Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 2023 (41)** : 1–9.
ISSN-0037-8712 in press
doi:10.5269/bspm.62662

On Periodic Solutions of a Recruitment Model with Iterative Terms and a Nonlinear Harvesting

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abstract: We consider a first-order delay differential equation involving iterative terms. We prove the existence of positive periodic and bounded solutions by utilizing the Schauder's fixed point theorem combined with the Green's functions method. Furthermore, by virtue of the Banach contraction principle, the uniqueness and stability of the solution are also analyzed. Our new results are illustrated with two examples that show the feasibility of our main findings.

Key Words: Existence, uniqueness, periodic solution, iterative functional differential equation, continuous dependence.

Contents

1 Introduction **1**

2 Relevant preliminaries **2**

3 Main results **3**
3.1 Existence 4
3.2 Uniqueness 6
3.3 Stability 6

4 Examples **8**

5 Conclusion **9**

1. Introduction

Our foremost concern in this work is to establish some sufficient criteria that assure the existence, uniqueness and stability of positive periodic and bounded solutions to the following first-order differential equation with a time-varying delay and iterative terms:

$$y'(t)+k(t)y(t)=ay^{[2]}(t)-b\left(y^{[2]}(t)\right)^2-qy^{[2]}(t)E\left(t,y(t),y(t-\tau(t))\right), \tag{1.1}$$

where $y^{[2]}(t)=y(y(t))$ is the second iterate of y, $a,b,q>0$, $k\in\mathcal{C}(\mathbb{R},(0,+\infty))$, $\tau\in\mathcal{C}(\mathbb{R},(0,+\infty))$ are two w-periodic functions and $E\in\mathcal{C}\left(\mathbb{R}^3,(0,+\infty)\right)$ is a w-periodic function with respect to the first variable and satisfies the following Lipschitz condition:

$$|E(t,y_1,y_2)-E(t,z_1,z_2)|\leq \ell_1|y_1-z_1|+\ell_2|y_2-z_2|. \tag{1.2}$$

It is worth noting here that equation (1.1) which involves the second iterate of the state variable can be seen as a special type of the following delayed differential equation:

$$\begin{aligned}y'(t)+k(t)y(t)&=ay(t-\tau_1(t,y(t)))-b\left(y(t-\tau_1(t,y(t)))\right)^2\\&\quad-qy(t-\tau_1(t,y(t)))E\left(t,y(t),y(t-\tau(t))\right),\end{aligned}$$

where $\tau_1(t,y(t))=t-y(t)$ can denote the gestation period, the life cycle, the time taken from birth to maturity or between oviposition and eclosion of adults and so on. The dependence on the population

2010 *Mathematics Subject Classification:* 34K13, 34A12, 47H10.
Submitted February 23, 2022. Published May 22, 2022

Typeset by $\mathcal{B}^{\mathcal{S}}_{\mathcal{M}}\mathcal{P}$ style.
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density may have been due to the competition for food during larval stages in insect populations or to the fact that some density-dependent factors such as hormones and growth factors affect the cell population dynamics, just to name a few.

These equations are widely used for describing various phenomena in diverse areas of sciences such as classical electrodynamics, biology, epidemiology, ecology, population dynamics and so on and so forth. For instance, equation (1.1) is usually relevant to the dynamics of single-species population growth with harvesting strategy where $y(t)$ represents the population density at time t, $k(t)$ is per capita daily adult mortality rate, $ay^{[2]}(t) - b\left(y^{[2]}(t)\right)^2$ is the recruitment term and $qy\left(t-\tau_1\left(t, y(t)\right)\right) E\left(t, y(t), y\left(t-\tau(t)\right)\right)$ is the harvesting term which may be due to live capture, fishing, hunting or trapping individuals where q is the catchability coefficient, $\tau(t)$ is referred to as the time delay required for harvesting mature individuals and E is the harvesting or fishery effort that depends on both the current and the past densities. So, our new results shed lights on an important question which is about the effect of the harvesting strategy on the population dynamics. Indeed, such external factor which reduces the population affects the mathematical model as it plays a key role in the dynamics of the population and, in some cases, can even lead to the eventual extinction.

Despite the long history of iterative differential equations, there were very little works available in the literature that dealt with these equations. But although the authors generally face some difficulties in studying them, such equations have recently attracted considerable attention that led to several recent contributions including (see [1]- [12]).

The purpose of this work is twofold: first, it aims to contribute to the emerging literature on this topic, and, secondly, to highlight the impact of the harvesting strategy on the population dynamics as our new findings highlights on this effect where the harvesting term involves two delays, the first lag depends on time while the second one which gives the second iterate $y^{[2]}(t)$, depends not only on the time but also depends on the population density.

The plan of this manuscript is organized as follows. In Section 2, we present some definitions and materials needed to establish our main results. In Section 3, we give certain conditions for which the Schauder's fixed point theorem could be applied and hence could guarantee the existence of at least one positive periodic and bounded solution of equation (1.1). Further, by means of the Banach contraction principle, we are also able to derive the existence and uniqueness result and also establish the continuous dependence of the unique solution on parameters. In Section 4, two examples are included to show the validity of the assumptions presented in this paper. Finally, the conclusion has been presented in the last section.

2. Relevant preliminaries

In this section, we shall recall some relevant preliminaries, which are crucial in our arguments.

For $w > 0$ and $c_1, c_2 \geq 0$, let us consider the Banach space

$$P_w = \left\{ y \in \mathcal{C}(\mathbb{R}, \mathbb{R}), y(t+w) = y(t) \right\},$$

equipped with the sup norm and

$$K_w(c_1, c_2) = \left\{ y \in P_w,\ 0 < y(t) \leq c_1,\ \left|y(t_2) - y(t_1)\right| \leq c_2 \left|t_2 - t_1\right|, \forall t_1, t_2 \in \mathbb{R} \right\},$$

a compact and convex subset of P_w.

Lemma 2.1. *[12]If $y_1, y_2 \in K_w(c_1, c_2)$, then*

$$\left\| y_1^{[2]} - y_2^{[2]} \right\| \leq (1 + c_2) \left\| y_1 - y_2 \right\|.$$

Lemma 2.2. *[12]It holds*

$$K_w(c_1, c_2) = \left\{ y \in P_w,\ 0 < y(t) \leq c_1,\ \left|y(t_2) - y(t_1)\right| \leq c_2 \left|t_2 - t_1\right|, \forall t_1, t_2 \in [0, w] \right\}.$$

Remark 2.3. *It follows from Lemma 2.1 that*

$$\left\| \left(y_1^{[2]}\right)^2 - \left(y_2^{[2]}\right)^2 \right\| \leq 2c_1 (1 + c_2) \left\| y_1 - y_2 \right\|,$$

for all $y_1, y_2 \in K_w(c_1, c_2)$.

3. Main results

We first begin by giving an equivalence between our equation (1.1) and an integral one.

Lemma 3.1. *$y \in K_w(c_1, c_2) \cap C^1(\mathbb{R}, \mathbb{R})$ is a solution of equation (1.1) if and only if $y \in K_w(c_1, c_2)$ is a solution of the following integral equation:*

$$y(t) = \int_t^{t+w} \mathcal{G}(t,s) \left\{ a y^{[2]}(s) - b\left(y^{[2]}(s)\right)^2 - q y^{[2]}(s) E(s, y(s), y(s-\tau(s))) \right\} ds, \tag{3.1}$$

where

$$\mathcal{G}(t,s) = \frac{\exp\left(\int_t^s k(u)\,du\right)}{\exp\left(\int_0^w k(u)\,du\right) - 1}. \tag{3.2}$$

Proof. Let $y \in K_w(c_1, c_2) \cap C^1(\mathbb{R}, \mathbb{R})$ be a solution of equation (1.1). Multiplying both sides of this equation by $\exp\left(\int_0^t k(u)\,du\right)$ we arrive at

$$\begin{aligned}
&\frac{d}{ds}\left[y(s)\exp\left(\int_0^s k(u)\,du\right)\right] ds \\
&= \left[a y^{[2]}(t) - b\left(y^{[2]}(t)\right)^2 - q y^{[2]}(t) E(t, y(t), y(t-\tau(t)))\right] \exp\left(\int_0^t k(u)\,du\right).
\end{aligned}$$

Integrating from t to $t+w$ we have

$$\begin{aligned}
&\int_t^{t+w} \frac{d}{ds}\left[y(s)\exp\left(\int_0^s k(u)\,du\right)\right] ds \\
&= \int_t^{t+w} \left[a y^{[2]}(s) - b\left(y^{[2]}(s)\right)^2 - q y^{[2]}(s) E(s, y(s), y(s-\tau(s)))\right] \exp\left(\int_0^s k(u)\,du\right) ds.
\end{aligned}$$

By the periodic properties we obtain that

$$\begin{aligned}
&\int_t^{t+w} \frac{d}{ds}\left[y(s)\exp\left(\int_0^s k(u)\,du\right)\right] ds \\
&= y(t)\left[\exp\left(\int_0^{t+w} k(u)\,du\right) - \exp\left(\int_0^t k(u)\,du\right)\right] \\
&= y(t)\left[\exp\left(\int_0^t k(u)\,du\right)\left(\exp\left(\int_t^{t+w} k(u)\,du\right) - 1\right)\right].
\end{aligned}$$

Thus

$$y(t) = \int_t^{t+w} \mathcal{G}(t,s) \left\{ a y^{[2]}(s) - b\left(y^{[2]}(s)\right)^2 - q y^{[2]}(s) E(s, y(s), y(s-\tau(s))) \right\} ds.$$

Conversely, assume that y satisfies the integral equation (3.1), by differentiation one can easily verify that y is a solution of equation (1.1). □

Remark 3.2. *If*

$$\frac{1}{\exp\left(\int_0^w k(u)\,du\right) - 1} = \alpha_1, \quad \frac{\exp\left(\int_0^w k(u)\,du\right)}{\exp\left(\int_0^w k(u)\,du\right) - 1} = \alpha_2,$$

then $\mathcal{G}(w+t, w+s) = \mathcal{G}(t,s)$ for all $s, t \in \mathbb{R}$ and

$$0 < \alpha_1 \le \mathcal{G}(t,s) \le \alpha_2. \tag{3.3}$$

Furthermore, function $\mathcal{G}$ satisfies

$$\int_{t_1}^{t_1+w} \left|\mathcal{G}(t_2, s) - \mathcal{G}(t_1, s)\right| ds \le wk\alpha_2 \left|t_2 - t_1\right|, \tag{3.4}$$

for all $t, s, t_1, t_2 \in \mathbb{R}$ *where*

$$k = \sup_{t\in[0,w]} k(t).$$

From the last lemma, we define an operator $\mathcal{F} : K_w(c_1, c_2) \to P_w$ as follows:

$$(\mathcal{F}y)(t) = \int_t^{t+w} \mathcal{G}(t,s) \left\{ ay^{[2]}(s) - b\left(y^{[2]}(s)\right)^2 - qy^{[2]}(s) E(s, y(s), y(s-\tau(s))) \right\} ds. \tag{3.5}$$

So equation (1.1) associated with the periodic properties can be converted into a fixed point problem. In other words, if y is a fixed point of the operator $\mathcal{F}$ then y is a solution of equation (1.1) and vice versa.

3.1. Existence

Now, we intend to state and prove our first main result. For this purpose, we will use Schauder's fixed point theorem to prove that operator $\mathcal{F}$ has at least one fixed point in $K_w(c_1, c_2)$ which means that equation (1.1) has at least one positive periodic and bounded solution.

Lemma 3.3. *Operator* $\mathcal{F}$ *is continuous.*

Proof. It is not difficult to show that $\mathcal{F}(t+w) = \mathcal{F}(t)$. For $y_1, y_2 \in K_w(c_1, c_2)$, we have

$$\begin{aligned}
|(\mathcal{F}y_1)(t) - (\mathcal{F}y_2)(t)| \le\, & a\int_t^{t+w} \mathcal{G}(t,s) \left| y_1^{[2]}(s) - y_2^{[2]}(s) \right| ds \\
& + b\int_t^{t+w} \mathcal{G}(t,s) \left| \left(y_1^{[2]}(s)\right)^2 - \left(y_2^{[2]}(s)\right)^2 \right| ds \\
& + q\int_t^{t+w} \mathcal{G}(t,s) \Big| y_1^{[2]}(s) E(s, y_1(s), y_1(s-\tau(s))) \, ds \\
& - y_2^{[2]}(s) E(s, y_2(s), y_2(s-\tau(s))) \Big| ds.
\end{aligned}$$

But

$$\begin{aligned}
& \left| y_1^{[2]}(s) E(s, y_1(s), y_1(s-\tau(s))) - y_2^{[2]}(s) E(s, y_2(s), y_2(s-\tau(s))) \right| \\
& \le |E(s, y_1(s), y_1(s-\tau(s)))| \left| y_1^{[2]}(s) - y_2^{[2]}(s) \right| \\
& + y_2^{[2]}(s) \left| E(s, y_1(s), y_1(s-\tau(s))) - E(s, y_2(s), y_2(s-\tau(s))) \right|.
\end{aligned}$$

By using (1.2) and Remark 2.3 we get

$$|E(s, y_1(s), y_1(s-\tau(s)))| \le E_0 + (\ell_1 + \ell_2) c_1, \tag{3.6}$$

where

$$E_0 = \max_{t\in[0,w]} E(t, 0, 0),$$

and

$$\begin{aligned}
& \left| y_1^{[2]}(s) E(s, y_1(s), y_1(s-\tau(s))) - y_1^{[2]}(s) E(s, y_2(s), y_2(s-\tau(s))) \right| \\
& \le (E_0 + (\ell_1 + \ell_2) c_1)(c_2 + 1) \|y_1 - y_2\| + c_1 (\ell_1 + \ell_2) \|y_1 - y_2\| \\
& \le (E_0 (c_2 + 1) + c_1 (c_2 + 2)(\ell_1 + \ell_2)) \|y_1 - y_2\|.
\end{aligned} \tag{3.7}$$

It follows from (3.3), (3.7), Lemma 2.1 and Remark 2.3 that

$$\|\mathcal{F}y_1 - \mathcal{F}y_2\| \le \lambda \|y_1 - y_2\|,$$

where

$$\lambda = \alpha_2 w ((c_2 + 1)(a + 2bc_1) + q (E_0 (c_2 + 1) + c_1 (c_2 + 2)(\ell_1 + \ell_2))),$$

from which we infer that $\mathcal{F}$ is a Lipschitz continuous operator and hence continuous. $\square$

Lemma 3.4. *If* $aw\alpha_2 \leq 1$ *and*

$$ay^{[2]}(s) - b\left(y^{[2]}(s)\right)^2 - qy^{[2]}(s)\,E\,(s, y(s), y(s-\tau(s))) > 0. \tag{3.8}$$

Then

$$0 < (\mathcal{F}y)(t) \leq c_1,$$

for all $y \in K_w(c_1, c_2)$ *and* $t \in \mathbb{R}$.

Proof. Let $y \in K_w(c_1, c_2)$. Since $aw\alpha_2 \leq 1$, it follows from (3.3) that

$$\begin{aligned}(\mathcal{F}y)(t) &\leq a\int_t^{t+w} \mathcal{G}(t,s)\,y^{[2]}(s)\,ds\\ &\leq aw\alpha_2 c_1\\ &\leq c_1,\end{aligned}$$

and by taking into account (3.3), (3.6) and (3.8) we get

$$\begin{aligned}(\mathcal{F}y)(t) &= \int_t^{t+w} \mathcal{G}(t,s)\left\{ay^{[2]}(s) - b\left(y^{[2]}(s)\right)^2 - qy^{[2]}(s)\,E\,(s, y(s), y(s-\tau(s)))\right\}\\ &> w\alpha_1 \min_{s\in[0,w]}\left\{ay^{[2]}(s) - b\left(y^{[2]}(s)\right)^2 - qy^{[2]}(s)\,E\,(s, y(s), y(s-\tau(s)))\right\}\\ &> 0.\end{aligned}$$

Consequently, $0 < (\mathcal{F}y)(t) \leq c_1$ for all $y \in K_w(c_1, c_2)$ and $t \in \mathbb{R}$. □

Lemma 3.5. *If*

$$\alpha_2 c_1 (kw+2)(qE_0 + a + bc_1 + q\ell_1 c_1 + q\ell_2 c_1) \leq c_2, \tag{3.9}$$

then

$$|(\mathcal{F}y)(t_2) - (\mathcal{F}y)(t_1)| \leq c_2\,|t_2 - t_1|,$$

for all $t_1, t_2 \in \mathbb{R}$ *and* $y \in K_w(c_1, c_2)$.

Proof. Let $t_1, t_2 \in \mathbb{R}$ and $y \in K_w(c_1, c_2)$. We have

$$\begin{aligned}|(\mathcal{F}y)(t_2) - (\mathcal{F}y)(t_1)| &\leq a\int_{t_2}^{t_1} y^{[2]}(s)\,\mathcal{G}(t_2,s)\,ds + a\int_{t_1+w}^{t_2+w} y^{[2]}(s)\,\mathcal{G}(t_2,s)\,ds\\ &\quad + a\int_{t_1}^{t_1+w} y^{[2]}(s)\,|\mathcal{G}(t_2,s) - \mathcal{G}(t_1,s)|\,ds\\ &\quad + b\int_{t_2}^{t_1}\left(y^{[2]}(s)\right)^2 \mathcal{G}(t_2,s)\,ds + b\int_{t_1+w}^{t_2+w}\left(y^{[2]}(s)\right)^2 \mathcal{G}(t_2,s)\,ds\\ &\quad + b\int_{t_1}^{t_1+w} |\mathcal{G}(t_2,s) - \mathcal{G}(t_1,s)|\left(y^{[2]}(s)\right)^2 ds\\ &\quad + q\int_{t_2}^{t_1} \mathcal{G}(t_2,s)\,y^{[2]}(s)\,E\,(s, y(s), y(s-\tau(s)))\,ds\\ &\quad + q\int_{t_1+w}^{t_2+w} \mathcal{G}(t_2,s)\,y^{[2]}(s)\,E\,(s, y(s), y(s-\tau(s)))\,ds\\ &\quad + q\int_{t_1}^{t_1+w} |\mathcal{G}(t_2,s) - \mathcal{G}(t_1,s)|\,y^{[2]}(s)\,E\,(s, y(s), y(s-\tau(s)))\,ds.\end{aligned}$$

From (3.3), (3.4) and (3.6) we arrive at

$$\begin{aligned}\left|(\mathcal{F}y)(t_2)-(\mathcal{F}y)(t_1)\right| &\leq 2ac_1\alpha_2\left|t_2-t_1\right|+c_1awk\alpha_2\left|t_2-t_1\right|\\ &+2bc_1^2\alpha_2\left|t_2-t_1\right|+bc_1^2wk\alpha_2\left|t_2-t_1\right|\\ &+2q\alpha_2c_1\left(E_0+\left(\ell_1+\ell_2\right)c_1\right)\left|t_2-t_1\right|\\ &+qc_1\left(E_0+\left(\ell_1+\ell_2\right)c_1\right)wk\alpha_2\left|t_2-t_1\right|\\ &\leq \alpha_2c_1\left(kw+2\right)\left(qE_0+a+bc_1+q\ell_1c_1+q\ell_2c_1\right)\left|t_2-t_1\right|.\end{aligned}$$

It follows from (3.9) and Lemma 2.2 that

$$\left|(\mathcal{F}y)(t_2)-(\mathcal{F}y)(t_1)\right|\leq c_2\left|t_2-t_1\right|,$$

for all $t_1, t_2\in\mathbb{R}$ and $y\in K_w(c_1,c_2)$. □

Theorem 3.6. *Suppose that conditions (3.8), (3.9) and $aw\alpha_2\leq 1$ hold, then equation (1.1) has at least one positive periodic and bounded solution in $K_w(c_1,c_2)$.*

Proof. From Lemmas 3.4 and 3.5 we conclude that operator $\mathcal{F}$ maps the compact subset $K_w(c_1,c_2)$ into itself and since Lemma 3.3 guarantees the continuity of the operator $\mathcal{F}$, then all conditions of Schauder's fixed point theorem are satisfied. Accordingly, $\mathcal{F}$ has at least one fixed point $y\in K_w(c_1,c_2)$ such that $\mathcal{F}y=y$. Thanks to Lemma 3.1, equation (1.1) has at least one positive periodic and bounded solution. □

3.2. Uniqueness

Theorem 3.7. *Suppose that conditions (3.8), (3.9) and $aw\alpha_2\leq 1$ are fulfilled. If $\lambda<1$, then equation (1.1) has a unique positive periodic and bounded solution $y\in K_w(c_1,c_2)$.*

Proof. Let $y_1,y_2\in K_w(c_1,c_2)$. From the proof of Lemma 3.3 we have

$$\left\|\mathcal{F}y_1-\mathcal{F}y_2\right\|\leq\lambda\left\|y_1-y_2\right\|.$$

Since $\lambda<1$, then $\mathcal{F}$ is a contraction. So, by the Banach fixed point theorem, $\mathcal{F}$ has a unique fixed point which is the unique positive periodic and bounded solution of equation (1.1). □

3.3. Stability

Theorem 3.8. *The unique solution obtained in Theorem 3.7 depends continuously on the death rate k and the harvesting effort E.*

Proof. Let

$$y_1(t)=\int_t^{t+w}\mathcal{G}_1(t,s)\left\{ay_1^{[2]}(s)-b\left(y_1^{[2]}(s)\right)^2-qy_1^{[2]}(s)E_1\left(s,y_1(s),y_1(s-\tau(s))\right)\right\}ds,$$

and

$$y_2(t)=\int_t^{t+w}\mathcal{G}_2(t,s)\left\{ay_2^{[2]}(s)-b\left(y_2^{[2]}(s)\right)^2-qy_2^{[2]}(s)E_2\left(s,y_2(s),y_2(s-\tau(s))\right)\right\}ds,$$

where

$$\mathcal{G}_1(t,s)=\frac{\exp\left(\int_t^s k_1(u)\,du\right)}{\exp\left(\int_0^w k_1(u)\,du\right)-1}\text{ and }\mathcal{G}_2(t,s)=\frac{\exp\left(\int_t^s k_2(u)\,du\right)}{\exp\left(\int_0^w k_2(u)\,du\right)-1},$$

are two different solutions of equation (1.1). We have

$$
\begin{aligned}
|y_1(t) - y_2(t)| \leq a \int_t^{t+w} & \left| y_1^{[2]}(s)\, \mathcal{G}_1(t,s) - y_2^{[2]}(s)\, \mathcal{G}_2(t,s) \right| ds \\
& + b \int_t^{t+w} \left| \left(y_1^{[2]}(s) \right)^2 \mathcal{G}_1(t,s) - y_2^{[2]}(s)\, \mathcal{G}_2(t,s) \right| ds \\
& + q \int_t^{t+w} \left| y_1^{[2]}(s)\, E_1(s, y_1(s), y_1(s-\tau(s)))\, \mathcal{G}_1(t,s) \right. \\
& \left. - y_2^{[2]}(s)\, E_2(s, y_2(s), y_2(s-\tau(s)))\, \mathcal{G}_2(t,s) \right| ds.
\end{aligned}
$$

Thus

$$
\begin{aligned}
& |y_1(t) - y_2(t)| \\
& \leq a \int_t^{t+w} \mathcal{G}_1(t,s) \left| y_1^{[2]}(s) - y_2^{[2]}(s) \right| ds + a \int_t^{t+w} y_2^{[2]}(s) \left| \mathcal{G}_1(t,s) - \mathcal{G}_2(t,s) \right| ds \\
& + b \int_t^{t+w} \mathcal{G}_1(t,s) \left| \left(y_1^{[2]}(s) \right)^2 - \left(y_2^{[2]}(s) \right)^2 \right| ds + b \int_t^{t+w} \left(y_2^{[2]}(s) \right)^2 \left| \mathcal{G}_1(t,s) - \mathcal{G}_2(t,s) \right| ds \\
& + q \int_t^{t+w} \mathcal{G}_1(t,s)\, y_1^{[2]}(s) \left| E_1(s, y_1(s), y_1(s-\tau(s))) - E_2(s, y_1(s), y_1(s-\tau(s))) \right| ds \\
& + q \int_t^{t+w} y_1^{[2]}(s)\, E_2(s, y_1(s), y_1(s-\tau(s))) \left| \mathcal{G}_1(t,s) - \mathcal{G}_2(t,s) \right| ds \\
& + q \int_t^{t+w} y_1^{[2]}(s)\, \mathcal{G}_2(t,s) \left| E_2(s, y_1(s), y_1(s-\tau(s))) - E_2(s, y_2(s), y_2(s-\tau(s))) \right| ds \\
& + q \int_t^{t+w} \left| y_1^{[2]}(s) - y_2^{[2]}(s) \right| \mathcal{G}_2(t,s)\, E_2(s, y_2(s), y_2(s-\tau(s)))\, ds.
\end{aligned}
$$

The mean value theorem leads to

$$
\int_t^{t+w} \left| G_1(t,s) - G_2(t,s) \right| ds \leq \sigma \left\| k_1 - k_2 \right\|, \tag{3.10}
$$

where

$$
\sigma = \frac{w^2 e^{w(\|k_2\| + \max(\|k_1\|, \|k_2\|))}}{\left(\exp\left(\int_0^w k_1(u)\, du \right) - 1 \right) \left(\exp\left(\int_0^w k_2(u)\, du \right) - 1 \right)} + \frac{w^2 e^{w \max(\|k_1\|, \|k_2\|)}}{\exp\left(\int_0^w k_1(u)\, du \right) - 1}.
$$

It follows from (1.2), (3.3), (3.6), (3.10), Lemma 2.1 and Remark 2.3 that

$$
\begin{aligned}
\|y_1 - y_2\| \leq\ & a w \alpha_2 (1 + c_2) \|y_1 - y_2\| + a c_1 \sigma \|k_1 - k_2\| \\
& + 2 b w \alpha_2 c_1 (1 + c_2) \|y_1 - y_2\| + b c_1^2 \sigma \|k_1 - k_2\| \\
& + q c_1 w \alpha_2 \|E_1 - E_2\| + q c_1 (E_0 + (\ell_1 + \ell_2) c_1) \sigma \|k_1 - k_2\| \\
& + q c_1 w \alpha_2 (\ell_1 + \ell_2) \|y_1 - y_2\| \\
& + q w \alpha_2 (E_0 + (\ell_1 + \ell_2) c_1)(1 + c_2) \|y_1 - y_2\|.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|y_1 - y_2\| \leq & \frac{1}{1 - \lambda} \left[c_1 \sigma \left(a + b c_1 + q \left(E_0 + (\ell_1 + \ell_2) c_1 \right) \right) \|k_1 - k_2\| \right. \\
& \left. + q c_1 w \alpha_2 \|E_1 - E_2\| \right].
\end{aligned}
$$

This completes the proof. □

4. Examples

In this section, we give two examples to demonstrate the findings obtained in the previous section.

Example 4.1. *Consider the following recruitment model:*

$$\begin{aligned} & y'(t)+\left(0.022+0.02\cos^2\left(\frac{2\pi}{35}t\right)\right)y(t) \\ & =(0.018)\,y^{[2]}(t)-(0.0012)\left(y^{[2]}(t)\right)^2-(0.003)\,y^{[2]}(t)\left((0.0025)\cos^2\left(\frac{2\pi}{35}t\right)\right. \\ & \left.+\frac{1}{27}\cos^2\left(\frac{2\pi}{35}t\right)y(t)+\frac{1}{29}\cos^2\left(\frac{2\pi}{35}t\right)y(t-\tau(t))\right). \end{aligned} \tag{4.1}$$

Here

$$\begin{aligned} & k(t)=0.022+0.02\cos^2\left(\frac{2\pi}{35}t\right),\ a=0.018,\ b=0.0012, \\ & E(t,y(t),y(t-\tau(t)))=(0.0025)\cos^2\left(\frac{2\pi}{35}t\right)+\frac{1}{27}\cos^2\left(\frac{2\pi}{35}t\right)y(t) \\ & +\frac{1}{29}\cos^2\left(\frac{2\pi}{35}t\right)y(t-\tau(t)),\ \ell_1=\frac{1}{27},\ \ell_2=\frac{1}{29},\ E_0=\max_{t\in[0,w]}E(t,0,0)=(0.0025). \end{aligned}$$

We choose

$$K_w(c_1,c_2)=\{y\in P_w,\ 0<c_0\le y(t)\le c_1,\ \ |y(t_2)-y(t_1)|\le c_2\,|t_2-t_1|\,,\forall t_1,t_2\in\mathbb{R}\}\,,$$

where $w=35$, $c_0=1.5$, $c_1=2.5$ *and* $c_2=0.5$.
So, we have

$$\begin{aligned} & aw\alpha_2=0.93511\le 1, \\ & ay^{[2]}(s)-b\left(y^{[2]}(s)\right)^2-qy^{[2]}(s)\,E(s,y(s),y(s-\tau(s)))=0.01814>0, \\ & \alpha_2c_1(kw+2)(qE_0+a+bc_1+q\ell_1c_1+q\ell_2c_1)=0.27741\le c_2=0.5, \end{aligned}$$

and

$$\lambda=\alpha_2w\left((c_2+1)(a+2bc_1)+q\left(E_0(c_2+1)+c_1(c_2+2)(\ell_1+\ell_2)\right)\right)\simeq 1.9405>1.$$

Condition $\lambda<1$ *is not satisfied but all requirements of Theorem 3.6 are fulfilled. Hence, equation (4.1) has at least one positive periodic and bounded solution in* $K_w(2.5,0.5)$ *which is not necessarily unique.*

Remark 4.2. *With the choice* $a=\rho\beta$ *and* $b=\rho^2\delta$, *equation (4.1) models the population dynamics of the housefly Musca domestica, where* $k(t)$ *is the death rate of fly adults,* ρ *denotes the number of eggs laid per adult,* $\beta>0$ *is the maximum egg-adult survival rate,* δ *stands for the reduction in survival produced by each additional egg and* $y^{[2]}(s)$ *results from a time and state dependent delay representing the life cycle of this fly.*

Example 4.3. *We consider the same previous recruitment model (4.1) with* $c_0=2$ *and* $a=0.005$.
In this case we have

$$\begin{aligned} & aw\alpha_2=0.25975\le 1, \\ & ay^{[2]}(s)-b\left(y^{[2]}(s)\right)^2-qy^{[2]}(s)\,E(s,y(s),y(s-\tau(s)))\simeq 1.1403\times 10^{-3}>0, \\ & \alpha_2c_1(kw+2)(qE_0+a+bc_1+q\ell_1c_1+q\ell_2c_1)=0.11001\le c_2=0.5. \end{aligned}$$

Moreover, we get

$$\lambda=\alpha_2w\left((c_2+1)(a+2bc_1)+q\left(E_0(c_2+1)+c_1(c_2+2)(\ell_1+\ell_2)\right)\right)=0.92743<1.$$

We infer that all hypotheses of Theorem 3.7 hold. So, equation (4.1) with $c_0=2$ *and* $a=0.005$ *has a unique positive periodic and bounded solution in* $K_w(2.5,0.5)$ *that depends continuously on the death rate* k *and the harvesting effort* E.

5. Conclusion

In the present work, we were interested in providing new existence, uniqueness and stability results for a first order differential equation with iterative recruitment and harvesting terms and a delayed harvesting effort.

Our first task we have set ourselves was to choose a suitable Banach space and a subset of it. Indeed this choice were the cornerstone of our technique as, on one hand, they facilitated the study of our equation including the application of the chosen fixed point theorems and the control of the iterative terms and, on the other hand, they ensured some basic biological facts such as the periodicity, positivity and boundedness of the sought solutions. The proofs have hinged on an efficient approach based on the fixed point theory with the help of some properties of a Green's kernel where the existence of the solutions of equation (1.1) was equivalent to the existence of fixed points of an integral operator obtained after reformulation of our equation as an equivalent integral one. The existence of positive bounded and periodic solutions was established by virtue of the Schauder's fixed point theorem and under an additional hypothesis, Banach contraction principle has guaranteed the existence and continuous dependence on parameters of the unique solution. Moreover, we have also applied the obtained outcomes to two models.

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