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On Full and Nearly Full Operators in Complex Banach Spaces

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ABSTRACT: A bounded linear operator T on a complex Banach space $\mathcal{X}$ is said to be full if $\overline{T\mathcal{M}} = \mathcal{M}$ for every invariant subspace $\mathcal{M}$ of $\mathcal{X}$. It is nearly full if $\overline{T\mathcal{M}}$ has finite codimension in $\mathcal{M}$. In this paper, we focus our attention to characterize full and nearly full operators in complex Banach spaces, showing that some valid results in complex Hilbert spaces can be generalized to this context.

Key Words: Full operator, nearly full operator, invariant subspace, quasi-nilpotent operator, bounded below operator.

Contents

1. Introduction

It is known that if T is an invertible operator on a space of finite dimension, then T^{-1} is a polynomial in T, i.e., there exists a polynomial p such that $p(T) = T^{-1}$ (see [11,22]). The analogue of this fact is false if the vector space is not of finite dimension (see [8]). There are examples in which T^{-1} is not even the limit of polynomials in T. The bilateral shift operator on $l_2(\mathbb{Z})$ is an example on this case, see [6,7,9,10] for a related results.

Let $\mathcal{H}$ be a complex Hilbert space and $T \in L(\mathcal{H})$ be a bounded linear operator on $\mathcal{H}$. Let $\mathtt{A}_T$ be the weak algebra generated by T and the identity operator. If T^{-1} belongs to $\mathtt{A}_T$, that is, T^{-1} can be weakly approximated by polynomials in T, and therefore, for a finite set of points $\mathbf{x}, \mathbf{y} \in \mathcal{H}$ and a given $\varepsilon > 0$ there exists a complex polynomial p such that

$$\left\langle (p(T) - T^{-1})\mathbf{x}, \mathbf{y} \right\rangle < \varepsilon, \tag{1.1}$$

then any invariant subspace for T is also invariant for T^{-1}. Thus, if $\mathrm{lat}(T)$ denotes the lattice of all subspaces invariant under T, the above argument ensures that $T\mathcal{M} = \mathcal{M}$, for all $\mathcal{M} \in \mathrm{lat}(T)$ (see [4,6,7,8]).

An operator that satisfies $\overline{T\mathcal{M}} = \mathcal{M}$, for all $\mathcal{M} \in \mathrm{lat}(T)$, is called a *full operator*, where the bar indicates the closure in the topology induced by the norm. Consequently, if T^{-1} belongs to $\mathtt{A}_T$, then necessarily T must be a full operator. The concept of full operator is introduced in [6] by Erdos, who studied dissipative operators and their relationship with the problem of approximating the inverse of an operator by polynomials in the operator. J. Bravo [3] in his doctoral thesis studied conditions for T^{-1} to belong to $\mathtt{A}_T$, also he characterized the full operators.

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Karanasios in [12,13] studied full operators in uniformly convex and reflexive Banach spaces and their relationship with the approximation problem of the inverse of an operator by polynomials in the operator. In [14], Karanasios and Pappas gave necessary and sufficient conditions for which the generalized inverse T^{+}, of T, can be approximated by a polynomial in T by connecting the problem with a nearly full operator; a concept introduced by Erdos in [7].

The main results in this work are related to full and nearly full operators in complex Banach spaces. We give characterizations of such operators, we then show that some results on full operators in complex Hilbert spaces can be extended to full operators in complex Banach spaces. Finally, we conclude with a section where we propose some problems related to the theory of full and nearly full operators.

2. Preliminaries and Notation

Throughout this paper, we will denote the complex Banach and Hilbert spaces with the letter $\mathcal{X}$ and $\mathcal{H}$ respectively. If $\mathcal{X}$ is a separable complex Banach space, $L(\mathcal{X})$ will denote the algebra of all bounded linear operators on $\mathcal{X}$. With $\mathcal{X}^*$ we will denote the dual space of $\mathcal{X}$ with standard norm, i.e., $\mathcal{X}^* = L(\mathcal{X}, \mathbb{C})$. If $(\mathcal{X}_1, \|\cdot\|_1)$ and $(\mathcal{X}_2, \|\cdot\|_2)$ are two complex Banach spaces, we will denote with $\mathcal{X}_1 \oplus \mathcal{X}_2$ the following vector space

$$\mathcal{X}_1 \oplus \mathcal{X}_2 = \{(\mathbf{x}_1, \mathbf{x}_2) : \mathbf{x}_1 \in \mathcal{X}_1,\ \mathbf{x}_2 \in \mathcal{X}_2\}\,, \tag{2.1}$$

which is a Banach space with the norm

$$\|(\mathbf{x}_1, \mathbf{x}_2)\| = \|\mathbf{x}_1\|_1 + \|\mathbf{x}_2\|_2\,. \tag{2.2}$$

Also $\mathcal{X}^{(n)}$ will denote the complex Banach space direct sum of n copies of $\mathcal{X}$. If $S \subseteq \mathcal{X}$, then the *annihilator* of S is defined by

$$S^{\perp} = \{f \in \mathcal{X}^* : f(\mathbf{x}) = 0,\ \forall \mathbf{x} \in S\}\,. \tag{2.3}$$

In complex Hilbert space we have

$$S^{\perp} = S^{\perp} = \{\mathbf{y} \in \mathcal{H} : \langle \mathbf{y}, \mathbf{x}\rangle = 0,\ \forall \mathbf{x} \in S\}\,. \tag{2.4}$$

For $T \in L(\mathcal{X}_1)$ and $S \in L(\mathcal{X}_2)$, $T \oplus S$ denotes the operator in $L(\mathcal{X}_1 \oplus \mathcal{X}_2)$ defined by

$$T \oplus S\,((\mathbf{x}_1, \mathbf{x}_2)) = (T\mathbf{x}_1, S\mathbf{x}_2)\,, \quad \text{for } \mathbf{x}_1 \in \mathcal{X}_1,\ \mathbf{x}_2 \in \mathcal{X}_2. \tag{2.5}$$

Also we recall (see [11,22]) that the subspace S is invariant under the operator T if $Tx \in S$ for every $x \in S$. For $T \in L(\mathcal{X})$, $\mathrm{lat}(T)$ denotes the set of all subspaces invariant under T, that is,

$$\mathrm{lat}(T) = \{\mathcal{M} \subseteq \mathcal{X} : \mathcal{M} \text{ is a closed subspace and } T\mathcal{M} \subseteq \mathcal{M}\}\,. \tag{2.6}$$

If $T \in L(\mathcal{X})$ the adjoint of T is defined as the only operator $T' \in L(\mathcal{X}^*)$ that satisfies

$$f(T\mathbf{x}) = T'f(\mathbf{x}), \tag{2.7}$$

for all $\mathbf{x} \in \mathcal{X}$ and $f \in \mathcal{X}^*$. The weak algebra generated by T and the identity is denoted by $\mathtt{A}_T$, the commutant of T will be denoted by $\{T\}'$ and defines the set

$$\mathrm{alglat}(T) = \{S \in L(\mathcal{X}) : \mathrm{lat}(T) \subset \mathrm{lat}(S)\}\,. \tag{2.8}$$

Note that $\mathtt{A}_T \subseteq \mathrm{alglat}(T)$, being this generally a strict inclusion. We say that a subspace $\mathcal{M} \in \mathrm{lat}(T \oplus S)$ is split, if there are subspaces $\mathcal{N}_1 \in \mathrm{lat}(T)$ and $\mathcal{N}_2 \in \mathrm{lat}(S)$ such that $\mathcal{M} = \mathcal{N}_1 \oplus \mathcal{N}_2$, see [1,2].

Definition 2.1. *An operator $T \in L(\mathcal{X})$ is called nearly full, if for all $\mathcal{M} \in$lat(T) we have that $\overline{T\mathcal{M}}$ has finite codimension in $\mathcal{M}$, i.e., $\overline{T\mathcal{M}}$ has a complementary subspace in $\mathcal{M}$ of finite dimension.*

Definition 2.2. *Let $k \in \mathbb{N} \cup \{0\}$. An operator $T \in L(\mathcal{X})$ is called nearly full of order k, if T is nearly full and for all $\mathcal{M} \in$ lat(T) we have that $\overline{T\mathcal{M}}$ has a complementary subspace in $\mathcal{M}$ of dimension less than $k+1$.*

Definition 2.3. *An operator $T \in L(\mathcal{X})$ is called full, if for all $\mathcal{M} \in lat(T)$ we have that $\overline{T\mathcal{M}} = \mathcal{M}$.*

Note that an operator being full is equivalent of being a nearly full operator of order 0. The shift operator of multiplicity n, with $n \in \mathbb{N}$ (unilateral or bilateral) is nearly full operator of order n (see [17] where the authors characterize the invariant subspaces of these operators).

Let $\{e_n : n \in \mathbb{Z}\}$ be an orthonormal basis for $\mathcal{H} = l_2(\mathbb{Z})$ and U the bilateral shift operator on $\mathcal{H}$ defined by $Ue_n = e_{n+1}$ for all $n \in Z$. In this case $U^* = U^{-1}$. Also, if $\mathcal{M}$ is the subspace of $\mathcal{H}$ spanned by $\{e_n : n \in \mathbb{N}\}$, then $\mathcal{M} \in$lat(U), but $\mathcal{M}$ is not in lat(U^{-1}), so U is not a full operator. In [23], Wermer characterized the unitary full operators as those whose constraints to any proper invariant subspace are not the bilateral shift operator.

Recall that the resolvent set $\rho(T)$ of $T \in L(\mathcal{X})$ is the set of all regular values of T, i.e.,

$$\rho(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is invertible}\} \tag{2.9}$$

The spectrum of T, denoted by $\sigma(T)$, is the complement of the resolvent set. It is known that $\sigma(T)$ is a compact set in $\mathbb{C}$, hence $\rho(T)$ is an open set, see [18] for further reading.

Definition 2.4. *An operator $T \in L(\mathcal{X})$ is called:*

1. *Quasi-nilpotent, if $\lim\limits_{n\to\infty} \|T^n\|^{\frac{1}{n}} = 0$, or equivalently $\sigma(T) = \{0\}$, i.e., $T - \lambda I$ is invertible for all $\lambda \in \mathbb{C}$, $\lambda \neq 0$.*
2. *Power bounded, if there exists $k > 0$ such that $\|T^n\| \leq k$ for all $n \in \mathbb{N}$.*
3. *Bounded below, if there exists $k > 0$ such that $\|T\mathbf{x}\| \geq k\|\mathbf{x}\|$ for all $\mathbf{x} \in \mathcal{X}$.*

Note that every closed-range nearly full operator is a Fredholm operator. If T is bounded below, then $T\mathcal{M}$ is closed for all $\mathcal{M} \in$ lat(T). Thus, an operator which is bounded below and nearly full is not only a Fredholm operator, but is a hereditarily Fredholm operator, that is, the restriction of T to any subspace $\mathcal{M} \in$ lat(T) is a Fredholm operator.

Let $\mathbf{x} \in \mathcal{M}$, and $T \in L(\mathcal{X})$. Recall that the T-cyclic subspace generated by $\mathbf{x}$ is the subspace generated by the set span$\{T^n\mathbf{x} : n \geq 0\}$. A subspace A of $\mathcal{X}$ is called a cyclic subspace for T if span$\{T^nA : n \geq 0\} = \mathcal{X}$, see [19] for further readings. For each $n \in \mathbb{N}$, by $\mathcal{M}(n, \mathbf{x}, T)$ we denote the closure of the T-cyclic subspace spanned by $T^n\mathbf{x}$. Note that

$$\mathcal{M}(n, \mathbf{x}, T) = \overline{T(\mathcal{M}(n-1, \mathbf{x}, T))} \in \text{lat}(T). \tag{2.10}$$

In general, If A is a subset of $\mathcal{X}$, we will denote by $\mathcal{M}_A$ the closure of the T-cyclic subspace generated by A. Clearly $\mathcal{M}_A \in$ lat(T), also if $\mathbf{x} \in A$, then

$$\mathcal{M}_A = [\mathbf{x}] \oplus T\mathcal{M}_{(A \setminus \{\mathbf{x}\}) \cup \{T\mathbf{x}\}} \tag{2.11}$$

where $[\mathbf{x}]$ denote the subspace spanned by $\mathbf{x}$ and $T\mathcal{M}_{(A \setminus \{\mathbf{x}\}) \cup \{T\mathbf{x}\}}$ the image of $\mathcal{M}_{(A \setminus \{\mathbf{x}\}) \cup \{T\mathbf{x}\}}$ by T. Finally note that

$$\overline{T\mathcal{M}_A} \subseteq \mathcal{M}_{TA}. \tag{2.12}$$

In [21] Sarason proved the following theorem which relates lat$(T - \lambda I)^{-1}$ in the different connected components of the resolvent of an operator T.

Theorem 2.5. *Let $T \in L(\mathcal{X})$. If $\rho(T)$ is the resolvent of T and $\rho_\infty(T)$ is the unbounded connected component of $\rho(T)$, then:*

(1) If λ and λ_1 belong to the same connected component of $\rho(T)$, then

$$lat(T - \lambda I)^{-1} = lat(T - \lambda_1 I)^{-1} \tag{2.13}$$

(2) If $\lambda \in \rho_\infty(T)$, then

$$lat(T - \lambda I)^{-1} = lat(T). \tag{2.14}$$

Note that as a consequence of this theorem it follows that if T is invertible and full, the spectrum of T does not separate the plane (for example $\sigma(T)$ is not a simple closed curve) and λ belongs to the resolvent of T, then

$$\mathrm{lat}(T-\lambda I)^{-1} = \mathrm{lat}(T-\lambda I). \tag{2.15}$$

Thus, $T-\lambda I$ is full for every λ in the resolvent of T. Therefore, under these conditions, if $T-\lambda I$ is not full, then λ is necessarily in spectrum of T. More discussions and details on the theory about the issues described above can be found in [3,4,5]

3. Full Operators in Complex Banach Spaces

We begin this section by characterizing the full operators which are bounded below in complex Banach spaces.

Theorem 3.1. *Let $T \in L(\mathcal{X})$ and $\mathcal{M} \in lat(T)$, such that $\overline{T\mathcal{M}} \neq \mathcal{M}$. Then there exists $\mathbf{x} \in \mathcal{M}$, $\|\mathbf{x}\| = 1$, and $f \in \mathcal{M}(1,\mathbf{x},T)^{\perp}$, with $f(\mathbf{x}) = 1$. If also T is bounded below, then*

$$T^{n-1}\mathbf{x} \notin \mathcal{M}(n,\mathbf{x},T), \tag{3.1}$$

for all $n \in \mathbb{N}$. In particular, $dim(\mathcal{M}(n,\mathbf{x},T)) = \infty$ for all $n \in \mathbb{N}$.

Proof. Since $\overline{T\mathcal{M}} \neq \mathcal{M}$, then there exists $\mathbf{x} \in \mathcal{M} \setminus \overline{T\mathcal{M}}$, with $\|\mathbf{x}\| = 1$. By Hahn-Banach theorem, there exists $f \in \mathcal{X}^*$ such that $f(\mathbf{x}) = 1$ and $f(\overline{T\mathcal{M}}) = 0$.

Furthermore, since $\mathcal{M}(1,\mathbf{x},T) \subseteq \overline{T\mathcal{M}}$, then $\overline{T\mathcal{M}}^{\perp} \subseteq \mathcal{M}(1,\mathbf{x},T)^{\perp}$, so $f \in \mathcal{M}(1,\mathbf{x},T)^{\perp}$.

On the other hand, assume that T is bounded below, we now show that $T^{n-1}\mathbf{x} \notin \mathcal{M}(n,\mathbf{x},T)$, for all $n \in \mathbb{N}$. Indeed, suppose there exist a polynomial p_k with $p_k(0) = 0$ such that if $T^{n-1}q_k(T) = p_k(T^n)$, then

$$\left\|T^{n-1}(p_k(T)\mathbf{x} - \mathbf{x})\right\| = \left\|T^{n-1}q_k(T)\mathbf{x} - T^{n-1}\mathbf{x}\right\| < \frac{1}{k}. \tag{3.2}$$

Thus,

$$\lim_{k\to\infty} T^{n-1}(q_k(T)\mathbf{x} - \mathbf{x}) = 0. \tag{3.3}$$

But since T is bounded below, we have

$$\lim_{k\to\infty} q_k(T)\mathbf{x} = \mathbf{x}. \tag{3.4}$$

Therefore $\mathbf{x} \in \mathcal{M}(1,\mathbf{x},T)$, and as a consequence, we get

$$0 = \lim_{k\to\infty} f(q_k(T)\mathbf{x}) = f(\mathbf{x}) = 1, \tag{3.5}$$

which is a clear contradiction. □

In the complex Hilbert space case this Theorem 3.1 can be written as

Theorem 3.2. *Let $T \in L(\mathcal{H})$ and $\mathcal{M} \in lat(T)$, such that $\overline{T\mathcal{M}} \neq \mathcal{M}$. Then there exists $\mathbf{x} \in \mathcal{M}$, $\|\mathbf{x}\| = 1$, and $\mathbf{x} \in \mathcal{M}(1,\mathbf{x},T)^{\perp}$. If also T is bounded below, then*

$$T^{n-1}\mathbf{x} \notin \mathcal{M}(n,\mathbf{x},T), \tag{3.6}$$

for all $n \in \mathbb{N}$. In particular, $dim(\mathcal{M}(n,\mathbf{x},T)) = \infty$ for all $n \in \mathbb{N}$.

Corollary 3.3. *Let $T \in L(\mathcal{X})$ and let $\mathbf{x}$ be as in Theorem 3.1. If also T is bounded below, then*

$$\mathcal{M}(n-1,\mathbf{x},T) = [T^{n-1}\mathbf{x}] \oplus \mathcal{M}(n,\mathbf{x},T), \tag{3.7}$$

for all $n \in \mathbb{N}$.

Lemma 3.4. *Let $T \in L(\mathcal{X})$ be injective and $\mathcal{M} \in lat(T)$ such that*

$$\bigcap_{n\in\mathbb{N}} T^n(\mathcal{M}) = \{\mathbf{0}\}\,. \tag{3.8}$$

Then T has no eigenvectors in $\mathcal{M}$. In particular, T does not have invariant subspaces of finite dimension in $\mathcal{M}$.

Proof. Suppose that $\mathbf{y} \in \mathcal{M}$ is an eigenvector of T, so $T\mathbf{y} = \lambda\mathbf{y}$ for some nonzero $\lambda \in \mathbb{C}$. Let k be the largest integer such that $\mathbf{y} \in T^k(\mathcal{M})$. Then $\mathbf{y} \notin T^{k+1}(\mathcal{M})$, but

$$\lambda\mathbf{y} = T\mathbf{y} \in T^{k+1}(\mathcal{M}), \tag{3.9}$$

which is a clear contradiction. □

Theorem 3.5. *Let $T \in L(\mathcal{X})$ be bounded below. Then the following are equivalent:*

1. *T is full.*
2. *$lat(0 \oplus T) = lat(0) \oplus lat(T)$.*
3. *There exist no $\mathbf{x} \in \mathcal{X}$, $f \in \mathcal{X}^*$, with $f(\mathbf{x}) = 1$ and $f(T^n\mathbf{x}) = 0$, for all $n \in \mathbb{N}$.*

Proof. ($1 \implies 2$) Let $\mathcal{M} \subseteq \mathcal{X} \oplus \mathcal{X}$ in $\mathrm{lat}(0 \oplus T)$ and let

$$\mathcal{M}_1 = \{\mathbf{x} \in \mathcal{X} : (\mathbf{x}, \mathbf{y}) \in \mathcal{M}, \text{ for some } \mathbf{y}\} \tag{3.10}$$

and

$$\mathcal{M}_2 = \{\mathbf{y} \in \mathcal{X} : (\mathbf{x}, \mathbf{y}) \in \mathcal{M}, \text{ for some } \mathbf{x}\}\,. \tag{3.11}$$

Then $\mathcal{M}_1$ and $\mathcal{M}_2$ are closed subspaces of $\mathcal{X}$. Clearly $\mathcal{M}_1 \in \mathrm{lat}(0)$. Also note that $\mathcal{M}_2 \in \mathrm{lat}(T)$, because if $\mathbf{y} \in \mathcal{M}_2$, then there exists $\mathbf{x}$ such that $(\mathbf{x}, \mathbf{y}) \in \mathcal{M}$, and since $\mathcal{M} \in \mathrm{lat}(0 \oplus T)$, we will have

$$(\mathbf{0}, T\mathbf{y}) = (0 \oplus T)(\mathbf{x}, \mathbf{y}) \in \mathcal{M}, \tag{3.12}$$

hence $T\mathbf{y} \in \mathcal{M}_2$.

Since T is full, we have $\overline{T\mathcal{M}_2} = \mathcal{M}_2$, so the set $\{T\mathbf{y} \in \mathcal{X} : (\mathbf{x}, \mathbf{y}) \in \mathcal{M}, \text{ for some } \mathbf{x}\}$ is dense in $\mathcal{M}_2$. Hence, if $(\mathbf{x}, \mathbf{y}) \in \mathcal{M}$ and $\varepsilon > 0$, then there exists $(\mathbf{x}_0, \mathbf{y}_0) \in \mathcal{M}$ such that

$$\|T\mathbf{y}_0 - \mathbf{y}\| < \varepsilon. \tag{3.13}$$

Thus,

$$\|(0 \oplus T)(\mathbf{x}_0, \mathbf{y}_0) - (\mathbf{0}, \mathbf{y})\| = \|(\mathbf{0}, T\mathbf{y}_0) - (\mathbf{0}, \mathbf{y})\| = \|T\mathbf{y}_0 - \mathbf{y}\| < \varepsilon. \tag{3.14}$$

On the other hand, since $(0 \oplus T)(\mathbf{x}_0, \mathbf{y}_0) = (\mathbf{0}, T\mathbf{y}_0) \in \mathcal{M}$, and since $\mathcal{M}$ is closed we have that $(\mathbf{0}, \mathbf{y}) \in \mathcal{M}$ and $\mathcal{M}$ split into subspaces

$$N_1 = \{\mathbf{x} \equiv (\mathbf{x}, \mathbf{0}) \in \mathcal{X} : (\mathbf{x}; \mathbf{y}) \in \mathcal{M} \;\text{ for some } \mathbf{y}\} \tag{3.15}$$

and

$$N_2 = \{\mathbf{y} \equiv (\mathbf{0}, \mathbf{y}) \in \mathcal{X} : (\mathbf{x}; \mathbf{y}) \in \mathcal{M} \;\text{ for some } \mathbf{x}\}\,, \tag{3.16}$$

which are 0 and T invariant respectively.

($2 \implies 3$) Suppose there exist some $\mathbf{x} \in \mathcal{X}$ and $f \in \mathcal{M}^*$ with $f(\mathbf{x}) = 1$ and $f(T^n\mathbf{x}) = 0$, for all $n \in \mathbb{N}$, and let $\mathcal{M}_1 \subset \mathcal{X} \oplus \mathcal{X}$ be the subspace generated by $(\mathbf{x}, \mathbf{x})$ and $\mathcal{M}_2 = \{(\mathbf{0}, \mathbf{y}) : \mathbf{y} \in \mathcal{M}(1, \mathbf{x}, T)\}$ and

$$\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2. \tag{3.17}$$

Then $\mathcal{M} \in \mathrm{lat}(0 \oplus T)$. But $\mathcal{M}$ does not split, because $(\mathbf{x}, \mathbf{0}) \notin \mathcal{M}$.

($3 \implies 1$) If $\mathbf{x}$ and f satisfy with $f(\mathbf{x}) = 1$ and $f(T^n\mathbf{x}) = 0$, for all $n \in \mathbb{N}$, then $f \in \mathcal{M}(1, \mathbf{x}, T)^\perp$ and by Theorem 3.1 $\mathbf{x} \notin \mathcal{M}(1, \mathbf{x}, T) = T\mathcal{M}$, so T is not full, completing the proof. □

The next corollary is a direct consequence of Theorem 3.5

Corollary 3.6. *Let $T \in L(\mathcal{X})$. If $T^{(n)}$ is full for all $n \in \mathbb{N}$, then*

$$\mathtt{A}_{0\oplus T} = \mathtt{A}_0 \oplus \mathtt{A}_T. \tag{3.18}$$

In addition, If T is invertible, then $T^{-1} \in \mathtt{A}_T$.

Theorem 3.7. *Let $T \in L(\mathcal{X})$ be bounded below, p a nonzero polynomial, and $Q = T^2 p(T)$. Then T and Q are full.*

Proof. Suppose that T is not full, then there exists $\mathbf{x} \in \mathcal{X}$, with $\|\mathbf{x}\| = 1$, and $f \in \mathcal{M}(1, \mathbf{x}, T)^{\perp}$, with $f(\mathbf{x}) = 1$.

Let $N = \mathcal{M}(0, \mathbf{x}, Q)$. Then $N \in \mathrm{lat}(Q) = \mathrm{lat}(T)$. Since $T\mathbf{x} \in N$, for each $n \in \mathbb{N}$, take a complex polynomial g_n such that

$$\|g_n(Q)\mathbf{x} - T\mathbf{x}\| < \frac{1}{n\,\|f\|}. \tag{3.19}$$

Since $f \in \mathcal{M}(1, \mathbf{x}, T)^{\perp}$ we have

$$|g_n(0)| = \|f(g_n(Q)\mathbf{x} - T\mathbf{x})\| \le \|f\|\,\|g_n(Q)\mathbf{x} - T\mathbf{x}\| < \frac{1}{n}. \tag{3.20}$$

Hence, $\|g_n(0)\mathbf{x}\| < \frac{1}{n}$. Taking $h_n = g_n - g_n(0)$, we have

$$\|h_n(Q)\mathbf{x} - T\mathbf{x}\| \le \|g_n(Q)\mathbf{x} - T\mathbf{x}\| + \|g_n(0)\mathbf{x}\| < \frac{1}{n}\left(1 + \frac{1}{\|f\|}\right), \tag{3.21}$$

and $h_n(0) = 0$.

By the definition of Q, we have

$$h_n(Q)\mathbf{x} \in \mathcal{M}(2, \mathbf{x}, T). \tag{3.22}$$

Therefore $T\mathbf{x} \in \mathcal{M}(2, \mathbf{x}, T)$ which contradicts Theorem 3.1, hence, T is full.

On the other hand, if $\mathcal{M} \in \mathrm{lat}(Q) = \mathrm{lat}(T)$, then since T is bounded below we have $T(\mathcal{M}) = \mathcal{M}$, and since Q is a polynomial in T, then $Q(\mathcal{M}) = \mathcal{M}$. Thus, Q is full. $\square$

3.1. Full and Quasi-nilpotent Operators

The results in this section generalize similar results proved by Karanasios in [12,13] for reflexive complex Banach spaces.

Theorem 3.8. *Let $T \in L(\mathcal{X})$ be bounded below and suppose that alglat(T) contains a full Quasi-nilpotent operator Q. Then T is full.*

Proof. Suppose that T is not full, then there exists $\mathbf{x} \in \mathcal{X}$, with $\|\mathbf{x}\| = 1$ and $f \in \mathcal{M}(1, \mathbf{x}, T)^{\perp}$, with $f(\mathbf{x}) = 1$. Since $\mathcal{M}(0, \mathbf{x}, T) \in lat(Q)$, we have that $Q\mathbf{x} = \alpha\mathbf{x} + \mathbf{y}$, where $\mathbf{y} \in \mathcal{M}(1, \mathbf{x}, T)$.

For every $n \in \mathbb{N}$, we have

$$Q^n\mathbf{x} = \alpha^n\mathbf{x} + \mathbf{y}_n, \tag{3.23}$$

where $\mathbf{y}_n \in \mathcal{M}(1, \mathbf{x}, T)$. Then

$$|\alpha| = |f(\alpha^n\mathbf{x} + \mathbf{y}_n)|^{\frac{1}{n}} = |f(Q^n\mathbf{x})|^{\frac{1}{n}} \le (\|f\|\,\|Q^n\|)^{\frac{1}{n}}. \tag{3.24}$$

Since Q is Quasi-nilpotent, then $\alpha = 0$, therefore $f(Q^n\mathbf{x}) = 0$, i.e., $f \in \mathcal{M}(1, \mathbf{x}, Q)^{\perp}$ and by Theorem 3.5, Q is not full, which contradicts the hypothesis. $\square$

Theorem 3.9. *If $Q \in L(\mathcal{X})$ is full Quasi-nilpotent, $\mathcal{M}$ and $\mathcal{N} \in lat(Q)$, $\mathcal{M} \subset N$, then*

$$\dim \frac{\mathcal{N}}{\mathcal{M}} \ne 1. \tag{3.25}$$

Proof. If $\dim \frac{\mathcal{N}}{\mathcal{M}} = 1$, then $\mathcal{N} = [\mathbf{x}] \oplus \mathcal{M}$ for some $\mathbf{x} \in \mathcal{N} \setminus \mathcal{M}$. Since $\overline{QN} = N$, then $Q\mathbf{x} = \alpha\mathbf{x} + \mathbf{y}$ with $\mathbf{y} \in \mathcal{M}$.

If $\alpha = 0$, then $Q\mathbf{x} \in \mathcal{M}$ which implies $QN \subset \mathcal{M}$, so $\mathcal{M} = \mathcal{N}$, which is a contradiction. Thus, $\alpha \neq 0$ and therefore

$$(Q - \alpha I)\mathcal{N} \subset \mathcal{M}. \tag{3.26}$$

On the other hand,

$$(Q - \alpha I)^{-1} = \sum_{n=1}^{\infty} \alpha^{-n} Q^n., \tag{3.27}$$

So $(Q - \alpha I)^{-1} \in \mathtt{A}_Q$, hence $(Q - \alpha I)^{-1}\mathcal{N} \subset \mathcal{N}$. Thus

$$\mathcal{N} = (Q - \alpha I)(Q - \alpha I)^{-1}\mathcal{N} \subset (Q - \alpha I)\mathcal{N} \subset \mathcal{M}, \tag{3.28}$$

which is again a contradiction. Therefore, $\dim \frac{N}{\mathcal{M}} \neq 1$. □

For the resolvent set in (2.9) we let $\rho_0(T)$ be the connected component of $\rho(T)$ that contain 0.

Lemma 3.10. *Let $T \in L(\mathcal{X})$ be invertible and not full, $\lambda \in \rho_0(T)$, $\lambda \neq 0$, and $\varepsilon > 0$. If $\{A_\alpha\} \subset \mathtt{A}_T$ is a net converging weakly to T, then there exists $\mathcal{M} \in$lat(T) such that*

$$\sigma(A_\alpha|_{\mathcal{M}}) \cap B(\lambda, \varepsilon) \neq \emptyset, \tag{3.29}$$

if α is large enough. In particular, $\sigma(A_\alpha|_{\mathcal{M}})$ contains non-null elements if α is large enough.

Proof. First note that since T is not full then $\mathrm{lat}(T) \neq \mathrm{lat}(T^{-1})$. Also, since T is invertible then $0 \in \rho(T)$. For $\lambda \in \rho_0(T)$, we have

$$\mathrm{lat}(T - \lambda I) = \mathrm{lat}(T) \neq \mathrm{lat}(T^{-1}) = \mathrm{lat}(T - \lambda I)^{-1}. \tag{3.30}$$

In particular $T - \lambda I$ is invertible and not full for all $\lambda \in \rho_0(T)$.

Let $\mathbf{x} \in \mathcal{X}$, with $\|\mathbf{x}\| = 1$ and $f \in \mathcal{M}(1, \mathbf{x}, T - \lambda I)^{\perp}$, with $f(\mathbf{x}) = 1$ and consider the decomposition

$$\mathcal{M}(0, \mathbf{x}, T - \lambda I) = [\mathbf{x}] \oplus \mathcal{M}(1, \mathbf{x}, T - \lambda I). \tag{3.31}$$

In such decomposition we have,

$$A_\alpha|_{\mathcal{M}(0,\mathbf{x},T-\lambda I)} = \begin{bmatrix} \lambda_\alpha & \mathbf{0} \\ B_\alpha & C_\alpha \end{bmatrix} \tag{3.32}$$

and

$$T|_{\mathcal{M}(0,\mathbf{x},T-\lambda I)} = \begin{bmatrix} \lambda & \mathbf{0} \\ T_1 & T_2 \end{bmatrix}. \tag{3.33}$$

Hence, as A_α converges weakly to T, we have

$$\lambda_\alpha = f\left(\begin{bmatrix} \lambda_\alpha & 0 \\ B_\alpha & C_\alpha \end{bmatrix}\begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix}\right) \longrightarrow f\left(\begin{bmatrix} \lambda & \mathbf{0} \\ T_1 & T_2 \end{bmatrix}\begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix}\right) = \lambda. \tag{3.34}$$

Clearly $\lambda_\alpha \in \sigma\left(A_\alpha|_{\mathcal{M}(0,\mathbf{x},T-\lambda I)}\right)$. Thus, given $\varepsilon > 0$ the above weakly convergence assures that for large α we will have $\lambda_\alpha \in B(\lambda, \varepsilon)$, which completes the proof of the lemma. □

With this Lemma in our hands we will show the analogue of the theorem for complex Banach spaces due to Ortuñes [15] in the complex Hilbert space case. This theorem is similar to one proved by Feintuch in [9] which is also in the complex Hilbert space case.

Theorem 3.11. *Let $T \in L(\mathcal{X})$ be invertible. If the Quasi-nilpotent operators in $\mathtt{A}_T$ are weakly dense in $\mathtt{A}_T$, then $T^{-1} \in \mathtt{A}_T$.*

Proof. Since the hypotheses hold for direct sums of the form $T^{(n)}$, we just need to prove that $\mathrm{lat}(T) = \mathrm{lat}(T^{-1})$. Suppose otherwise, then T is not full. Let $\lambda \in \rho_0(T)$, $\lambda \neq 0$ and $\{A_\alpha\} \subset \mathtt{A}_T$ is a net of Quasi-nilpotent operators that converges weakly to T, then by Lemma 3.10, there exists $\mathcal{M} \in \mathrm{lat}(T)$ and $\lambda_\alpha \neq 0$ such that $\lambda_\alpha \in \sigma(A_\alpha|_{\mathcal{M}})$ for some an A_α, which contradicts the fact that A_α is a Quasi-nilpotent operator. □

3.2. Power Bounded Operators

The following results were proved by Bravo in [4], we now give new proofs of them.

Lemma 3.12. *Let $T \in L(\mathcal{X})$ be a power bounded operator, $f \in \mathcal{X}^*$, and $\mathbf{x}_0 \in \mathcal{X}$, such that*

$$\lim_{n\to\infty} f(T^n \mathbf{x}_0) = \alpha \neq 0. \tag{3.35}$$

Then T' has a fixed point.

Proof. Since T is power bounded, then for each $\mathbf{x} \in \mathcal{X}$ we have that

$$\{f(T^n \mathbf{x})\} \in l_\infty\left(\mathbb{Z}^+\right). \tag{3.36}$$

If L is a complex Banach limit at l_∞, then $\theta : \mathcal{X} \to \mathbb{C}$ defined by $\theta(\mathbf{x}) = L(\{f(T^n \mathbf{x})\})$ is linear and bounded.

Since

$$L(\{f(T^n \mathbf{x}_0)\}) = \lim_{n\to\infty} f(T^n \mathbf{x}_0) = \alpha \neq 0, \tag{3.37}$$

it follows that $\theta \neq 0$. By the properties of the complex Banach limits we have,

$$L(\{f(T^n \mathbf{x})\}) = L(\left\{f(T^{n+1}\mathbf{x})\right\}), \text{ for all x}, \tag{3.38}$$

hence, we have $T'(\theta \mathbf{x}) = \theta(T\mathbf{x}) = \theta(\mathbf{x})$, for all $\mathbf{x} \in \mathcal{X}$, therefore $T'\theta = \theta$. $\square$

Corollary 3.13. *If $\mathcal{X}$ is a reflexive complex Banach space and $T \in L(\mathcal{X})$ is power bounded, then $T\mathbf{x} = \mathbf{x}$ for some $\mathbf{x} \neq 0$ if and only if $T'\theta = \theta$, for some $\theta \neq 0$.*

Proof. The direct conclusion comes out using the previous lemma. The other direction of the fact yields from the fact of being $A'' = A$ because $\mathcal{X}$ is a reflexive complex Banach space. $\square$

Corollary 3.14. *If $T \in L(\mathcal{X})$ is power bounded and T' has no eigenvalues of absolute value 1, then $T - \lambda I$ is full for every $\lambda \in \mathbb{C}$, with $|\lambda| = 1$.*

Proof. Suppose that $T - \lambda I$ is not full for some λ with $|\lambda| = 1$. We can assume without loss of generality that $\lambda = 1$.

Let $\mathcal{M} \in \operatorname{lat}(T)$ such that $\overline{(T - I)\mathcal{M}} \subsetneq \mathcal{M}$. By the Hahn-Banach theorem, there exists $\mathbf{x}_0 \in \mathcal{M}$, $f \in \mathcal{X}^*$ such that $f(\mathbf{x}_0) \neq 0$ and $f((T - I)\mathcal{M}) = 0$.

Since $f((T - I)\mathcal{M}) = 0$, then $f(T\mathbf{x}_0) = f(\mathbf{x}_0)$ and using induction we get

$$f(T^n \mathbf{x}_0) = f(\mathbf{x}_0) \neq 0. \tag{3.39}$$

Now, using Lemma 3.12 it follows that 1 is an eigenvalue of T', which contradicts the hypothesis. $\square$

The proof of the following theorem is an improvement of the one given by Bravo in [3].

Theorem 3.15. *Let $T \in L(\mathcal{X})$ be invertible and let $S \in (alglat(T)) \cap \{T\}'$ be a non-scalar, power bounded and full operator with spectrum properly contained in the unit circumference, i.e., $|\lambda| = 1$ for all $\lambda \in \sigma(S)$. Then T is full or S' has an eigenvalue.*

Proof. If T is not full, then there exists $\mathcal{M} \in \operatorname{lat}(T)$, such that $\overline{T\mathcal{M}} \subsetneq \mathcal{M}$. Let $\mathbf{x} \in \mathcal{X}$ with $\|\mathbf{x}\| = 1$ and $f \in \mathcal{M}(1, \mathbf{x}, T)^\perp$, with $f(\mathbf{x}) = 1$. Let us consider the decomposition

$$\mathcal{M}(0, \mathbf{x}, T) = [\mathbf{x}] \oplus \mathcal{M}(1, \mathbf{x}, T), \tag{3.40}$$

then we have

$$S\mathbf{x} = \alpha \mathbf{x} + z, \qquad z \in \mathcal{M}(1, \mathbf{x}, T). \tag{3.41}$$

If $\alpha = 0$, then $S\mathbf{x} = z \in \mathcal{M}(1, \mathbf{x}, T)$, so

$$\mathbf{x} = S^{-1}z \in \mathcal{M}(1, \mathbf{x}, T), \tag{3.42}$$

because S is a full operator, so $f(\mathbf{x}) = 0$, which is a contradiction.

Now, note that $(S - \alpha I)\mathbf{x} \in \mathcal{M}(1, \mathbf{x}, T)$. Since by the hypothesis $S \in (\text{alglat}(T)) \cap \{T\}'$ not scalar, this implies that $\mathcal{M}(1, \mathbf{x}, T)$ is invariant under $S - \alpha I$. Hence,

$$(S - \alpha I)^n \mathbf{x} \in \mathcal{M}(1, \mathbf{x}, T), \tag{3.43}$$

for all $n \in \mathbb{N}$, since $f(\mathbf{x}) \neq 0$ and $f((S - \alpha I)^n \mathbf{x}) = 0$. Therefore, we concluded that $S - \alpha I$ is not a full operator. Thus, By Theorem 2.5, we have $|\alpha| = 1$, and by Corollary 3.14, S' has an eigenvalue. □

4. Nearly Full Operators

Erdos in [7] proposes some problems where nearly full operators appear. In this section we will characterize nearly full operators and we will see that some results on full operators are preserved for these operators. In what follows we will initially work on complex Banach spaces, the results in the case of complex Hilbert spaces will still be stated without proof unless one is not given for its equivalent in complex Banach spaces. For all natural number m, we also use the notation $[m]$ to denote the set $\{1, 2, \cdots, m\}$.

Lemma 4.1. *Let $\mathcal{M}$ be a closed subspace of a complex Banach space $\mathcal{X}$. Then $\mathcal{M}$ is infinite codimensional if and only if there exists a linearly independent set $A = \{\mathbf{x}_n \in \mathcal{X} : n \in \mathbb{N}\}$ and $B = \left\{f_n \in \mathcal{M}^\perp : n \in \mathbb{N}\right\}$, such that $f_i(\mathbf{x}_j) = \delta_{ij}$ for all $i, j \in \mathbb{N}$.*

Proof. Suppose there are A and B as in the statement. Let $n \in \mathbb{N}$. If $\mathbf{y} \in \mathcal{W} = [\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n]$, $\mathbf{y} \neq \mathbf{0}$, then there is a subset $I \subset \{1, 2, ..., n\}$ so that

$$\mathbf{y} = \sum_{i \in I} \alpha_i \mathbf{x}_i, \tag{4.1}$$

and $\alpha_i \neq 0$ for all $i \in I$. Let

$$f = \frac{1}{|I|} \sum_{i=1}^{n} \frac{f_i}{\alpha_i}, \tag{4.2}$$

where $|I|$ represents the number of elements of I, thus $f \in \mathcal{M}^\perp$ and $f(\mathbf{y}) = 1$, so $\mathbf{y} \notin \mathcal{M}$ and $\mathcal{M} \cap \mathcal{W} = \{0\}$, i.e., the codimension of $\mathcal{M}$ is greater than n for all $n \in N$, hence, $\mathcal{M}$ is an infinite codimensional.

Conversely, assume that $\mathcal{M}$ is infinite codimensional, and choose $\mathbf{x}_1 \notin \mathcal{M}$. By Hahn-Banach theorem, there exists $f_1 \in \mathcal{M}^\perp$ such that $f_1(\mathbf{x}_1) = 1$. Since $\mathcal{M}_1 = \mathcal{M} \oplus [\mathbf{x}_1] \neq \mathcal{X}$, as above we can choose $\mathbf{x}_2 \notin \mathcal{M}_1$ and $f_2 \in \mathcal{M}_1^\perp$, such that $f_2(\mathbf{x}_2) = 1$. Following this process, the required sets A and B in the statement are constructed.

□

Corollary 4.2. *Let $\mathcal{M}$ be a closed subspace of a complex Banach space $\mathcal{X}$. Then $\mathcal{M}$ is of codimension less than or equal to k if and only if there does not exist a linearly independent set $A = \{\mathbf{x}_n \in \mathcal{X} : n \in [k+1]\}$ and $B = \left\{f_n \in \mathcal{M}^\perp : n \in [k+1]\right\}$, such that $f_i(\mathbf{x}_j) = \delta_{ij}$ for all $1 \leq i, j \leq k+1$.*

Proof. The proof is similar to the proof of Lemma 4.1. □

In the case of complex Hilbert spaces, Lemma 4.1 and Corollary 4.2 are stated as follows:

Lemma 4.3. *Let $\mathcal{M}$ be a closed subspace of a complex Hilbert space $\mathcal{H}$. Then $\mathcal{M}$ is infinite codimension if and only if there is a linearly independent set $A = \{\mathbf{x}_n \in \mathcal{H} : n \in \mathbb{N}\}$ in $\mathcal{M}^\perp$.*

Corollary 4.4. *Let $\mathcal{M}$ be a closed subspace of a complex Hilbert space $\mathcal{H}$. Then $\mathcal{M}$ is of codimension less than or equal to k if and only if there is no linearly independent set $A = \{\mathbf{x}_n \in \mathcal{H} : n \in [k+1]\}$ in $\mathcal{M}^\perp$.*

4.1. Characterization of Nearly Full Operators

Theorem 4.5. *Let $T \in L(\mathcal{X})$. Then the following are equivalent*

1. *T is nearly full.*
2. *There exist no sets $A = \{\mathbf{x}_n \in \mathcal{X} : n \in \mathbb{N}\}$ linearly independent and $B = \left\{f_n \in \overline{T\mathcal{M}_A}^{\perp} : n \in \mathbb{N}\right\}$ such that $f_i(\mathbf{x}_j) = \delta_{ij}$, for all $i, j \in \mathbb{N}$.*

Proof. ($1 \Longrightarrow 2$) If there are $A = \{\mathbf{x}_n \in \mathcal{X} : n \in \mathbb{N}\}$ linearly independent and $B = \left\{f_n \in \overline{T\mathcal{M}_A}^{\perp} : n \in \mathbb{N}\right\}$ such that $f_i(\mathbf{x}_j) = \delta_{ij}$, for all $i, j \in \mathbb{N}$, then $\mathbf{x}_n \notin \overline{T\mathcal{M}_A}$ for all $n \in \mathbb{N}$, thus the subspace $\overline{T\mathcal{M}_A}$ is infinite codimensional in $\mathcal{M}_A$ contradicting that T is nearly full, because $\mathcal{M}_A \in \operatorname{lat}(T)$.

($2 \Longrightarrow 1$) Suppose that T is not nearly full, then there exists $\mathcal{M} \in \operatorname{lat}(T)$, such that $\overline{T\mathcal{M}}$ is infinite codimensional in $\mathcal{M}$, hence, by Lemma 4.1, there existsets $A = \{\mathbf{x}_n \in \mathcal{M} : n \in \mathbb{N}\}$ linearly independent and $B = \left\{f_n \in \overline{T\mathcal{M}_A}^{\perp} : n \in \mathbb{N}\right\}$ such that $f_i(\mathbf{x}_j) = \delta_{ij}$, for all $i, j \in \mathbb{N}$. □

Theorem 4.6. *Let $T \in L(\mathcal{X})$. Then the following are equivalent:*

1. *T is nearly full of order k*
2. *There exist no sets*
$$A = \{\mathbf{x}_n \in \mathcal{X} : n \in [k+1]\}$$
linearly independent and
$$B = \left\{f_n \in \overline{T\mathcal{M}_A}^{\perp} : n \in [k+1]\right\}$$
such that $f_i(\mathbf{x}_j) = \delta_{ij}$ for all $1 \leq i, j \leq k+1$.

Proof. The proof is similar to the proof of the previous Theorem using Corollary 4.2 instead of Lemma 4.1. □

Remark 4.7. *Item (2) in the previous Theorems is equivalent to: There exist no sets*
$$A = \{\mathbf{x}_n \in \mathcal{X} : n \in I\}$$
linearly independent and
$$B = \left\{f_n \in \overline{T\mathcal{M}_A}^{\perp} : n \in I\right\},$$
such that
$$f_i(\mathbf{x}_j) = \delta_{ij} \tag{4.3}$$
for all $i, j \in I$, and
$$f_i(T^k \mathbf{x}_j) = 0, \tag{4.4}$$
for all $i, j \in I$, $k \in \mathbb{N}$, where I is $\mathbb{N}$ or the finite set $[k+1]$ according the case.

The statements of the previous theorems in the case of complex Hilbert spaces are given below.

Theorem 4.8. *Let $T \in L(\mathcal{H})$. Then the following are equivalent:*

1. *T is nearly full*
2. *There exist no set $A = \{\mathbf{x}_n \in \mathcal{X} : n \in \mathbb{N}\}$ linearly independent such that $A \subset (\overline{T\mathcal{M}_A})^{\perp}$.*

Theorem 4.9. *Let $T \in L(\mathcal{H})$. Then the following are equivalent:*

1. *T is nearly full of order k*
2. *There exist no set $A = \{\mathbf{x}_n \in \mathcal{X} : n \in [k+1]\}$ linearly independent such that $A \subset (\overline{T\mathcal{M}_A})^{\perp}$.*

4.2. Nearly Full and Quasi-nilpotent Operators

If T is not nearly full and A and B are as in Theorem 4.8, then for all $n \in \mathbb{N}$ we have:

1. $\mathcal{M}_A = [\mathbf{x}_n] \oplus (\mathcal{M}_A \cap \ker f_n)$, where f_n can be considered as an element of $(\mathcal{M}_A)^*$
2. $\mathcal{M}_A \cap \ker f_n$ is T-invariant. Indeed, if $\mathbf{y} \in \mathcal{M}_A \cap \ker f_n$ then $T\mathbf{y} \in \mathcal{M}_A$, moreover $T\mathbf{y} \in T\mathcal{M}_A \subset \overline{T\mathcal{M}_A}$ and since $f_n \in \left(\overline{T\mathcal{M}_A}\right)^{\perp}$, we have that $f_n(T\mathbf{y}) = 0$.

Theorem 4.10. *Let $T \in L(\mathcal{X})$ be bounded below and suppose that alglat(T) contains an operator Q Quasi-nilpotent nearly full. Then T is nearly full. If Q is nearly full of order k, then T is nearly full of order less than or equal to k*

Proof. Suppose T is not nearly full, then there are sets $A = \{\mathbf{x}_n \in \mathcal{X} : n \in \mathbb{N}\}$ linearly independent and $B = \left\{ f_n \in \overline{T\mathcal{M}_A}^{\perp} : n \in \mathbb{N} \right\}$ such that $f_i(\mathbf{x}_j) = \delta_{ij}$, for all $i, j \in \mathbb{N}$.

Let $i \in \mathbb{N}$. Since $\mathcal{M}_A \in$ lat(Q), we have that $Q\mathbf{x}_i = \alpha_i \mathbf{x}_i + \mathbf{y}_1$, where $\mathbf{y}_n \in (\mathcal{M}_A \cap \ker f_i)$. Therefore, for every $n \in \mathbb{N}$, we have

$$Q^n \mathbf{x}_i = \alpha_i^n \mathbf{x}_i + \mathbf{y}_n, \tag{4.5}$$

where $\mathbf{y}_i \in (\mathcal{M}_A \cap \ker f_i)$. Hence

$$|\alpha_i| = |f_i(\alpha_i^n \mathbf{x}_i + \mathbf{y}_n)|^{\frac{1}{n}} = |f_i(Q^n \mathbf{x}_i)|^{\frac{1}{n}} \leq (\|f_i\| \, \|\mathbf{x}_i\| \, \|Q^n\|)^{\frac{1}{n}} . \tag{4.6}$$

Taking limit when n tends to infinity, we get $\alpha_i = 0$, since Q is Quasi-nilpotent. Thus, $f_i(Q^n \mathbf{x}_i) = 0$, On the other hand, since $\mathbf{x}_j \in (\mathcal{M}_A \cap \ker f_i)$ for $j \neq i$ and $(\mathcal{M}_A \cap \ker f_i) \in$ lat(Q), then

$$f_i(Q^n \mathbf{x}_j) = 0. \tag{4.7}$$

Consequently, $\overline{Q\mathcal{M}_A}$ is infinite codimensional in $\mathcal{M}_A$ and Q is not nearly full which contradicts the hypothesis. Analogously, the case in which Q is nearly full of order k is proved analogously. □

In the case of complex Hilbert spaces, Rosales [20] proved that changing the hypothesis of Q being full Quasi-nilpotent in alglat(T) of Theorem 4.9 by Q nearly full Quasi-nilpotent in alglat$(T) \cap \{T\}'$ will yield that T is full operator.

5. Final Comments and Problems

We will dedicate this section for making some final observations about possible open problems to study related to these topics.

Recall that Theorem 3.1 guarantees that if $T \in L(\mathcal{X})$ is bounded below and not full, then there exists $\mathbf{x} \in \mathcal{X}$, such that $\beta = \left\{\mathbf{x}, T\mathbf{x}, T^2\mathbf{x}, \cdots, T^n\mathbf{x}, \cdots\right\}$ is a Minkowski basis of $\mathcal{M}(0, \mathbf{x}, T)$. If β was a Schauder basis of $\mathcal{M}(0, \mathbf{x}, T)$, It is not difficult to prove that

$$\bigcap_{n\in\mathbb{N}} T^n(\mathcal{M}(0, \mathbf{x}, T)) = \{\mathbf{0}\}, \tag{5.1}$$

and some theorems proved in the case of complex Hilbert spaces could be proved in the most general context of complex Banach spaces. Hence our first questions in this order are:

Problem 5.1. *Under what conditions for T, is β a Schauder basis of $\mathcal{M}(0, \mathbf{x}, T)$?*

Problem 5.2. *Is it suffices that β be a Minkowski basis of $\mathcal{M}(0, \mathbf{x}, T)$ to prove that*

$$\bigcap_{n\in\mathbb{N}} T^n(\mathcal{M}(0, \mathbf{x}, T)) = \{\mathbf{0}\}, \tag{5.2}$$

Pacheco in [16] used the fact that compressions of compact or Quasi-nilpotent operators in semi-invariant subspaces are also compact or Quasi-nilpotent operators, respectively, and some spectral properties of these operators, as is the fact that the eigenspaces associated with non-zero eigenvalues are of finite dimension. It is known that Riesz operators contain both compact and Quasi-nilpotent operators and have the same spectral properties as these compact operators (see [5] for related results), hence, we address our next problem on complex Hilbert space operator theory.

Problem 5.3. *Are the compressions of Riesz operators to semi-invariant subspaces also Riesz operators?*

Karanasios and Pappas in [14] proposed as definition of almost full operator of an operator T such that $\overline{T\mathcal{M}} = \mathcal{M}$ for all $\mathcal{M}$ in $\operatorname{lat}(T)$, except perhaps for $\mathcal{M} = \mathcal{H}$ and for $\ker T$. Using this definition they proved that if T is normal with closed range and almost full in the previous sense, then the generalized inverse of T can be approximated by polynomials in T. Now, in the definition proposed by Karanasios and Pappas, one must necessarily have $\ker T = \{0\}$. Otherwise, if $\mathcal{M}$ belongs to $\operatorname{lat}(T)$ and $\ker T \subsetneq \mathcal{M}$, then the subspace $\mathcal{M}+\ker T$ is in $\operatorname{lat}(T)$ and cannot satisfy the proposed condition. Thus, in the definition of Karanasios and Pappas, every almost full operator is injective, but an injective normal operator of closed range is necessarily invertible and what is proven by Karanasios and Pappas is a well known result.

Our question related to the definition above is the following.

Problem 5.4. *Are the almost full operators of Karanasios and Pappas need to be full operators?*

Finally, the examples of nearly full operators presented in this work are all nearly full operators of order k for some integer k. Therefore, it is natural to ask if there is a nearly full operator T such that for all $n \in \mathbb{N}$ there is a subspace $\mathcal{M}_n \in \operatorname{lat}(T)$ for which the codimension of $\overline{T\mathcal{M}_n}$ in $\mathcal{M}_n$ is greater than or equal to n? Or equivalently:

Problem 5.5. *Are there nearly full operators that are not nearly full operators of order k for some integer k?*

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