [\lambda\overline{u}(r),\lambda\underline{u}(r)], & \lambda<0\end{cases}$$

where $0\leq r\leq 1$.

Definition 2.4. *Suppose that fuzzy valued function $y(t)\in C^F(I)\cap L^F(I)$, then its parametric interval form can be shown as:*

$$[y(t)]_r=\left[\underline{y}(t,r),\overline{y}(t,r)\right],\quad 0\leq r\leq 1.$$

Definition 2.5. *(Generalized derivative) Then the gH-derivative of function y can be defined in the following form,*

$$_{gH}y'(\tau)=\lim_{h\to 0^+}\frac{y(\tau+h)\ominus_{gH}y(\tau)}{h}=\lim_{h\to 0^+}\frac{y(\tau)\ominus_{gH}y(\tau+h)}{h}. \tag{2.1}$$

where $\left.\frac{dy(\tau)}{dt}\right|_{gH}\in C^F(I)\cap L^F(I)$. And

$$y(\tau+h)\ominus_{gH}y(\tau)=g(\tau)\Leftrightarrow\begin{cases}i) & y(\tau+h)=y(\tau)\oplus g(\tau)\\ ii) & y(\tau)=y(\tau+h)\oplus(-1)g(\tau)\end{cases} \tag{2.2}$$

The situation (i) is equivalent to the Hukuhara difference definition $y(\tau+h)\ominus y(\tau)$.
The derivative in the following interval is formed by taking an r-cut of both sides of the above derivative

(2.1) and considering the definition of gH-difference (2.2).
Case 1. (i-Differentiability)

$$ {}_{i-gh}\left[y'\right]_r = \left[\underline{y}'(t,r), \overline{y}'(t,r)\right], \quad 0 \le r \le 1 $$

Case 2. (ii-Differentiability)

$$ {}_{ii-gh}\left[y'\right]_r = \left[\overline{y}'(t,r), \underline{y}'(t,r)\right], \quad 0 \le r \le 1. $$

To define the ABC derivative in interval parametric form, we must first define the Lebesgue integral of y (t) in interval parametric form, which is as follows:

$$ \begin{aligned} \left[\int_0^t y'(\tau)d\tau\right]_r &= \int_0^t \left[y'(\tau)\right]_r d\tau \\ &= \begin{cases} \left[\int_0^t \underline{y}'(t,r)d\tau, \int_0^t \overline{y}'(t,r)d\tau\right], & \text{in case (1)} \\ \left[\int_0^t \overline{y}'(t,r)d\tau, \int_0^t \underline{y}'(t,r)d\tau\right], & \text{in case (2).} \end{cases} \end{aligned} $$

3. ABC-type fuzzy fractional derivatives

In this section, we introduced our definition of the Atangana-Baleanu-type fuzzy fractional derivatives under the generalized Hukuhara difference. The definition is similar to the concept of the ABC-type derivative in the crisp case and a direct extension of strongly generalized H-differentiability to the fractional context.

Definition 3.1. *[2] Let $y \in C^{\mathbb{F}}(I) \cap L^{\mathbb{F}}(I)$, be a fuzzy valued function. then the Atangana-Baleanu derivative in the Caputo sense of y is given by*

$$ {}^{ABC}D^{\alpha}y(t) = \frac{B(\alpha)}{1-\alpha}\int_0^t ({}_{gH}y'(s))E_\alpha\big(-\frac{\alpha}{1-\alpha}(t-s)^\alpha\big)ds. $$

Such as $B(\alpha)$ is a normalized function defined by $B(\alpha) = 1-\alpha+\frac{\alpha}{\Gamma(\alpha)}$ and satisfied $B(0) = B(1) = 1$.

Theorem 3.2. *[2] The ABC fractional derivative of y in the sense of Caputo is defined in two cases as follow*

$$ \left[{}^{ABC}D^\alpha y(t)\right]^r = \left[{}^{ABC}D^\alpha \underline{y}(t,r), {}^{ABC}D^\alpha \overline{y}(t,r)\right], \qquad case(1). $$
$$ \left[{}^{ABC}D^\alpha y(t)\right]^r = \left[{}^{ABC}D^\alpha \overline{y}(t,r), {}^{ABC}D^\alpha \underline{y}(t,r)\right], \qquad case(2). $$

Where

$$ {}^{ABC}D^\alpha \underline{y}(t,r) = \frac{B(\alpha)}{1-\alpha}\int_0^t ({}_{i-gH})\underline{y}'(s)E_\alpha\big(-\frac{\alpha}{1-\alpha}(t-s)^\alpha\big)ds, $$
$$ {}^{ABC}D^\alpha \overline{y}(t,r) = \frac{B(\alpha)}{1-\alpha}\int_0^t ({}_{ii-gH})\overline{y'}(s)E_\alpha\big(-\frac{\alpha}{1-\alpha}(t-s)^\alpha\big)ds. $$

Proof. Let us consider y is an ABC-type fuzzy fractional differentiable function. Then we have the following:

$$ {}^{ABC}D^{\alpha}y(t) = \frac{B(\alpha)}{1-\alpha}\int_0^t ({}_{gH}y'(s))E_\alpha\big(-\frac{\alpha}{1-\alpha}(t-s)^\alpha\big)ds. $$

Taking a r -cut of both side of the above

$$ \left[{}^{ABC}D^\alpha y(t)\right]_r = \frac{B(\alpha)}{1-\alpha}\int_0^t \left[y'(\tau)\right]_r E_\alpha\left(-\frac{\alpha}{1-\alpha}(t-\tau)^\alpha\right)d\tau. $$

Therefore, According to the differentiability of y, we have

$$ \left[{}^{ABC}D^\alpha y(t)\right]_r = \frac{B(\alpha)}{1-\alpha}\int_0^t \left[\underline{y}'(\tau), \overline{y'}(\tau)\right] E_\alpha\left(-\frac{\alpha}{1-\alpha}(t-\tau)^\alpha\right)d\tau, \quad \text{in case (1).} $$

$$\left[{}^{ABC}D^{\alpha}y(t)\right]_r = \frac{B(\alpha)}{1-\alpha}\int_0^t \left[\overline{y}'(\tau), \underline{y}'(\tau)\right] E_{\alpha}\left(-\frac{\alpha}{1-\alpha}(t-\tau)^{\alpha}\right) d\tau, \quad \text{in case (2).}$$

Then in case (1)

$$\left[{}^{ABC}D^{\alpha}y(t)\right]_r = \left[\frac{B(\alpha)}{1-\alpha}\int_0^t \underline{y}'(\tau)E_{\alpha}\left(\frac{-\alpha}{1-\alpha}(t-\tau)^{\alpha}\right) d\tau, \frac{B(\alpha)}{1-\alpha}\int_0^t \overline{y}'(\tau)E_{\alpha}\left(\frac{-\alpha}{1-\alpha}(t-\tau)^{\alpha}\right) d\tau\right],$$

And in case (2)

$$\left[{}^{ABC}D^{\alpha}y(t)\right]_r = \left[\frac{B(\alpha)}{1-\alpha}\int_0^t \overline{y}'(\tau)E_{\alpha}\left(\frac{-\alpha}{1-\alpha}(t-\tau)^{\alpha}\right) d\tau, \frac{B(\alpha)}{1-\alpha}\int_0^t \underline{y}'(\tau)E_{\alpha}\left(\frac{-\alpha}{1-\alpha}(t-\tau)^{\alpha}\right) d\tau\right].$$

So the proof is completed. □

On the other hand, the fractional integral operator associated with the ABC fractional derivative operator on fuzzy valued functions in the interval parametric form is given by the following definition.

Definition 3.3. *Let $y \in C^{\mathbb{F}}(I) \cap L^{\mathbb{F}}(I)$. Then the fuzzy fractional integral of Atangana-Baleanu of y is defined by*

$$\left[({}^{AB}I^{\alpha})(y(t))\right]_r = \left[({}^{AB}I^{\alpha})\underline{y}(t,r), ({}^{AB}I^{\alpha})\overline{y}(t,r)\right].$$

Where,

$${}^{AB}I^{\alpha}\underline{y}(t,r) = \frac{1-\alpha}{B(\alpha)}\underline{y}(t,r) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\underline{y}(\tau,r)d\tau.$$

$${}^{AB}I^{\alpha}\overline{y}(t,r) = \frac{1-\alpha}{B(\alpha)}\overline{y}(t,r) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\overline{y}(\tau,r)d\tau.$$

4. Solving fuzzy differential equation of fractional order

In this section, the Adams-Bashforth-Moulton method for solving fuzzy fractional differential equations under the Atangana-Baleanu derivative in the sense of Caputo will be present. This method is based on the Lagrange formula used as a prediction at each step and the modified trapezoidal rule used to obtain the finite value at each step. To this end, consider the following FFDE

$$\begin{cases} {}^{ABC}D^{\alpha}y(t) = f(t, y(t)) \quad t \in (0,1) \quad \alpha \in (0,1) \\ y(0) = y_0. \end{cases} \tag{4.1}$$

The problem (4.1) can be considered equivalent to the following initial value problems
In case(1),

$$\begin{cases} {}^{ABC}D^{\alpha}\underline{y}(t,r) = \underline{f}(t, \underline{y}(t), \overline{y}(t), r) \quad t \in (0,1) \quad \alpha \in (0,1) \\ \underline{y}(0) = \underline{y}_0. \end{cases} \tag{4.2}$$

$$\begin{cases} {}^{ABC}D^{\alpha}\overline{y}(t,r) = \overline{f}(t, y(t), r) \quad t \in (0,1) \quad \alpha \in (0,1) \\ \overline{y}(0) = \overline{y}_0. \end{cases} \tag{4.3}$$

In case(2),

$$\begin{cases} {}^{ABC}D^{\alpha}\underline{y}(t,r) = \overline{f}(t, \underline{y}(t), \overline{y}(t), r) \quad t \in (0,1) \quad \alpha \in (0,1) \\ \underline{y}(0) = \underline{y}_0. \end{cases} \tag{4.4}$$

$$\begin{cases} {}^{ABC}D^{\alpha}\overline{y}(t,r) = \underline{f}(t, y(t), r) \quad t \in (0,1) \quad \alpha \in (0,1) \\ \overline{y}(0) = \overline{y}_0. \end{cases} \tag{4.5}$$

The problems (4.2)-(4.3) can be equivalent to the following integral equations
In case (1)

$$\begin{cases} \underline{y}(t,r) - \underline{y}(0,r) = {}^{ABC}I^{\alpha}\underline{f}(t, \underline{y}(t), \overline{y}(t), r) \quad t \in (0,1) \quad \alpha \in (0,1) \\ \underline{y}(0) = \underline{y}_0. \end{cases} \tag{4.6}$$

$$\begin{cases} \overline{y}(t,r) - \overline{y}(0,r) = {}^{ABC} I^{\alpha}\overline{f}(t, y(t), r) \quad t \in (0,1) \quad \alpha \in (0,1) \\ \overline{y}(0) = \overline{y}_0. \end{cases} \tag{4.7}$$

In case(2),

$$\begin{cases} \underline{y}(t,r) - \underline{y}(0,r) = {}^{ABC} I^{\alpha}\overline{f}(t, \underline{y}(t), \overline{y}(t), r) \quad t \in (0,1) \quad \alpha \in (0,1) \\ \underline{y}(0) = \underline{y}_0. \end{cases} \tag{4.8}$$

$$\begin{cases} \overline{y}(t,r) - \overline{y}(0,r) = {}^{ABC} I^{\alpha}\underline{f}(t, \underline{y}(t), \overline{y}, r) \quad t \in (0,1) \quad \alpha \in (0,1) \\ \overline{y}(0) = \overline{y}_0. \end{cases} \tag{4.9}$$

Let the interval $[0,1]$ be subdivided into N subintervals $[t_i, t_{i+1}]$ of step size $h = 1/N$ using the nodes $t_i = ih$ for $i = 0; 1; ...; N$.
Let us first find the numerical solution for the case (1).
Consider the Atangana-Baleanu integral associated with equations (4.6) and (4.7).

$$\underline{y}(t,r) - \underline{y}(0,r) = \frac{1-\alpha}{B(\alpha)}\underline{f}(t, \underline{y}(t), \overline{y}(t), r) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\underline{f}(s, \underline{y}(s), \overline{y}(s), r)ds$$

$$\overline{y}(t,r) - \overline{y}(0,r) = \frac{1-\alpha}{B(\alpha)}\overline{f}(t, \underline{y}(t), \overline{y}(t), r) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\overline{f}(s, \underline{y}(s), \overline{y}(s), r)ds$$

Taking t = tn = nh. We have,

$$\underline{y}(t_n,r) - \underline{y}(0,r) = \frac{1-\alpha}{B(\alpha)}\underline{f}(t_n, \underline{y}(t_n), \overline{y}(t_n), r) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^{t_n} (t_n-s)^{\alpha-1}\underline{f}(s, \underline{y}(s), \overline{y}(s), r)ds$$

$$\overline{y}(t_n,r) - \overline{y}(0,r) = \frac{1-\alpha}{B(\alpha)}\overline{f}(t_n, \underline{y}(t_n), \overline{y}(t_n), r) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^{t_n} (t_n-s)^{\alpha-1}\overline{f}(s, \underline{y}(s), \overline{y}(s), r)ds$$

By applying the quadrature rule, we obtain:

$$\underline{y}(t_n,r) - \underline{y}(0,r) = \frac{1-\alpha}{B(\alpha)}\underline{f}(t_n, \underline{y}(t_n), \overline{y}(t_n), r) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} (t_n-s)^{\alpha-1}\underline{f}(s, \underline{y}(s), \overline{y}(s), r)ds \tag{4.10}$$

$$\overline{y}(t_n,r) - \overline{y}(0,r) = \frac{1-\alpha}{B(\alpha)}\overline{f}(t_n, \underline{y}(t_n), \overline{y}(t_n), r) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} (t_n-s)^{\alpha-1}\overline{f}(s, \underline{y}(s), \overline{y}(s), r)ds \tag{4.11}$$

We approach the integrals by the modified trapezoid technique by making the Lagrange approximation of order 1 for $\underline{f}(s, \underline{y}(s), \overline{y}(s), r)$ *and* $\overline{f}(s, \underline{y}(s), \overline{y}(s), r)$.
So,

$$\underline{f}(s, \underline{y}(s), \overline{y}(s), r) = \underline{f}(t_i, \underline{y}(t_i), \overline{y}(t_i), r)\frac{t_{i+1}-s}{t_{i+1}-t_i} + \underline{f}(t_{i+1}, \underline{y}(t_{i+1}), \overline{y}(t_{i+1}), r)\frac{s-t_i}{t_{i+1}-t_i},$$

The same

$$\overline{f}(s, \underline{y}(s), \overline{y}(s), r) = \overline{f}(t_i, \underline{y}(t_i), \overline{y}(t_i), r)\frac{t_{i+1}-s}{t_{i+1}-t_i} + \overline{f}(t_{i+1}, \underline{y}(t_{i+1}), \overline{y}(t_{i+1}), r)\frac{s-t_i}{t_{i+1}-t_i}.$$

We replace $\underline{f}(s, \underline{y}(s), \overline{y}(s), r)$ and $\overline{f}(s, \underline{y}(s), \overline{y}(s), r)$ with these values in (4.10) and (4.11), we obtain

$$\begin{aligned}\underline{y}(t_n,r) - \underline{y}(0,r) = \frac{1-\alpha}{B(\alpha)}\underline{f}(t_n, \underline{y}(t_n), \overline{y}(t_n), r) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} (t_n-s)^{\alpha-1} \\ \times \left(\underline{f}(t_{i+1}, \underline{y}(t_{i+1}), \overline{y}(t_{i+1}), r)\frac{s-t_i}{t_{i+1}-t_i} + \underline{f}(t_i, \underline{y}(t_i), \overline{y}(t_i), r)\frac{s-t_{i+1}}{t_i-t_{i+1}}\right)ds,\end{aligned}$$

$$\overline{y}(t_n,r)-\overline{y}(0,r)=\frac{1-\alpha}{B(\alpha)}\overline{f}(t_n,\underline{y}(t_n),\overline{y}(t_n),r)+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}(t_n-s)^{\alpha-1}$$
$$\times\left(\overline{f}(t_{i+1},\underline{y}(t_{i+1}),\overline{y}(t_{i+1}),r)\frac{s-t_i}{t_{i+1}-t_i}+\overline{f}(t_i,\underline{y}(t_i),\overline{y}(t_i),r)\frac{s-t_{i+1}}{t_i-t_{i+1}}\right)ds.$$

So

$$\underline{y}(t_n,r)-\underline{y}(0,r)=\frac{1-\alpha}{B(\alpha)}\underline{f}(t_n,\underline{y}(t_n),\overline{y}(t_n),r)+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\sum_{i=0}^{n}a_{i,n}\underline{f}(t_i,\overline{y}(t_i),\overline{y}(t_i),r), \tag{4.12}$$

$$\overline{y}(t_n,r)-\overline{y}(0,r)=\frac{1-\alpha}{B(\alpha)}\overline{f}(t_n,\underline{y}(t_n),\overline{y}(t_n),r)+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\sum_{i=0}^{n}a_{i,n}\overline{f}(t_i,\underline{y}(t_i),\overline{y}(t_i),r). \tag{4.13}$$

With

$$a_{i,n}=\begin{cases}\int_{t_0}^{t_1}(t_n-s)^{\alpha-1}\frac{t_1-s}{t_1-t_0}ds & \text{if} \quad i=0,\\ \int_{t_i}^{t_{i+1}}(t_n-s)^{\alpha-1}\frac{t_{i+1}-s}{t_{i+1}-t_i}ds+\int_{t_{i-1}}^{t_i}(t_n-s)^{\alpha-1}\frac{s-t_{i-1}}{t_i-t_{i-1}}ds & \text{if} \quad i\in[1,n-1],\\ \int_{t_{n-1}}^{t_n}(t_n-s)^{\alpha-1}\frac{s-t_{n-1}}{t_n-t_{n-1}}ds & \text{if} \quad i=n.\end{cases}$$

Consequently, we simplify the writing of $a_{i,n}$ by the following calculations:

For i=0:

We have

$$\int_{t_0}^{t_1}(t_n-s)^{\alpha-1}\frac{t_1-s}{t_1-t_0}ds=\frac{1}{h}\int_{t_0}^{t_1}(t_n-s)^{\alpha-1}(t_1-s)ds,$$

Or $t_1=t_n-(n-1)h$, then

$$\int_{t_0}^{t_1}(t_n-s)^{\alpha-1}\frac{t_1-s}{t_1-t_0}ds=\frac{1}{h}\int_{t_0}^{t_1}(t_n-s)^{\alpha-1}(t_n-(n-1)h-s)ds,$$

Therefore

$$\int_{t_0}^{t_1}(t_n-s)^{\alpha-1}\frac{t_1-s}{t_1-t_0}ds=\frac{1}{h}\left(\int_{t_0}^{t_1}(t_n-s)^{\alpha-1}(t_n-s)ds-\int_{t_0}^{t_1}(t_n-s)^{\alpha-1}h(n-1)ds\right),$$

So

$$a_{0,n}=\frac{1}{h}\left(\int_{t_0}^{t_1}(t_n-s)^{\alpha}ds-h(n-1)\int_{t_0}^{t_1}(t_n-s)^{\alpha-1}ds\right).$$

Then

$$a_{0,n}=\frac{h^\alpha}{\alpha(\alpha+1)}\left[(n-1)^{\alpha+1}-(n-1-\alpha)n^\alpha\right].$$

For i=n:

In the same way, we calculate $a_{n,n}$, we have:

$$a_{n,n}=\int_{t_{n-1}}^{t_n}(t_n-s)^{\alpha-1}\frac{s-t_{n-1}}{t_n-t_{n-1}}ds,$$

So

$$a_{n,n}=\frac{1}{h}\int_{t_{n-1}}^{t_n}(t_n-s)^{\alpha-1}(s-t_{n-1})ds,$$

Since $t_n-t_{n-1}=h$, then

$$a_{n,n}=\frac{1}{h}\int_{t_{n-1}}^{t_n}(t_n-s)^{\alpha-1}(s-t_n+h),$$

By a simple calculation, we have:

$$a_{n,n}=\frac{h^\alpha}{\alpha(\alpha+1)}.$$

For $1 \leqslant i \leqslant n-1$:
We have

$$a_{i,n} = \int_{t_i}^{t_{i+1}} (t_n - s)^{\alpha-1} \frac{t_{i+1} - s}{t_{i+1} - t_i} ds + \int_{t_{i-1}}^{t_i} (t_n - s)^{\alpha-1} \frac{s - t_{i-1}}{t_i - t_{i-1}} ds.$$

By a simple calculation we have

$$a_{i,n} = \frac{h^\alpha}{\alpha(\alpha+1)} \Big((n-i+1)^{\alpha+1} + (n-i-1)^{\alpha+1} - 2(n-i)^{\alpha+1} \Big).$$

Then

$$a_{i,n} = \frac{h^\alpha}{\alpha(\alpha+1)} \begin{cases} (n-1)^{\alpha+1} - (n-1-\alpha)n^\alpha & \text{if} \quad i = 0, \\ (n-i+1)^{\alpha+1} + (n-i-1)^{\alpha+1} - 2(n-i)^{\alpha+1} & \text{if} \quad i \in [1, n-1], \\ 1 & \text{if} \quad i = n. \end{cases}$$

Consequently, the equalities (4.12) and (4.13) becomes

$$\begin{aligned} \underline{y}(t_n, r) - \underline{y}(0, r) = & \frac{1-\alpha}{B(\alpha)} \underline{f}(t_n, \underline{y}(t_n), \overline{y}(t_n), r) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \frac{h^\alpha}{\alpha(\alpha+1)} a_{0,n} \underline{f}(t_0, \underline{y}(t_0), \overline{y}(t_0), r) \\ & + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \frac{h^\alpha}{\alpha(\alpha+1)} \sum_{i=1}^{n} a_{i,n} \underline{f}(t_i, \underline{y}(t_i), \overline{y}(t_i), r). \end{aligned}$$

$$\begin{aligned} \overline{y}(t_n, r) - \overline{y}(0, r) = & \frac{1-\alpha}{B(\alpha)} \overline{f}(t_n, \underline{y}(t_n), \overline{y}(t_n), r) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \frac{h^\alpha}{\alpha(\alpha+1)} a_{0,n} \overline{f}(t_0, y_l(t_0), y_u(t_0)) \\ & + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \frac{h^\alpha}{\alpha(\alpha+1)} \sum_{i=1}^{n} a_{i,n} \overline{f}(t_i, \underline{y}(t_i), \overline{y}(t_i), r). \end{aligned}$$

Or $a_{0,n} = (n-1)^{\alpha+1} - (n-1-\alpha)n^\alpha$, then

$$\begin{aligned} \underline{y}(t_n, r) - \underline{y}(0, r) = & \frac{1-\alpha}{B(\alpha)} \underline{f}(t_n, \underline{y}(t_n), \overline{y}(t_n), r) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \frac{h^\alpha}{\alpha(\alpha+1)} \Big((n-1)^{\alpha+1} - (n-1-\alpha)n^\alpha \Big) \\ & \times \underline{f}(t_0, \underline{y}(t_0), \overline{y}(t_0), r) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \frac{h^\alpha}{\alpha(\alpha+1)} \sum_{i=1}^{n-1} a_{i,n} \underline{f}(t_i, \underline{y}(t_i), \overline{y}(t_i), r) \\ & + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \frac{h^\alpha}{\alpha(\alpha+1)} a_{n,n} \underline{f}(t_n, \underline{y}(t_n), \overline{y}(t_n), r). \end{aligned}$$

$$\begin{aligned} \overline{y}(t_n, r) - \overline{y}(0, r) = & \frac{1-\alpha}{B(\alpha)} \overline{f}(t_n, \underline{y}(t_n), \overline{y}(t_n), r) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \frac{h^\alpha}{\alpha(\alpha+1)} \Big((n-1)^{\alpha+1} - (n-1-\alpha)n^\alpha \Big) \\ & \times \overline{f}(t_0, \underline{y}(t_0), \overline{y}(t_0), r) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \frac{h^\alpha}{\alpha(\alpha+1)} \sum_{i=1}^{n-1} a_{i,n} \overline{f}(t_i, \underline{y}(t_i), \overline{y}(t_i), r) \\ & + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \frac{h^\alpha}{\alpha(\alpha+1)} a_{n,n} \overline{f}(t_n, \underline{y}(t_n), \overline{y}(t_n), r). \end{aligned}$$

Since $a_{n,n} = 1$, then

$$\begin{aligned} \underline{y}(t_n, r) = & \underline{y}(0, r) + \frac{\alpha h^\alpha}{B(\alpha)} \frac{1}{\alpha(\alpha+1)\Gamma(\alpha)} \Big((n-1)^{\alpha+1} - (n-1-\alpha)n^\alpha \Big) \underline{f}(t_0, \underline{y}(t_0), \overline{y}(t_0), r) + \frac{\alpha h^\alpha}{B(\alpha)} \\ & \times \left(\frac{1}{\alpha(\alpha+1)\Gamma(\alpha)} + \frac{1-\alpha}{\alpha h^\alpha} \right) \underline{f}(t_n, \underline{y}(t_n), \overline{y}(t_n), r) + \frac{\alpha h^\alpha}{B(\alpha)} \frac{1}{\alpha(\alpha+1)\Gamma(\alpha)} \\ & \times \sum_{i=1}^{n-1} a_{i,n} \underline{f}(t_i, \underline{y}(t_i), \overline{y}(t_i), r), \end{aligned}$$

And

$$\overline{y}(t_n,r)=\overline{y}(0,r)+\frac{\alpha h^{\alpha}}{B(\alpha)}\frac{1}{\alpha(\alpha+1)\Gamma(\alpha)}\Big((n-1)^{\alpha+1}-(n-1-\alpha)n^{\alpha}\Big)\overline{f}(t_0,\underline{y}(t_0),\overline{y}(t_0),r)+\frac{\alpha h^{\alpha}}{B(\alpha)}$$
$$\times\left(\frac{1}{\alpha(\alpha+1)\Gamma(\alpha)}+\frac{1-\alpha}{\alpha h^{\alpha}}\right)\overline{f}(t_n,\underline{y}(t_n),\overline{y}(t_n),r)+\frac{\alpha h^{\alpha}}{B(\alpha)}\frac{1}{\alpha(\alpha+1)\Gamma(\alpha)}$$
$$\times\sum_{i=1}^{n-1}a_{i,n}\overline{f}(t_i,\underline{y}(t_i),\overline{y}(t_i),r).$$

Or $\alpha(\alpha+1)\Gamma(\alpha)=\Gamma(\alpha+2)$, So

$$\underline{y}(t_n,r)=\underline{y}(0,r)+\frac{\alpha h^{\alpha}}{B(\alpha)}\left(\frac{(n-1)^{\alpha+1}-(n-1-\alpha)n^{\alpha}}{\Gamma(\alpha+2)}\right)\underline{f}(t_0,\underline{y}(t_0),\overline{y}(t_0),r)+\frac{\alpha h^{\alpha}}{B(\alpha)}\Bigg(\Big(\frac{1}{\Gamma(\alpha+2)}$$
$$+\frac{1-\alpha}{\alpha h^{\alpha}}\Big)\underline{f}(t_n,\underline{y}(t_n),\overline{y}(t_n),r)+\sum_{i=1}^{n-1}\frac{a_{i,n}}{\Gamma(\alpha+2)}\underline{f}(t_i,\underline{y}(t_i),\overline{y}(t_i),r)\Bigg),$$

And

$$\overline{y}(t_n,r)=\overline{y}(0,r)+\frac{\alpha h^{\alpha}}{B(\alpha)}\left(\frac{(n-1)^{\alpha+1}-(n-1-\alpha)n^{\alpha}}{\Gamma(\alpha+2)}\right)\overline{f}(t_0,\underline{y}(t_0),\overline{y}(t_0),r)+\frac{\alpha h^{\alpha}}{B(\alpha)}\Bigg(\Big(\frac{1}{\Gamma(\alpha+2)}$$
$$+\frac{1-\alpha}{\alpha h^{\alpha}}\Big)\overline{f}(t_n,\underline{y}(t_n),\overline{y}(t_n),r)+\sum_{i=1}^{n-1}\frac{a_{i,n}}{\Gamma(\alpha+2)}\overline{f}(t_i,\underline{y}(t_i),\overline{y}(t_i),r)\Bigg).$$

Then

$$\underline{y}(t_n,r)=\underline{y}(0,r)+\frac{\alpha h^{\alpha}}{B(\alpha)}\left(\xi_n\underline{f}(t_0,\underline{y}(t_0),\overline{y}(t_0),r)+\sum_{i=1}^{n}\mu_{n-i}\underline{f}(t_i,\underline{y}(t_i),\overline{y}(t_i),r)\right),$$

And

$$\overline{y}(t_n,r)=\overline{y}(0,r)+\frac{\alpha h^{\alpha}}{B(\alpha)}\left(\xi_n\overline{f}(t_0,\underline{y}(t_0),\overline{y}(t_0),r)+\sum_{i=1}^{n}\mu_{n-i}\overline{f}(t_i,\underline{y}(t_i),\overline{y}(t_i),r)\right).$$

Thus, we found the numerical solution for the case (1)

$$\begin{cases}\underline{y}_n=\underline{y}_0+\dfrac{\alpha h^{\alpha}}{B(\alpha)}\left(\xi_n\underline{f}(t_0,\underline{y}_0,\overline{y}_0,r)+\displaystyle\sum_{i=1}^{n}\mu_{n-i}\underline{f}(t_i,\underline{y}_i,\overline{y}_i,r)\right),\\ \overline{y}_n=\overline{y}_0+\dfrac{\alpha h^{\alpha}}{B(\alpha)}\left(\xi_n\overline{f}(t_0,\underline{y}_0,\overline{y}_0,r)+\displaystyle\sum_{i=1}^{n}\mu_{n-i}\overline{f}(t_i,\underline{y}_i,\overline{y}_i,r)\right).\end{cases}\tag{4.14}$$

With the coefficients ξ_n and μ_{n-i}, are given by the following formulas:

$$\xi_n=\frac{(n-1)^{\alpha+1}-(n-1-\alpha)n^{\alpha}}{\Gamma(\alpha+2)},$$

$$\mu_{n-i}=\begin{cases}\frac{1}{\Gamma(\alpha+2)}+\frac{1-\alpha}{\alpha h^{\alpha}} & \text{if} \quad i=n,\\ \frac{(n-i+1)^{\alpha+1}+(n-i-1)^{\alpha+1}-2(n-i)^{\alpha+1}}{\Gamma(\alpha+2)} & \text{if} \quad i\in[1,n-1].\end{cases}$$

Now, in the same way and following the same steps, we obtain the numerical solution for the case (2)

$$\underline{y}(t_n,r)=\underline{y}(0,r)+\frac{\alpha h^{\alpha}}{B(\alpha)}\left(\xi_n\overline{f}(t_0,\underline{y}(t_0),\overline{y}(t_0),r)+\sum_{i=1}^{n}\mu_{n-i}\overline{f}(t_i,\underline{y}(t_i),\overline{y}(t_i),r)\right),$$

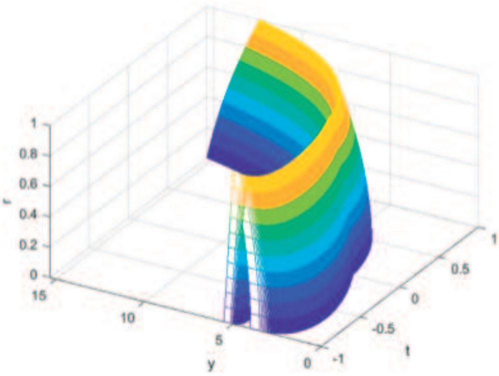

Figure 1: The approximate solution to problem (5.1) for $\alpha = 0.35$.

$$\overline{y}(t_n, r) = \overline{y}(0, r) + \frac{\alpha h^\alpha}{B(\alpha)}\Big(\xi_n \underline{f}(t_0, \underline{y}(t_0), \overline{y}(t_0), r) + \sum_{i=1}^{n} \mu_{n-i}\underline{f}(t_i, \underline{y}(t_i), \overline{y}(t_i), r)\Big).$$

Therefor

$$\begin{cases} \underline{y}_n = \underline{y}_0 + \dfrac{\alpha h^\alpha}{B(\alpha)}\Big(\xi_n \overline{f}(t_0, \underline{y}_0, \overline{y}_0, r) + \sum\limits_{i=1}^{n} \mu_{n-i}\overline{f}(t_i, \underline{y}_i, \overline{y}_i, r)\Big), \\ \overline{y}_n = \overline{y}_0 + \dfrac{\alpha h^\alpha}{B(\alpha)}\Big(\xi_n \underline{f}(t_0, \underline{y}_0, \overline{y}_0, r) + \sum\limits_{i=1}^{n} \mu_{n-i}\underline{f}(t_i, \underline{y}_i, \overline{y}_i, r)\Big). \end{cases} \tag{4.15}$$

5. Examples

In this section, we present two examples of solving the Fuzzy Fractional Differential Equation under the Atangana-Baleanu-type fuzzy fractional derivatives.

Example 5.1. *Consider the following Fuzzy Fractional Differential Equation*

$$\begin{cases} {}^{ABC}D^\alpha y(t) = ty(t), \quad t \in (-1, 1), \quad \alpha \in (0, 1), \\ \quad y(0) = y_0. \end{cases} \tag{5.1}$$

subject to initial values $y_0[r] = \left[\underline{y_0}(r), \overline{y_0}(r)\right] = \left[r^2 + 4, 5.5 - 0.5r\right],\ 0 \le r \le 1.$

We analyze two possibilities for t *in this example.*

- $-1 \le t < 0:$ *We only have the second case of derivative in this case.*

$$y(t) = y(-1) \ominus (-1)\frac{t(1-\alpha)}{B(\alpha)}y(t) \ominus (-1)\frac{t\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}y(\tau)d\tau.$$

- $0 \le t \le 1:$ *The unique solution is as.*

$$y(t) = y(0) \oplus \frac{(1-\alpha)}{B(\alpha)}ty(t) \oplus \frac{t\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}y(\tau)d\tau.$$

According to ${}^{ABC}D^\alpha y(t)$ *and using the Predictor-Corrector method, the approximate solution of problem (5.1) with* $h = 0.1$ *is shown in Figure 1 and Table 1.*

Table 1: The approximate solution of problem (5.1).

	$\alpha = 0.35$		$\alpha = 0.5$		$\alpha = 0.7$		$\alpha = 0,95$	
r	$\underline{y}(t)$	$\overline{y}(t)$	$\underline{y}(t)$	$\overline{y}(t)$	$\underline{y}(t)$	$\overline{y}(t)$	$\underline{y}(t)$	$\overline{y}(t)$
0	26.2938	36.1540	38.3493	52.7303	67.8871	93.3448	135.4255	186.2100
0.1	26.3595	36.4827	38.4452	53.2097	68.0568	94.1934	135.7641	187.9029
0.2	26.5567	36.8113	38.7328	53.6890	68.5660	95.0419	136.7797	189.5957
0.3	26.8854	37.1400	39.2122	54.1684	69.4146	95.8905	138.4726	191.2885
0.4	27.3456	37.4687	39.8833	54.6478	70.6026	96.7391	140.8425	192.9813
0.5	27.9372	37.7973	40.7461	55.1271	72.1300	97.5877	143.8896	194.6741
0.6	28.6602	38.1260	41.8008	55.6065	73.9969	98.4363	147.6138	196.3670
0.7	29.5148	38.4547	43.0471	56.0859	76.2033	99.2849	152.0151	198.0598
0.8	30.5008	38.7834	44.4852	56.5652	78.7490	100.1335	157.0936	199.7526
0.9	31.6183	39.1120	46.1151	57.0446	81.6342	100.9821	162.8492	201.4454
1.0	32.8673	39.4407	47.9366	57.5240	84.8589	101.8306	169.2819	203.1382

Table 2: The approximate solution of problem (5.2) where $\lambda = 1$.

	$\alpha = 0.9$		$\alpha = 0.95$	
r	$\underline{y}(t)$	$\overline{y}(t)$	$\underline{y}(t)$	$\overline{y}(t)$
0	0	3148.0	0	3734.5
0.1	104.9	2938.1	124.5	3485.5
0.2	209.9	2728.3	249.0	3236.6
0.3	314.8	2518.4	373.4	2987.6
0.4	419.7	2308.5	497.9	2738.6
0.5	524.7	2098.7	622.4	2489.7
0.6	629.6	1888.8	746.9	2240.7
0.7	734.5	1678.9	871.4	1991.7
0.8	839.5	1469.1	995.9	174.28
0.9	944.4	1259.2	1120.3	1493.8
1.0	1049.3	1049.3	1244.8	1244.8

Example 5.2. *Let us consider the following FFDE*

$$\begin{cases} {}^{ABC}D^{\alpha}y(t) = \lambda y(t), \quad t \in (0,1), \quad \alpha \in (0,1), \\ y(0) = y_0. \end{cases} \tag{5.2}$$

According to ${}^{ABC}D^{\alpha}y(x)$ and using the PC method, the approximate solution to the FFDE (5.2) with $h = 0.1$ is shown in Table 2 and Figure 2 for $\lambda = 1$.
And for $\lambda = -1$ the approximation solution of problem (5.2) is shown in Table 3 and Figure 3. Please note that, in this example, $\underline{y}_0 = r$ and $\overline{y}_0 = 3 - 2r, \quad 0 \le r \le 1$.

Table 3: The approximate solution of problem (5.2) where $\lambda = -1$.

	$\alpha = 0.9$		$\alpha = 0.95$	
r	$\underline{y}(t)$	$\overline{y}(t)$	$\underline{y}(t)$	$\overline{y}(t)$
0	0	0.0428	0	0.0186
0.1	0.0014	0.0399	0.0006	0.0173
0.2	0.0029	0.0371	0.0012	0.0161
0.3	0.0043	0.0342	0.0019	0.0148
0.4	0.0057	0.0314	0.0025	0.0136
0.5	0.0071	0.0285	0.0031	0.0124
0.6	0.0086	0.0257	0.0037	0.0111
0.7	0.0100	0.0228	0.0043	0.099
0.8	0.0114	0.0200	0.0049	0.0089
0.9	0.0128	0.0171	0.0056	0.0074
1.0	0.0143	0.0143	0.0062	0.0062

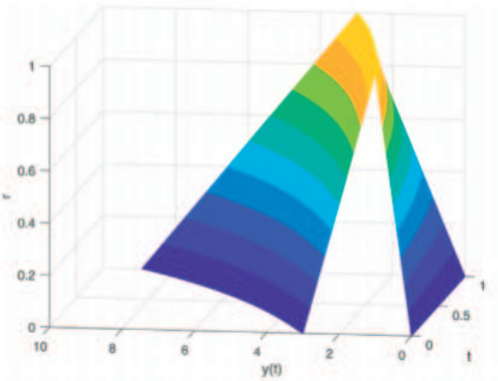

Figure 2: The approximate solution to problem (5.2) for $\lambda = 1$.

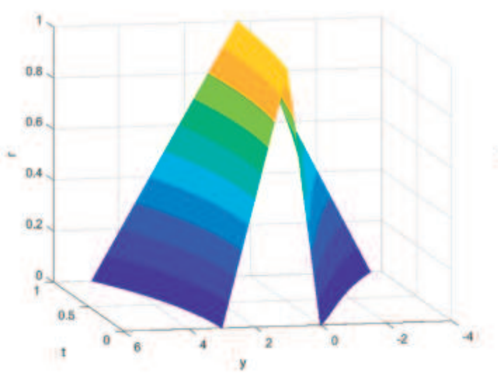

Figure 3: The approximate solution to problem (5.2) for $\lambda = -1$.

6. Conclusion

In this paper, the Adams-Bashforth-Moulton method is considered for solving FFDE under the Atangana-Baleanu Caputo-type fuzzy fractional derivatives of order $\alpha \in (0,1)$. Modeling physical, electrical, and chemical systems and the like based on FDEs in the presence of fuzzy uncertainty involves solving an initial value problem whose differential equations are of fractional order and usually nonlinear. Moreover, the initial conditions in many of these systems are of integer order. The application of FFDEs under Riemann–Liouville derivatives is restricted, in some ways, because their initial conditions are of fractional order. Consequently, the Atangana-Baleanu in the sense of Caputo-type fuzzy fractional derivatives are applied here. The definition can be extended $\alpha \in (n-1,n)$ with $n \in \mathbb{N}$. Additionally, the Predictor-Corrector method, as a known and simpler method is preferred for solving FFDE nonlinear. In future studies, an effort will be made to use the Collocation method for examining FFDEs under the ABC-type fuzzy fractional derivatives of order $\alpha \in (0,1)$.

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Said Melliani,
Laboratory of Applied Mathematics and Scientific Calculus,
Sultan Moulay Slimane University,
Morocco.
E-mail address: said.melliani@gmail.com

and

Fouziya Zamtain,
Laboratory of Applied Mathematics and Scientific Calculus,
Sultan Moulay Slimane University,
Morocco.
E-mail address: fouziyazzamtainz@gmail.com

and

M'hamed Elomari,
Laboratory of Applied Mathematics and Scientific Calculus,
Sultan Moulay Slimane University,
Morocco.
E-mail address: m.elomari@usms.ma

and

Lalla Saadia Chadli,
Laboratory of Applied Mathematics and Scientific Calculus,
Sultan Moulay Slimane University,
Morocco.
E-mail address: sa.chadli@yaho.fr