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Variable Exponent $p(\cdot)$-Kirchhoff Type Problem with Convection in Variable Exponent Sobolev Spaces

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Abstract: We establish the existence of weak solution for a class of $p(x)$-Kirchhoff type problem for the $p(x)$-Laplacian-like operator with Dirichlet boundary condition and with gradient dependence (convection) in the reaction term. Our result is obtained using the topological degree for a class of demicontinuous operators of generalized (S_+) type and the theory of the variable exponent Sobolev spaces. Our results extend and generalize several corresponding results from the existing literature.

Key Words: $p(x)$-Kirchhoff type problems, weak solution, variable exponent Sobolev spaces.

Contents

1. Introduction and motivation

Let Ω be a bounded domain in $\mathbb{R}^N(N>1)$ with smooth boundary denoted by $\partial\Omega$, and let θ,μ and λ be three real parameters and $p(\cdot),\delta(\cdot)\in C_+(\overline{\Omega})$.

In this research, We consider the following nonlinear $p(x)$-Kirchhoff type problem with Dirichlet boundary condition and with a reaction term depending also on the gradient (convection) and on three real parameters

$$\begin{cases} -\mathcal{K}\Big(\mathcal{A}(u)\Big)\Big(\Delta^l_{p(x)}u-|u|^{p(x)-2}u\Big)+\theta\,|u|^{\delta(x)-2}u=\mu\,\mathcal{B}(x,u)+\lambda\,\mathcal{C}(x,u,\nabla u) & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where

$$\mathcal{A}(u):=\int_\Omega \frac{1}{p(x)}\Big(|\nabla u|^{p(x)}+\sqrt{1+|\nabla u|^{2p(x)}}+|u|^{p(x)}\Big)dx,$$

and

$$\Delta^l_{p(x)}u:=\operatorname{div}\Big(|\nabla u|^{p(x)-2}\nabla u+\frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\Big)$$

is the $p(x)$-Laplacian-like operators, $\mathcal{B}:\Omega\times\mathbb{R}\to\mathbb{R}$ and $\mathcal{C}:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ are Carathéodory functions that satisfy the growth assumption, and $\mathcal{K}(t):\mathbb{R}^+\to\mathbb{R}^+$ is a continuous function.

The study of differential equations and variational problems with nonstandard $p(x)$-growth conditions (or nonstandard $(p(x),q(x))-$growth conditions) is an attractive topic and has been the object of considerable attention in recent years (see [16,22]). One of the motivations for studying (1.1) comes from the application of similar models in physics to represent the behavior of elasticity [25] and electrorheological fluids (see [20,23]), which have the ability to modify their mechanical properties when exposed to an electric field (see [1,2,22,17,18,19]), specifically the phenomenon of capillarity, which depends on solid-liquid interfacial characteristics as surface tension, contact angle, and solid surface geometry.

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Problems related to (1.1) have been studied by many scholars, for example, Ni and Serrin [10,11] considered the following equation

$$-\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) = f(u) \quad \text{in } \mathbb{R}^N. \tag{1.2}$$

The operator $-\operatorname{div}\Big(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big)$ is most often denoted by the specified mean curvature operator and $\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}$ is the Kirchhoff stress term.

In case $\mathcal{K}\Big(\mathcal{A}(u)\Big) \equiv 1$, $\mu = \theta = 0$, $\lambda > 0$, $\mathcal{C}$ independent of ∇u and without the term $|u|^{p(x)-2}u$, we know that the problem (1.1) has a nontrivial solutions from [21].

Note that, in case $\mathcal{A}(u) = \displaystyle\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx$, $\mu = \theta = 0$, $\lambda = 1$, $\mathcal{C}$ independent of ∇u and without the term $|u|^{p(x)-2}u$, then we obtain the following problem

$$\begin{cases} -\mathcal{K}\Big(\displaystyle\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx\Big)\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = \mathcal{C}(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

which is called the $p(x)$-Kirchhoff type problem. In this case, Dai et al. [4], by a direct variational approach, established conditions ensuring the existence and multiplicity of solutions to (1.3). Furthermore, the problem (1.3) is a generalization of the stationary problem of a model introduced by Kirchhoff [7] of the following form:

$$\rho\frac{\partial^2 u}{\partial t^2} - \Big(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \Big|\frac{\partial u}{\partial x}\Big|^2 dx\Big)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.4}$$

where ρ, ρ_0, h, E, L are all constants, which extends the classical D'Alembert's wave equation, by considering the effect of the changing in the length of the string during the vibration.

Lapa et al. [9] showed, by using a Fredholm-type result for a couple of nonlinear operators, and the theory of variable exponent Sobolev spaces, the existence of weak solutions for the problem (1.1), under no-flux boundary conditions, in case $\mu = \theta = 0$, $\lambda = 1$ and $\mathcal{C}$ independent of ∇u.

The remainder of the paper is organized as follows. In Section 2, we review some fundamental preliminaries about the functional framework where we will treat our problem. In Section 3, we introduce some classes of operators of generalized (S_+) type, as well as the Berkovits topological degrees. Finaly, in Section 4, we give our basic assumptions, some technical lemmas, and we will state and prove the main result of the paper.

2. Preliminaries

In the analysis of problem (1.1), we will use the theory of the generalized Lebesgue-Sobolev spaces $L^{p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$. For convenience, we only recall some basic facts with will be used later, we refer to [5,8,12,13,14,15] for more details.

Let Ω be a smooth bounded domain in $\mathbb{R}^N (N > 1)$, with a Lipschitz boundary denoted by $\partial\Omega$. Set

$$C_+(\overline{\Omega}) = \Big\{p : p \in C(\overline{\Omega}) \text{ such that } p(x) > 1 \text{ for any } x \in \overline{\Omega}\Big\}.$$

For each $p \in C_+(\overline{\Omega})$, we define

$$p^+ := \max\Big\{p(x),\ x \in \overline{\Omega}\Big\} \text{ and } p^- := \min\Big\{p(x),\ x \in \overline{\Omega}\Big\}.$$

For every $p \in C_+(\overline{\Omega})$, we define

$$L^{p(x)}(\Omega) = \Big\{u : \Omega \to \mathbb{R} \text{ is measurable such that } \int_\Omega |u(x)|^{p(x)}dx < +\infty\Big\},$$

equipped with the Luxemburg norm

$$|u|_{p(x)} = \inf\Big\{\lambda > 0 : \rho_{p(x)}\left(\frac{u}{\lambda}\right) \le 1\Big\},$$

where

$$\rho_{p(x)}(u) = \int_\Omega |u(x)|^{p(x)}dx, \ \forall \ u \in L^{p(x)}(\Omega).$$

Proposition 2.1. *[5] Let (u_n) and $u \in L^{p(\cdot)}(\Omega)$, then*

$$|u|_{p(x)} < 1\,(resp. = 1; > 1) \Leftrightarrow \rho_{p(x)}(u) < 1\,(resp. = 1; > 1)\,, \tag{2.1}$$

$$|u|_{p(x)} > 1 \ \Rightarrow \ |u|^{p^-}_{p(x)} \le \rho_{p(x)}(u) \le |u|^{p^+}_{p(x)}, \tag{2.2}$$

$$|u|_{p(x)} < 1 \ \Rightarrow \ |u|^{p^+}_{p(x)} \le \rho_{p(x)}(u) \le |u|^{p^-}_{p(x)}, \tag{2.3}$$

$$\lim_{n\to\infty} |u_n - u|_{p(x)} = 0 \ \Leftrightarrow \ \lim_{n\to\infty} \rho_{p(x)}\left(u_n - u\right) = 0. \tag{2.4}$$

Remark 2.2. *According to (2.2) and (2.3), we have*

$$|u|_{p(x)} \le \rho_{p(x)}(u) + 1, \tag{2.5}$$

$$\rho_{p(x)}(u) \le |u|^{p^-}_{p(x)} + |u|^{p^+}_{p(x)}. \tag{2.6}$$

Proposition 2.3. *[8] The spaces $\left(L^{p(x)}(\Omega), |\cdot|_{p(x)}\right)$ is a separable and reflexive Banach space.*

Proposition 2.4. *[8] The conjugate space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ for all $x \in \Omega$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have the following Hölder-type inequality*

$$\Big|\int_\Omega uv \ dx\Big| \le \left(\frac{1}{p-} + \frac{1}{p'-}\right)|u|_{p(x)}|v|_{p'(x)} \le 2|u|_{p(x)}|v|_{p'(x)}. \tag{2.7}$$

Remark 2.5. *If $p_1, p_2 \in C_+(\overline{\Omega})$ with $p_1(x) \le p_2(x)$ for any $x \in \overline{\Omega}$, then there exists the continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$.*

Now, let $p \in C_+(\overline{\Omega})$ and we define $W^{1,p(x)}(\Omega)$ as

$$W^{1,p(x)}(\Omega) = \Big\{u \in L^{p(x)}(\Omega) \text{ such that } |\nabla u| \in L^{p(x)}(\Omega)\Big\},$$

equipped with the norm

$$||u|| = |u|_{p(x)} + |\nabla u|_{p(x)}.$$

We also define $W_0^{1,p(\cdot)}(\Omega)$ as the subspace of $W^{1,p(\cdot)}(\Omega)$, which is the closure of $C_0^\infty(\Omega)$ with respect to the norm $||\cdot||$.

Proposition 2.6. *[5] If the exponent $p(\cdot)$ satisfies the log-Hölder continuity condition, i.e. there is $a > 0$ such that for every $x, \ y \in \Omega, \ x \ne y$ with $|x-y| \le \frac{1}{2}$ one has*

$$|p(x) - p(y)| \le \frac{a}{-\log|x-y|}, \tag{2.8}$$

then, there exists $C > 0$ depending only on Ω and the function p such that

$$|u|_{p(x)} \le C|\nabla u|_{p(x)}, \ \forall \ u \in W_0^{1,p(\cdot)}(\Omega). \tag{2.9}$$

In this paper we will use the following equivalent norm on $W_0^{1,p(\cdot)}(\Omega)$

$$|u|_{1,p(x)} = |\nabla u|_{p(x)},$$

which is equivalent to $||\cdot||$.
Furthermore, we have the compact embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$(see [8]).

Proposition 2.7. *[5,8] The spaces $\Big(W^{1,p(x)}(\Omega), |\cdot|_{1,p(x)}\Big)$ and $\Big(W_0^{1,p(x)}(\Omega), |\cdot|_{1,p(x)}\Big)$ are separable and reflexive Banach spaces.*

Remark 2.8. *The dual space of $W_0^{1,p(x)}(\Omega)$ denoted $W^{-1,p'(x)}(\Omega)$, is equipped with the norm*

$$|u|_{-1,p'(x)} = \inf\Big\{|u_0|_{p'(x)} + \sum_{i=1}^{N} |u_i|_{p'(x)}\Big\},$$

where the infinimum is taken on all possible decompositions $u = u_0 - \mathrm{div} F$ with $u_0 \in L^{p'(x)}(\Omega)$ and $F = (u_1, \ldots, u_N) \in (L^{p'(x)}(\Omega))^N$.

3. A review on topological degree theory

Now, we give some results and properties from the theory of topological degree. The readers can find more information about the history of this theory in [3,6].

In what follows, let X be a real separable reflexive Banach space and X^* be its dual space with dual pairing $\langle\cdot\,,\,\cdot\rangle$ and given a nonempty subset Ω of X. Strong (weak) convergence is represented by the symbol $\to$ ($\rightharpoonup$).

Definition 3.1. *Let Y be real Banach space. A operator $F : \Omega \subset X \to Y$ is said to be :*

1. *bounded, if it takes any bounded set into a bounded set.*
2. *demicontinuous, if for any sequence $(u_n) \subset \Omega$, $u_n \to u$ implies that $F(u_n) \rightharpoonup F(u)$.*
3. *compact, if it is continuous and the image of any bounded set is relatively compact.*

Definition 3.2. *A mapping $F : \Omega \subset X \to X^*$ is said to be :*

1. *of class (S_+), if for any sequence $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$ and $\limsup\limits_{n\to\infty}\langle Fu_n, u_n - u\rangle \leq 0$, we have $u_n \to u$.*
2. *quasimonotone, if for any sequence $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$, we have $\limsup\limits_{n\to\infty}\langle Fu_n,\ u_n - u\rangle \geq 0$.*

Definition 3.3. *Let $T : \Omega_1 \subset X \to X^*$ be a bounded operator such that $\Omega \subset \Omega_1$. For any operator $F : \Omega \subset X \to X$, we say that*

1. *F of class $(S_+)_T$, if for any sequence $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$, $y_n := Tu_n \rightharpoonup y$ and $\limsup\limits_{n\to\infty}\langle Fu_n,\ y_n - y\rangle \leq 0$, we have $u_n \to u$.*
2. *F has the property $(QM)_T$, if for any sequence $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$, $y_n := Tu_n \rightharpoonup y$, we have $\limsup\limits_{n\to\infty}\langle Fu_n, y - y_n\rangle \geq 0$.*

In the sequel, we consider the following classes of operators:

$$\mathcal{F}_1(\Omega) := \Big\{F : \Omega \to X^* : F \text{ is bounded, demicontinuous and of class } (S_+)\Big\},$$

$$\mathcal{F}_{T,B}(\Omega) := \Big\{F : \Omega \to X : F \text{ is bounded, demicontinuous and of class } (S_+)_T\Big\},$$

$$\mathcal{F}_T(\Omega) := \Big\{F : \Omega \to X : F \text{ is demicontinuous and of class } (S_+)_T\Big\},$$

for any $\Omega \subset D(F)$, where $D(F)$ denotes the domain of F, and any $T \in \mathcal{F}_1(\Omega)$.
Now, let $\mathcal{O}$ be the collection of all bounded open set in X and we define

$$\mathcal{F}(X) := \Big\{F \in \mathcal{F}_T(\overline{E}) : E \in \mathcal{O},\ \mathrm{T} \in \mathcal{F}_1(\overline{\mathrm{E}})\Big\},$$

where $\mathrm{T} \in \mathcal{F}_1(\overline{\mathrm{E}})$ is called an essential inner map to F.

Lemma 3.4. *[6, Lemma 2.3] Let $S : D(S) \subset X^* \to X$ be demicontinuous and $T \in \mathcal{F}_1(\overline{E})$ be continuous such that $T(\overline{E}) \subset D(S)$, where E is a bounded open set in a real reflexive Banach space X. Then the following statements are true:*

1. *If S is quasimonotone, then $I + S \circ T \in \mathcal{F}_T(\overline{E})$, where I denotes the identity operator.*
2. *If S is of class (S_+), then $S \circ T \in \mathcal{F}_T(\overline{E})$.*

Definition 3.5. *Suppose that E is bounded open subset of a real reflexive Banach space X, $T \in \mathcal{F}_1(\overline{E})$ is continuous and $F, S \in \mathcal{F}_T(\overline{E})$. The affine homotopy $\mathcal{H} : [0,1] \times \overline{E} \to X$ defined by*

$$\mathcal{H}(t,u) := (1-t)Fu + tSu, \quad \textit{for all} \quad (t,u) \in [0,1] \times \overline{E}$$

is called an admissible affine homotopy with the common continuous essential inner map T.

Remark 3.6. *[6, Lemma 2.5]* The above affine homotopy is of class $(S_+)_T$.

Next, as in [6] we give the topological degree for the class $\mathcal{F}(X)$.

Theorem 3.7. *Let*

$$M := \Big\{(F,E,h) : E \in \mathcal{O},\ T \in \mathcal{F}_1(\overline{E}),\ F \in \mathcal{F}_{T,B}(\overline{E}),\ h \notin F(\partial E)\Big\}.$$

Then, there exists a unique degree function $d : M \longrightarrow \mathbb{Z}$ that satisfies the following properties:

1. *(*Normalization*) For any $h \in E$, we have $d(I,E,h) = 1$.*
2. *(*Additivity*) Let $F \in \mathcal{F}_{T,B}(\overline{E})$. If E_1 and E_2 are two disjoint open subsets of E such that $h \notin F(\overline{E} \backslash (E_1 \cup E_2))$, then we have*

$$d(F,E,h) = d(F,E_1,h) + d(F,E_2,h).$$

3. *(*Homotopy invariance*) If $\mathcal{H} : [0,1] \times \overline{E} \to X$ is a bounded admissible affine homotopy with a common continuous essential inner map and h: $[0,1] \to X$ is a continuous path in X such that $h(t) \notin \mathcal{H}(t, \partial E)$ for all $t \in [0,1]$, then*

$$d(\mathcal{H}(t,\cdot), E, h(t)) = const \textit{ for all } t \in [0,1].$$

4. *(*Existence*) If $d(F,E,h) \neq 0$, then the equation $Fu = h$ has a solution in E.*

Definition 3.8. *[6, Definition 3.3] The above degree is defined as follows:*

$$d(F,E,h) \ := d_B(F|_{\overline{E}_0}, E_0, h),$$

where d_B is the Berkovits degree [3] and E_0 is any open subset of E with $F^{-1}(h) \subset E_0$ and F is bounded on $\overline{E}_0$.

4. Existence result

In this section, we will discuss the existence of weak solutions of (1.1). We assume that $\Omega \subset \mathbb{R}^N (N > 1)$ is a bounded domain with a Lipschitz boundary $\partial\Omega$, $p \in C_+(\overline{\Omega})$ satisfy the log-Hölder continuity condition (2.8), $\delta \in C_+(\overline{\Omega})$ with $2 \leq \delta^- \leq \delta(x) \leq \delta^+ < p^-$, $\mathcal{K} : \mathbb{R}^+ \to \mathbb{R}^+$, $\mathcal{B} : \Omega \times \mathbb{R} \to \mathbb{R}$ and $\mathcal{C} : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are functions such that:

(A_0) $\mathcal{K}(t) : [0, +\infty) \to (m_0, +\infty)$ is a continuous and increasing function with $m_0 > 0$.

(A_1) $\mathcal{C}$ is a Carathéodory function.

(A_2) There exists $C_1 > 0$ and $\gamma \in L^{p'(x)}(\Omega)$ such that

$$|\mathcal{C}(x, \zeta, \xi)| \leq C_1(\gamma(x) + |\zeta|^{q(x)-1} + |\xi|^{q(x)-1}).$$

(A_3) $\mathcal{B}$ is a Carathéodory function.

(A_4) There are $C_2 > 0$ and $\nu \in L^{p'(x)}(\Omega)$ such that

$$|\mathcal{B}(x, \zeta)| \leq C_2(\nu(x) + |\zeta|^{s(x)-1}),$$

for a.e. $x \in \Omega$ and all $(\zeta, \xi) \in \mathbb{R} \times \mathbb{R}^N$, where $q, s \in C_+(\overline{\Omega})$ with $2 \leq q^- \leq q(x) \leq q^+ < p^-$ and $2 \leq s^- \leq s(x) \leq s^+ < p^-$.

Remark 4.1.
- *Note that, for all $u, \vartheta \in W_0^{1,p(x)}(\Omega)$*

$$\mathcal{K}\Big(\mathcal{A}(u)\Big) \int_\Omega \bigg(\Big(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1 + |\nabla u|^{2p(x)}}}\Big)\nabla\vartheta + |u|^{p(x)-2}u\,\vartheta\bigg)dx$$

is well defined (see [9]).

- *$\theta\,|u|^{\delta(x)-2}u \in L^{p'(x)}(\Omega)$, $\mu\,\mathcal{B}(x,u) \in L^{p'(x)}(\Omega)$ and $\lambda\,\mathcal{C}(x,u,\nabla u) \in L^{p'(x)}(\Omega)$ under $u \in W_0^{1,p(x)}(\Omega)$, the assumptions (A_2) and (A_4) and the given hypotheses about the exponents p, δ, q and s because: $r(x) = (q(x) - 1)p'(x) \in C_+(\overline{\Omega})$ with $r(x) < p(x)$, $\beta(x) = (\delta(x) - 1)p'(x) \in C_+(\overline{\Omega})$ with $\beta(x) < p(x)$ and $\kappa(x) = (s(x) - 1)p'(x) \in C_+(\overline{\Omega})$ with $\kappa(x) < p(x)$.*
  Then, by Remark 2.5 we can conclude that

$$L^{p(x)} \hookrightarrow L^{r(x)},\ L^{p(x)} \hookrightarrow L^{\beta(x)} \text{ and } L^{p(x)} \hookrightarrow L^{\kappa(x)}.$$

Hence, since $\vartheta \in L^{p(x)}(\Omega)$, we have

$$\Big(-\theta\,|u|^{\delta(x)-2}u + \mu\,\mathcal{B}(x,u) + \lambda\,\mathcal{C}(x,u,\nabla u)\Big)\vartheta \in L^1(\Omega).$$

This implies that, the integral

$$\int_\Omega \Big(-\theta\,|u|^{\delta(x)-2}u + \mu\,\mathcal{B}(x,u) + \lambda\,\mathcal{C}(x,u,\nabla u)\Big)\vartheta dx$$

exist.

Then, we shall use the definition of weak solution for problem (1.1) in the following sense:

Definition 4.2. *We say that a function $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution of (1.1), if for any $\vartheta \in W_0^{1,p(x)}(\Omega)$, it satisfies the following:*

$$\mathcal{K}\Big(\mathcal{A}(u)\Big) \int_\Omega \bigg(\Big(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1 + |\nabla u|^{2p(x)}}}\Big)\nabla\vartheta + |u|^{p(x)-2}u\,\vartheta\bigg)dx$$

$$= \int_\Omega \Big(-\theta\,|u|^{\delta(x)-2}u + \mu\,\mathcal{B}(x,u) + \lambda\,\mathcal{C}(x,u,\nabla u)\Big)\vartheta dx.$$

Before giving our main result we first give two lemmas that will be used later.
First, let us consider the following functional:

$$\mathcal{J}(u) := \widehat{\mathcal{K}}\Big(\mathcal{A}(u)\Big), \quad \text{where} \ \ \widehat{\mathcal{K}}(s) = \int_0^s \mathcal{K}(\tau)\mathrm{d}\tau,$$

such that $\mathcal{K}(\tau)$ satisfies the assumption (A_0).
From [9], it is obvious that $\mathcal{J}$ is a continuously Gâteaux differentiable and $\mathcal{T} := \mathcal{J}'(u) \in W^{-1,p'(x)}(\Omega)$ such that

$$\langle \mathcal{T}u, \vartheta\rangle = \mathcal{K}\Big(\mathcal{A}(u)\Big)\int_\Omega \Big(\Big(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\Big)\nabla\vartheta + |u|^{p(x)-2}u\,\vartheta\Big)dx,$$

for all $u, \vartheta \in W_0^{1,p(x)}(\Omega)$ where $\langle\cdot,\cdot\rangle$ the duality pairing between $W^{-1,p'(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$. In addition, the following lemma summarizes the properties of the operator $\mathcal{T}$ (see [9]).

Lemma 4.3. *If (A_0) holds, then $\mathcal{T} : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ is a continuous, bounded and strictly monotone operator, and is a mapping of class (S_+).*

Lemma 4.4. *Assume that the assumptions $(A_1) - (A_4)$ hold, then the operator*

$$\begin{aligned} &\mathcal{S} : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)\\ &\langle \mathcal{S}u, \vartheta\rangle = -\int_\Omega \Big(-\theta\,|u|^{\delta(x)-2}u + \mu\,\mathcal{B}(x,u) + \lambda\,\mathcal{C}(x,u,\nabla u)\Big)\vartheta dx, \end{aligned}$$

for all $u, \vartheta \in W_0^{1,p(x)}(\Omega)$, is compact.

Proof. In order to prove this lemma, we proceed in four steps.
Step 1 : Let $\Upsilon : W_0^{1,p(x)}(\Omega) \to L^{p'(x)}(\Omega)$ be an operator defined by

$$\Upsilon u(x) := -\mu\,\mathcal{B}(x,u).$$

In this step, we prove that the operator Υ is bounded and continuous.
First, let $u \in W_0^{1,p(x)}(\Omega)$, bearing (A_4) in mind and using (2.5) and (2.6), we infer

$$\begin{aligned} |\Upsilon u|_{p'(x)} &\le \rho_{p'(x)}(\Upsilon u) + 1\\ &= \int_\Omega |\mu\,\mathcal{B}(x,u(x))|^{p'(x)}dx + 1\\ &= \int_\Omega |\mu|^{p'(x)}|\,\mathcal{B}(x,u(x)|^{p'(x)}dx + 1\\ &\le \Big(|\mu|^{p'^-} + |\mu|^{p'^+}\Big)\int_\Omega |C_2\Big(\nu(x) + |u|^{s(x)-1}\Big)|^{p'(x)}dx + 1\\ &\le const\Big(|\mu|^{p'^-} + |\mu|^{p'^+}\Big)\int_\Omega \Big(|\nu(x)|^{p'(x)} + |u|^{\kappa(x)}\Big)dx + 1\\ &\le const\Big(|\mu|^{p'^-} + |\mu|^{p'^+}\Big)\Big(\rho_{p'(x)}(\nu) + \rho_{\kappa(x)}(u)\Big) + 1\\ &\le const\Big(|\nu|_{p(x)}^{p'^+} + |u|_{\kappa(x)}^{\kappa^+} + |u|_{\kappa(x)}^{\kappa^-}\Big) + 1. \end{aligned}$$

Then, we deduce from (2.9) and $L^{p(x)} \hookrightarrow L^{\kappa(x)}$, that

$$|\Upsilon u|_{p'(x)} \le const\Big(|\nu|_{p(x)}^{p'^+} + |u|_{1,p(x)}^{\kappa^+} + |u|_{1,p(x)}^{\kappa^-}\Big) + 1,$$

that means Υ is bounded on $W_0^{1,p(x)}(\Omega)$.
Second, we show that the operator Υ is continuous. To this purpose let $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$. We need

to show that $\Upsilon u_n \to \Upsilon u$ in $L^{p'(x)}(\Omega)$. We will apply the Lebesgue's theorem.
Note that if $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$, then $u_n \to u$ in $L^{p(x)}(\Omega)$. Hence there exist a subsequence (u_k) of (u_n) and ϕ in $L^{p(x)}(\Omega)$ such that

$$u_k(x) \to u(x) \text{ and } |u_k(x)| \leq \phi(x), \tag{4.1}$$

for a.e. $x \in \Omega$ and all $k \in \mathbb{N}$.
Hence, from (A_2) and (4.1), we have

$$|\mathcal{B}(x, u_k(x))| \leq C_2(\nu(x) + |\phi(x)|^{s(x)-1}),$$

for a.e. $x \in \Omega$ and for all $k \in \mathbb{N}$.
On the other hand, thanks to (A_3) and (4.1), we get, as $k \longrightarrow \infty$

$$\mathcal{B}(x, u_k(x)) \to \mathcal{B}(x, u(x)) \text{ a.e. } x \in \Omega.$$

Seeing that

$$\nu + |\phi|^{s(x)-1} \in L^{p'(x)}(\Omega) \text{ and } \rho_{p'(x)}(\Upsilon u_k - \Upsilon u) = \int_\Omega |\mathcal{B}(x, u_k(x)) - \mathcal{B}(x, u(x))|^{p'(x)} dx,$$

then, from the Lebesgue's theorem and the equivalence (2.4), we have

$$\Upsilon u_k \to \Upsilon u \text{ in } L^{p'(x)}(\Omega),$$

and consequently

$$\Upsilon u_n \to \Upsilon u \text{ in } L^{p'(x)}(\Omega),$$

that is, Υ is continuous.
Step 2 : We define the operator $\Psi : W_0^{1,p(x)}(\Omega) \to L^{p'(x)}(\Omega)$ by

$$\Psi u(x) := \theta\, |u(x)|^{\delta(x)-2} u(x).$$

We will prove that Ψ is bounded and continuous.
It is clear that Ψ is continuous. Next we show that Ψ is bounded.
Let $u \in W_0^{1,p(x)}(\Omega)$ and using (2.5) and (2.6), we obtain

$$\begin{aligned}
|\Psi u|_{p'(x)} &\leq \rho_{p'(x)}(\Psi u) + 1 \\
&= \int_\Omega |\theta |u|^{\delta(x)-2} u|^{p'(x)} dx + 1 \\
&= \int_\Omega |\theta|^{p'(x)} |u|^{(\delta(x)-1)p'(x)} dx + 1 \\
&\leq \left(|\theta|^{p'^-} + |\theta|^{p'^+}\right) \int_\Omega |u|^{\beta(x)} dx + 1 \\
&= \left(|\theta|^{p'^-} + |\theta|^{p'^+}\right) \rho_{\beta(x)}(u) + 1 \\
&\leq \left(|\theta|^{p'^-} + |\theta|^{p'^+}\right) \left(|u|_{\beta(x)}^{\beta^-} + |u|_{\beta(x)}^{\beta^+}\right) + 1.
\end{aligned}$$

Hence, we deduce from $L^{p(x)} \hookrightarrow L^{\beta(x)}$ and (2.9) that

$$|\Psi u|_{p'(x)} \leq const\left(|u|_{1,p(x)}^{\beta^-} + |u|_{1,p(x)}^{\beta^+}\right) + 1,$$

and consequently, Ψ is bounded on $W_0^{1,p(x)}(\Omega)$.
Step 3 : Let us define the operator $\Phi : W_0^{1,p(x)}(\Omega) \to L^{p'(x)}(\Omega)$ by

$$\Phi u(x) := -\lambda\, \mathcal{C}(x, u(x), \nabla u(x)).$$

We will show that Φ is bounded and continuous.
Let $u \in W_0^{1,p(x)}(\Omega)$. According to (A_2) and the inequalities (2.5) and (2.6), we obtain

$$\begin{aligned}
|\Phi u|_{p'(x)} &\leq \rho_{p'(x)}(\Phi u) + 1 \\
&= \int_\Omega |\lambda\, \mathcal{C}(x, u(x), \nabla u(x))|^{p'(x)} dx + 1 \\
&= \int_\Omega |\lambda|^{p'(x)} |\, \mathcal{C}(x, u(x), \nabla u(x))|^{p'(x)} dx + 1 \\
&\leq \Big(|\lambda|^{p'^-} + |\lambda|^{p'^+}\Big) \int_\Omega |C_1\Big(\gamma(x) + |u|^{q(x)-1} + |\nabla u|^{q(x)-1}\Big)|^{p'(x)} dx + 1 \\
&\leq const\Big(|\lambda|^{p'^-} + |\lambda|^{p'^+}\Big) \int_\Omega \Big(|\gamma(x)|^{p'(x)} + |u|^{r(x)} + |\nabla u|^{r(x)}\Big) dx + 1 \\
&\leq const\Big(|\lambda|^{p'^-} + |\lambda|^{p'^+}\Big)\Big(\rho_{p'(x)}(\gamma) + \rho_{r(x)}(u) + \rho_{r(x)}(\nabla u)\Big) + 1 \\
&\leq const\Big(|\gamma|^{p'^+}_{p(x)} + |u|^{r^+}_{r(x)} + |u|^{r^-}_{r(x)} + |\nabla u|^{r^+}_{r(x)} + |\nabla u|^{r^-}_{r(x)}\Big) + 1.
\end{aligned}$$

Taking into account that $L^{p(x)} \hookrightarrow L^{r(x)}$ and (2.9), we have then

$$|\Phi u|_{p'(x)} \leq const\Big(|\gamma|^{p'^+}_{p(x)} + |u|^{r^+}_{1,p(x)} + |u|^{r^-}_{1,p(x)}\Big) + 1,$$

and consequently Φ is bounded on $W_0^{1,p(x)}(\Omega)$.
It remains to show that Φ is continuous. Let $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$, we need to show that $\Phi u_n \to \Phi u$ in $L^{p'(x)}(\Omega)$. We will apply the Lebesgue's theorem.
Note that if $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$, then $u_n \to u$ in $L^{p(x)}(\Omega)$ and $\nabla u_n \to \nabla u$ in $(L^{p(x)}(\Omega))^N$. Hence, there exist a subsequence (u_k) and ϕ in $L^{p(x)}(\Omega)$ and ψ in $(L^{p(x)}(\Omega))^N$ such that

$$u_k(x) \to u(x) \quad \text{and} \quad \nabla u_k(x) \to \nabla u(x), \tag{4.2}$$

$$|u_k(x)| \leq \phi(x) \quad \text{and} \quad |\nabla u_k(x)| \leq |\psi(x)|, \tag{4.3}$$

for a.e. $x \in \Omega$ and all $k \in \mathbb{N}$.
Hence, thanks to (A_1) and (4.2), we get, as $k \longrightarrow \infty$

$$\mathcal{C}(x, u_k(x), \nabla u_k(x)) \to \mathcal{C}(x, u(x), \nabla u(x)) \quad \text{a.e.} \quad x \in \Omega.$$

On the other hand, from (A_2) and (4.3), we can deduce the estimate

$$|\, \mathcal{C}(x, u_k(x), \nabla u_k(x))| \leq C_1(\gamma(x) + |\phi(x)|^{q(x)-1} + |\psi(x)|^{q(x)-1}),$$

for a.e. $x \in \Omega$ and for all $k \in \mathbb{N}$.
Seeing that

$$\gamma + |\phi|^{q(x)-1} + |\psi(x)|^{q(x)-1} \in L^{p'(x)}(\Omega),$$

and taking into account the equality

$$\rho_{p'(x)}(\Phi u_k - \Phi u) = \int_\Omega |\, \mathcal{C}(x, u_k(x), \nabla u_k(x)) - \mathcal{C}(x, u(x), \nabla u(x))|^{p'(x)} dx,$$

then, we conclude from the Lebesgue's theorem and (2.4) that

$$\Phi u_k \to \Phi u \quad \text{in} \quad L^{p'(x)}(\Omega)$$

and consequently

$$\Phi u_n \to \Phi u \quad \text{in} \quad L^{p'(x)}(\Omega),$$

and then Φ is continuous.
Step 4: Let $I^*: L^{p'(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ be the adjoint operator of the operator $I: W_0^{1,p(x)}(\Omega) \to L^{p(x)}(\Omega)$. We then define

$$I^* \circ \Upsilon : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega),$$
$$I^* \circ \Psi : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega),$$

and

$$I^* \circ \Phi : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega).$$

On another side, taking into account that I is compact, then I^* is compact. Thus, the compositions $I^* \circ \Upsilon$, $I^* \circ \Psi$ and $I^* o \Phi$ are compact, that means $\mathcal{S} = I^* \circ \Upsilon + I^* \circ \Psi + I^* \circ \Phi$ is compact. With this last step the proof of Lemma 4.4 is completed.
Consequently, we establish the following existence result.

Theorem 4.5. *If hypotheses (A_0), (A_1), (A_2), (A_3) and (A_4) hold, then problem (1.1) admits at least a weak solution u in $W_0^{1,p(x)}(\Omega)$.*

Proof.
We will reduce the problem (1.1) to a new one governed by a Hammerstein equation, and we will apply the theory of topological degree introduced in Section 3.

For all $u, \vartheta \in W_0^{1,p(x)}(\Omega)$, we define the operators $\mathcal{T}$ and $\mathcal{S}$ by

$$\begin{aligned} &\mathcal{T} : W_0^{1,p(x)}(\Omega) \longrightarrow W^{-1,p'(x)}(\Omega) \\ &\langle \mathcal{T}u, \vartheta \rangle = \mathcal{K}\Big(\mathcal{A}(u)\Big) \int_\Omega \Big(\Big(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\Big)\nabla\vartheta + |u|^{p(x)-2}u\,\vartheta\Big)dx, \end{aligned}$$

and

$$\begin{aligned} &\mathcal{S} : W_0^{1,p(x)}(\Omega) \longrightarrow W^{-1,p'(x)}(\Omega) \\ &\langle \mathcal{S}u, \vartheta \rangle = -\int_\Omega \Big(-\theta\,|u|^{\delta(x)-2}u + \mu\,\mathcal{B}(x,u) + \lambda\,\mathcal{C}(x,u,\nabla u)\Big)\vartheta dx. \end{aligned}$$

Consequently, the problem (1.1) is equivalent to the equation

$$\mathcal{T}u + \mathcal{S}u = 0, \quad u \in W_0^{1,p(x)}(\Omega). \tag{4.4}$$

Taking into account that, by Lemma 4.3, the operator $\mathcal{T}$ is a continuous, bounded, strictly monotone and of class (S_+), then, by [24, Theorem 26 A], the inverse operator

$$\mathcal{L} := \mathcal{T}^{-1} : W^{-1,p'(x)}(\Omega) \to W_0^{1,p(x)}(\Omega),$$

is also bounded, continuous, strictly monotone and of class (S_+).
On another side, according to Lemma 4.4, we have that the operator $\mathcal{S}$ is bounded, continuous and quasi-monotone.
Consequently, following Zeidler's terminology [24], the equation (4.4) is equivalent to the following abstract Hammerstein equation

$$u = \mathcal{L}\vartheta \ \text{ and } \ \vartheta + \mathcal{S} \circ \mathcal{L}\vartheta = 0, \quad u \in W_0^{1,p(x)}(\Omega) \text{ and } \vartheta \in W^{-1,p'(x)}(\Omega). \tag{4.5}$$

Seeing that (4.4) is equivalent to (4.5), then to solve (4.4) it is thus enough to solve (4.5). In order to solve (4.5), we will apply the Berkovits topological degree introduced in Section 3.
First, let us set

$$\mathcal{E} := \Big\{\vartheta \in W^{-1,p'(x)}(\Omega) : \exists\, t \in [0,1] \ \text{ such that } \ \vartheta + t\mathcal{S} \circ \mathcal{L}\vartheta = 0\Big\}.$$

Next, we show that $\mathcal{E}$ is bounded in $\in W^{-1,p'(x)}(\Omega)$.
Let us put $u := \mathcal{L}\vartheta$ for all $\vartheta \in \mathcal{E}$. Taking into account that $|\mathcal{L}\vartheta|_{1,p(x)} = |\nabla u|_{p(x)}$, then we have the

following two cases:
First case : If $|\nabla u|_{p(x)} \leq 1$. Then $|\mathcal{L}\vartheta|_{1,p(x)} \leq 1$, that means $\Big\{\mathcal{L}\vartheta : \vartheta \in \mathcal{E}\Big\}$ is bounded.
Second case : If $|\nabla u|_{p(x)} > 1$. Then, we deduce from (2.2), (A_2) and (A_4), the inequalities (2.7) and (2.6) and the Young's inequality that

$$\begin{aligned}
|\mathcal{L}\vartheta|^{p^-}_{1,p(x)} &= |\nabla u|^{p-}_{p(x)} \\
&\leq \rho_{p(x)}(\nabla u) \\
&\leq \langle \mathcal{T}u,\ u\rangle \\
&= \langle \vartheta,\ \mathcal{L}\vartheta\rangle \\
&= -t\langle \mathcal{S}\circ\mathcal{L}\vartheta,\ \mathcal{L}\vartheta\rangle \\
&= t\int_\Omega \Big(-\theta\,|u|^{\delta(x)-2}u + \mu\,\mathcal{B}(x,u) + \lambda\,\mathcal{C}(x,u,\nabla u)\Big)udx \\
&\leq t\max(|\theta|, C_2|\mu|, C_1|\lambda|)\Big(\int_\Omega |u|^{\delta(x)}dx + \int_\Omega |\nu(x)u(x)|dx + \int_\Omega |u(x)|^{s(x)}dx \\
&\qquad + \int_\Omega |\gamma(x)u(x)|dx + \int_\Omega |u(x)|^{q(x)}dx + \int_\Omega |\nabla u|^{q(x)-1}|u|dx\Big) \\
&= t\max(|\theta|, C_2|\mu|, C_1|\lambda|)\Big(\rho_{\delta(x)}(u) + \int_\Omega |\nu(x)u(x)|dx + \int_\Omega |\gamma(x)u(x)|dx \\
&\qquad + \rho_{s(x)}(u) + \rho_{q(x)}(u) + \int_\Omega |\nabla u|^{q(x)-1}|u|dx\Big) \\
&\leq const\Big(|u|^{\delta^-}_{\delta(x)} + |u|^{\delta^+}_{\delta(x)} + |\nu|_{p'(x)}|u|_{p(x)} + |\gamma|_{p'(x)}|u|_{p(x)} + |u|^{s^+}_{s(x)} + |u|^{s^-}_{s(x)} \\
&\qquad + |u|^{q^+}_{q(x)} + |u|^{q^-}_{q(x)} + \frac{1}{q'^-}\rho_{q(x)}(\nabla u) + \frac{1}{q^-}\rho_{q(x)}(u)\Big) \\
&\leq const\Big(|u|^{\delta^-}_{\delta(x)} + |u|^{\delta^+}_{\delta(x)} + |u|_{p(x)} + |u|^{s^+}_{s(x)} + |u|^{s^-}_{s(x)} + |u|^{q^+}_{q(x)} + |u|^{q^-}_{q(x)} \\
&\qquad + |\nabla u|^{q^+}_{q(x)}\Big).
\end{aligned}$$

Then, according to $L^{p(x)} \hookrightarrow L^{\delta(x)}$, $L^{p(x)} \hookrightarrow L^{s(x)}$ and $L^{p(x)} \hookrightarrow L^{q(x)}$, we get

$$|\mathcal{L}\vartheta|^{p^-}_{1,p(x)} \leq const\Big(|\mathcal{L}\vartheta|^{\delta^+}_{1,p(x)} + |\mathcal{L}\vartheta|_{1,p(x)} + |\mathcal{L}\vartheta|^{s^+}_{1,p(x)} + |\mathcal{L}\vartheta|^{q^+}_{1,p(x)}\Big),$$

what implies that $\Big\{\mathcal{L}\vartheta : \vartheta \in \mathcal{E}\Big\}$ is bounded.
On the other hand, we have that the operator is $\mathcal{S}$ is bounded, then $\mathcal{S}\circ\mathcal{L}\vartheta$ is bounded. Thus, thanks to (4.5), we have that $\mathcal{E}$ is bounded in $W^{-1,p'(x)}(\Omega)$.
However, $\exists\ a > 0$ such that

$$|\vartheta|_{-1,p'(x)} < a \quad \text{for all} \quad \vartheta \in \mathcal{E},$$

which leads to

$$\vartheta + t\mathcal{S}o\mathcal{L}\vartheta \neq 0, \quad \vartheta \in \partial\mathcal{E}_a(0) \quad \text{and} \quad t \in [0,1],$$

where $\mathcal{E}_a(0)$ is the ball of center 0 and radius a in $W^{-1,p'(x)}(\Omega)$.
Moreover, by Lemma 3.4, we conclude that

$$I + \mathcal{S}o\mathcal{L} \in \mathcal{F}_{\mathcal{L}}(\overline{\mathcal{E}_a(0)}) \quad \text{and} \quad I = \mathcal{T}\circ\mathcal{L} \in \mathcal{F}_{\mathcal{L}}(\overline{\mathcal{E}_a(0)}).$$

On another side, taking into account that I, $\mathcal{S}$ and $\mathcal{L}$ are bounded, then $I + \mathcal{S}\circ\mathcal{L}$ is bounded. Hence, we infer that

$$I + \mathcal{S}o\mathcal{L} \in \mathcal{F}_{\mathcal{L},B}(\overline{\mathcal{E}_a(0)}) \quad \text{and} \quad I = \mathcal{T}\circ\mathcal{L} \in \mathcal{F}_{\mathcal{L},B}(\overline{\mathcal{E}_a(0)}).$$

Now, we define the homotopy $\mathcal{H} : [0,1] \times \overline{\mathcal{E}_a(0)} \to W^{-1,p'(x)}(\Omega)$ by

$$\mathcal{H}(t,\vartheta) := \vartheta + t\mathcal{S}\circ\mathcal{L}\vartheta.$$

Applying the homotopy invariance and normalization property of the degree d seen in Theorem 3.7, we have

$$d\big(I+\mathcal{S}o\mathcal{L},\mathcal{E}_a(0),0\big)=d\big(I,\mathcal{E}_a(0),0\big)=1\neq 0.$$

Since $d\big(I+\mathcal{S}\circ\mathcal{L},\mathcal{E}_a(0),0\big)\neq 0$, then by the existence property of the degree d stated in Theorem 3.7, we conclude that there exists $\vartheta\in\mathcal{E}_a(0)$ which verifies

$$\big(I+\mathcal{S}\circ\mathcal{L}\big)(\vartheta)=0\Leftrightarrow\vartheta+\mathcal{S}\circ\mathcal{L}\vartheta=0\Leftrightarrow\mathcal{T}o\mathcal{L}\vartheta+\mathcal{S}\circ\mathcal{L}\vartheta=0.$$

Finally, we infer that $u=\mathcal{L}\vartheta$ is a weak solutions of (1.1). The proof is completed.

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