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Semiprime Rings with Generalized Homoderivations

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ABSTRACT: This study develops some results involving generalized homoderivation in semiprime rings and investigates the commutativity of semiprime rings admitting generalized homoderivations of ring R satisfying certain identities and some related results have also been discussed.

Key Words: Semiprime ring, ideal, homoderivation, generalized homoderivation.

Contents

1. Introduction

Let R be an associative ring with center Z. For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$ and the symbol $x \circ y$ denotes for the anti-commutator $xy + yx$. Recall that a ring R is semiprime if and only if $xRx = \{0\}$ implies $x = 0$. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. An additive mapping $h : R \to R$ is called a homoderivation if $h(xy) = h(x)h(y) + h(x)y + xh(y)$, for all $x, y \in R$ in [4]. An example of such mapping is to let $h(x) = f(x) - x$ for all $x \in R$ where f is an endomorphism on R. It is clear that a homoderivation h is also a derivation if $h(x)h(y) = 0$ for all $x, y \in R$. Inspired by the definition of a homoderivation, the notion of generalized homoderivation was extended as follows:

Definition 1.1. *An additive mapping $F : R \to R$ is called a right generalized homoderivation if there exists a homoderivation $h : R \to R$ such that*

$$F(xy) = F(x)h(y) + F(x)y + xh(y) \text{ for all } x, y \in R$$

and F is called a left generalized homoderiation if there exists a homoderiation $h : R \to R$ such that

$$F(xy) = h(x)F(y) + h(x)y + xF(y) \text{ for all } x, y \in R.$$

F is said to be a generalized homoderivation with associated homoderivation h if it is both a left and right generalized homoderivation with associated homoderivation h.

Example 1.2. *Suppose the ring $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \middle| a, b \in \mathbb{R} \right\}$. Define maps $F, h : R \to R$ as follows:*

$$F\left(\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix},$$
$$h\left(\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}.$$

Then it is easy to verify that F is a left generalized homoderivation of R associated with a homoderivation h but not a right generalized homoderivation associated with a homoderivation h.

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Example 1.3. *Suppose the ring* $R = \left\{ \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \middle| a, b \in \mathbb{R} \right\}$. *Define maps* $F, h : R \to R$ *as follows:*

$$F\left(\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix},$$
$$h\left(\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}\right) = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}.$$

Then it is easy to verify that F *is a right generalized homoderivation of* R *associated with a homoderivation* h*, which is not a left generalized homoderivation associated with a homoderivation* h*. Furthermore,* F *is not a generalized homoderivation of* R*.*

If $S \subset R$, then a mapping $f : R \to R$ preserves S if $f(S) \subset S$. A mapping $f : R \to R$ is zero-power valued on S if f preserves S and if for each $x \in S$, there exists a positive integer $n(x) > 1$ such that $f^{n(x)}(x) = 0$ (see [4]).

In [3], Daif and Bell proved that if R is a semiprime ring, U is a nonzero ideal of R and d is a derivation of R such that $d([x, y]) = \pm[x, y]$, for all $x, y \in U$, then $U \subset Z$. In [5], N. Rehman et al. has examined the above conditions for the homoderivation. Our first objective in this paper is to prove corresponding results for generalized homoderivations with ideals in semiprime rings.

M. Ashraf et al. [2] proved that a prime ring R must be commutative, if R satisfies any one of the following conditions: (i) $f(xy) = xy$, (ii) $f(x)f(y) = xy$, where f is a generalized derivation of R and I is a nonzero two-sided ideal of R. In [1], M. Ashraf and N. Rehman showed that a prime ring R with a nonzero ideal I must be commutative if it admits a derivation d satisfying either of the properties $d(xy) + xy \in Z$ or $d(xy) - xy \in Z$; for all $x, y \in I$. Our second objective of this note is to show the same conditions imposed on generalized homoderivations with ideals in semiprime rings. Precisely we shall prove that R contains a nonzero central ideal if any one of the following holds: $F(uv) + uv \in Z$, $F(uv) - uv \in Z$, for all $u, v \in I$.

Fact: Let R be a semiprime ring, then

i) The center of R contains no nonzero nilpotent elements.

ii) $P \neq R$ is a nonzero prime ideal of R and $a, b \in R$ such that $aRb \subseteq P$, if and only if either $a \in P$ or $b \in P$.

iii) The center of a nonzero one sided ideal is contained in the center of R. In particular, any commutative one sided ideal is contained in the center of R.

Lemma 1.4. *Let* R *be a semiprime ring and* I *a nonzero ideal of* R*. Suppose that* R *admits a generalized homoderivation* F *associated with a nonzero mapping* h *which* $Ih(I) \neq \{0\}$ *such that* $F([u, v]) = 0$ *for all* $u, v \in I$*, then* R *contains a nonzero central ideal.*

Proof. By the hypothesis, we have

$$F([u, v]) = 0 \text{ for all } u, v \in I.$$

Replacing v by vu in this equation, we get

$$F([u, v]\, u) = 0.$$

Since F is a generalized homoderivation, we have

$$F([u, v])\, h(u) + F([u, v])\, u + [u, v]\, h(u) = 0.$$

By the hypothesis, we get

$$[u, v]\, h(u) = 0, \text{ for all } u, v \in I. \tag{1.1}$$

Replacing v by rv, $r \in R$ in the last equation, we obtain that

$$r\, [u, v]\, h(u) + [u, r]\, vh(u) = 0, \text{ for all } u, v \in I, r \in R.$$

Using (1.1) in this equation, we have

$$[u,r]\,vh(u)=0,\ \text{for all } u,v\in I, r\in R. \tag{1.2}$$

That is,

$$[u,R]\,RIh(u)=(0)\ ,\ \text{for all } u\in I.$$

Since R is semiprime, we must contain a family $\wp=\{P_\alpha \mid \alpha\in\Lambda\}$ of prime ideals such that $\cap P_\alpha=\{0\}$. If P is a typical member of $\wp$ and $u\in I$, we have

$$[u,R]\subseteq P \quad \text{or} \quad Ih(u)\subseteq P$$

by fact (ii). Define two additive subgroups

$$A=\{u\in I \mid [u,R]\subseteq P\}\ \text{ and }\ B=\{u\in I\mid Ih(u)\subseteq P\}.$$

It is clear that $I=A\cup B$. Since a group cannot be a union of two of its subgroups, either $A=I$ or $B=I$. That is

$$[I,R]\subseteq P \quad \text{or} \quad Ih(I)\subseteq P.$$

Thus both cases together yield

$$[I,R]Ih(I)\subseteq P,\ \text{for any } P\in\wp.$$

Therefore

$$[I,R]Ih(I)\subseteq \cap P_\alpha=\{0\},$$

and so

$$[I,R]Ih(I)=\{0\}.$$

Hence, we get

$$[RIR,R]Ih(I)=\{0\}.$$

This implies that $[J,R]RJ=\{0\}$, where $J=RIh(I)R$ is a nonzero ideal of R, since $Ih(I)\neq\{0\}$. Then

$$[J,R]R[J,R]=\{0\}.$$

By the semiprimeness of R, we get $[J,R]=\{0\}$, and so $J\subseteq Z$. We conclude that R contains a nonzero central ideal. □

2. Results

Theorem 2.1. *Let R be a 2-torsion free semiprime ring and I a nonzero ideal of R. Suppose that R admits a right generalized homoderivation F associated with a nonzero mapping h which $Ih(I)\neq\{0\}$ such that*
i) $F([u,v])=\pm[u,v]$ for all $u,v\in I$, or
ii) $F([u,v])=(u\circ v)$ for all $u,v\in I$.
Then R contains a nonzero central ideal.

Proof. i) We get

$$F([u,v])=[u,v]\ \text{ for all } u,v\in I.$$

Taking v by vu in this equation, we obtain that

$$F([u,v])h(u)+F([u,v])u+[u,v]\,h(u)=[u,v]\,u.$$

Using the hypothesis, we get

$$[u,v]\,h(u)+[u,v]\,u+[u,v]\,h(u)=[u,v]\,u.$$

That is,

$$[u,v]\,h(u)+[u,v]\,h(u)=0,$$

and so

$$2\,[u,v]\,h(u)=0.$$

Since R is 2-torsion free ring, we have

$$[u,v]\,h(u)=0,\ \text{for all } u,v\in I.$$

Using the same arguments in the proof equation (1.1), we find that R contains a nonzero central ideal. We complete the proof.

ii) We have

$$F\left([u,v]\right)=(u\circ v)\ \text{for all } u,v\in I.$$

Replacing v by vu in the last equation, we obtain that

$$F\left([u,v]\right)h(u)+F([u,v])u+[u,v]\,h(u)=(u\circ v)u.$$

By the hypothesis, we get

$$(u\circ v)h(u)+(u\circ v)u+[u,v]\,h(u)=(u\circ v)u.$$

That is,

$$(u\circ v)h(u)+[u,v]\,h(u)=0. \tag{2.1}$$

Takin v by $rv, r\in R$ in the above equation, we have

$$r(u\circ v)h(u)+[u,r]vh(u)+r\,[u,v]\,h(u)+[u,r]vh(u)=0.$$

Using equation (2.1),we see that

$$2[u,r]vh(u)=0\ \text{for all } u,v\in I, r\in R.$$

Since R is $2-$torsion free ring, we have

$$[u,r]vh(u)=0\ \text{for all } u,v\in I, r\in R.$$

The rest of the proof is the same as equation (1.2). We conclude that R contains a nonzero central ideal. □

We can give the following useful corollaries by the preceding theorem.

Corollary 2.2. *Let R be a 2-torsion free semiprime ring and I a nonzero ideal of R. Suppose that R admits a right generalized homoderivation F associated with a nonzero mapping h such that $Ih(I)\neq(0)$. If one the following assertions holds :*

i) $F(uv)=uv$, for all $u,v\in I$,

ii) $F(uv)=-uv$, for all $u,v\in I$,

iii) For each $u,v\in I$, either $F(uv)=uv$ or $F(uv)=-uv$.

Then R contains a nonzero central ideal.

Proof. i) By the hypothesis, we get $F(uv)=uv$, for all $u,v\in I$. Thus, we have

$$F\left(uv-vu\right)=F\left(uv\right)-F\left(vu\right)=uv-vu.$$

Therefore $F\left([u,v]\right)=[u,v]$, for all $u,v\in I$. By Theorem 1, we arrive at R contains a nonzero central ideal.

ii) Using the same arguments in the proof of (i), we find the required result.

iii) For each $u\in I$, we put $I_u=\{v\in I\mid F(uv)=uv\}$ and $I_u^*=\{v\in I\mid F(uv)=-uv\}$. Then $(I,+)=I_u\cup I_u^*$. But a group cannot be the union of proper subgroups. Hence we get $I=I_u$ or $I=I_u^*$. By the same method in (i) or (ii), we complete the proof. □

The following theorem is the generalization of the above result.

Theorem 2.3. *Let R be a semiprime ring, I a nonzero ideal of R. Suppose that R admits a right generalized homoderivation F which is zero-power valued on I associated with a homoderivation h. If either:*
i) $F(uv)+uv\in Z$, for all $u,v\in I$, or
ii) $F(uv)-uv\in Z$, for all $u,v\in I$.
Then R contains a nonzero central ideal.

Proof. i) By the hypothesis, we get

$$F(uv)+uv\in Z \text{ for all } u,v\in I. \tag{2.2}$$

Since F is zero-power valued on I, then for all $u,v\in I$ there exists an integer $n(uv)>1$ such that $F^{n(uv)}(uv)=0$. Replacing uv by $uv-F(uv)+F^2(uv)+...+(-1)^{n(uv)-1}F^{n(uv)-1}(uv)$ in (2.2) to get immediately $uv\in Z(R)$ for all $u,v\in I$ and so $I^2\subseteq Z(R)$ and from semiprimeness R; we get that $I^2\neq 0$ since $I\neq 0$. Hence R has a nonzero central ideal (which is I^2). This proves our result.
ii) To avoid repetition in the proof, it is enough to use the same techniques used in (i) with small changes, we find the requested result. □

Theorem 2.4. *Let R be a 2-torsion free semiprime ring, I a nonzero ideal of R. Suppose that R admits a generalized homoderivation F which is zero-power valued on I associated with a homoderivation h such that $I^2h(I)\neq(0)$. If either:*
i) $F(u)F(v)=uv$ for all $u,v\in I$ or
ii) $F(u)F(v)=vu$ for all $u,v\in I$.
Then R contains a nonzero central ideal.

Proof. i) By the hypothesis, we get

$$F(u)F(v)=uv \text{ for all } u,v\in I.$$

Replacing v by $vw, w\in I$ in this equation, we have

$$F(u)F(v)h(w)+F(u)F(v)w+F(u)vh(w)=uvw, \text{ for all } u,v,w\in I.$$

Using the hypothesis, we see that

$$uvh(w)+F(u)vh(w)=0, \text{ for all } u,v,w\in I. \tag{2.3}$$

Taking u by $ru, r\in R$ in the above equation, we find that

$$ruvh(w)+h(r)F(u)vh(w)+rF(u)vh(w)+h(r)uvh(w)=0, \text{ for all } u,v,w\in I, r\in R.$$

Using equation (2.3), we get

$$h(r)F(u)vh(w)+h(r)uvh(w)=0, \text{ for all } u,v,w\in I, r\in R.$$

That is,

$$h(r)(F(u)+u)vh(w)=0.$$

Since F is zero-power valued on I, there exists an integer $n(x)>1$ such that $F^{n(x)}(x)=0$ for all $x\in I$. Replacing u by $u-F(u)+F^2(u)+...+(-1)^{n(u)-1}F^{n(u)-1}(u)$ in this equation, we get

$$h(r)uvh(w)=0.$$

Replacing r by w, $w\in I$ in this equation, we get

$$h(w)uvh(w)=0.$$

That is,

$$uvh(w)Ruvh(w) = (0).$$

Since R is semiprime ring, we have

$$uvh(w) = 0, \text{ for all } u, v, w \in I.$$

Replacing u by $[r, w]u$ in the last equation, we have

$$[r, w]uvh(w) = 0, \text{ for all } u, v, w \in I, \ r \in R,$$

and so

$$[R, w]I^2h(w) = 0, \text{ for all } w \in I. \tag{2.4}$$

That is,

$$[R, w]RI^2h(w) = 0, \text{ for all } w \in I.$$

By fact (ii), we have

$$[R, w] \subseteq P \text{ or } I^2h(w) \subseteq P.$$

Define two additive subgroups

$$A = \{w \in I | [R, w] \subseteq P\} \text{ and } B = \{w \in I | I^2h(w) \subseteq P\}.$$

It is clear that $I = A \cup B$. Since a group cannot be a union of two its subgroups, either $A = I$ or $B = I$, and so, we have

$$[R, I] \subseteq P \text{ or } I^2h(I) \subseteq P.$$

Thus both cases together yield

$$[R, I]I^2h(I) \subseteq P, \text{ for any } P \in \wp.$$

Therefore,

$$[R, I]I^2h(I) \subset \cap P_\alpha = (0),$$

and so

$$[R, I]I^2h(I) = (0).$$

Hence, we get

$$[R, RIR]I^2h(I) = (0).$$

This implies that $[R, J]RJ = (0)$ where $J = RI^2h(I)R$ is a nonzero ideal of R since $I^2h(I) \neq (0)$. Then

$$[R, J]R[R, J] = (0).$$

By the semiprimeness of R, we get $[R, J] = (0)$, and so $J \subseteq Z$.

ii) By the hypothesis, we have

$$F(u)F(v) = vu \text{ for all } u, v \in I.$$

Replacing v by $vw, w \in I$ in this equation, we get

$$F(u)F(v)h(w) + F(u)F(v)w + F(u)vh(w) = vwu,$$

Using the hypothesis, we obtain that

$$vuh(w) + F(u)vh(w) = -vuw + vwu.$$

That is,

$$F(u)vh(w) = -vuw + vwu - vuh(w). \tag{2.5}$$

Replacing v by $rv, r \in R$ in this equation, we obtain that

$$F(u)rvh(w) = -rvuw + rvwu - rvuh(w).$$

Using equation (2.5) in this equation, we get

$$F(u)rvh(w) = rF(u)vh(w), \text{for all } u, v, w \in I, r \in R.$$

Replacing r by $F(w)$ in the above equation, we find that

$$F(u)F(w)vh(w) = F(w)F(u)vh(w).$$

Using the hypothesis, we see that

$$wuvh(w) = uwvh(w), \text{ for all } u, v, w \in I,$$

That is,

$$[u, w]vh(w) = 0, \text{ for all } u, v, w \in I.$$

Replacing u by ru in the last equation and using this equation, we get

$$[r, w]uvh(w) = 0, \text{ for all } u, v, w \in I, \ r \in R,$$

and so,

$$[R, w]I^2h(w) = 0, \text{ for all } w \in I.$$

The rest of the proof is the same as equation (2.4). This completes proof. □

Theorem 2.5. *Let R be a 2-torsion free semiprime ring and I a nonzero ideal of R. Suppose that R admits a generalized homoderivation F associated with a nonzero homoderivation h which $Ih(I) \neq (0)$ such that*

i) $F(u \circ v) = 0$ for all $u, v \in I$, or

ii) $F(u \circ v) = (u \circ v)$ for all $u, v \in I$, or

iii) $F(u \circ v) = [u, v]$ for all $u, v \in I$.

Then R contains a nonzero central ideal.

Proof. i) We have

$$F(u \circ v) = 0 \text{ for all } u, v \in I.$$

Relacing v by vu in the above equation, we get

$$F(u \circ v)h(u) + F(u \circ v)u + (u \circ v)h(u) = 0.$$

Using the hypothesis, we get

$$(u \circ v)h(u) = 0 \text{ for all } u, v \in I. \tag{2.6}$$

Taking v by $rv, r \in R$ in the last equation, we get

$$[u, r]vh(u) = 0, \text{ for all } u, v \in I, \ r \in R.$$

Using the same arguments in the proof Theorem 1 (i), we find that R contains a nonzero central ideal.

ii) By the hypothesis, we have

$$F(u \circ v) = (u \circ v) \text{ for all } u, v \in I.$$

Replacing v by vu in this equation, we see that

$$F(u \circ v)h(u) + F(u \circ v)u + (u \circ v)h(u) = (u \circ v)u.$$

Using the hypothesis, we get

$$(u \circ v)h(u) + (u \circ v)h(u) = 0.$$

That is,

$$2(u \circ v)h(u) = 0, \text{for all } u, v \in I.$$

Since R is $2-$torsion free ring, we have

$$(u \circ v)\, h(u) = 0, \text{for all } u, v \in I.$$

This equation is the same as equation (2.6). Using the same arguments in the proof of (i), we find the required result.

iii) We get

$$F\,(u \circ v) = [u, v] \ \text{ for all } u, v \in I.$$

Substituting vu for v in the above equation, we get

$$F\,(u \circ v)\, h(u) + F\,(u \circ v)\, u + (u \circ v)\, h(u) = [u, v]\, u.$$

Using the hypothesis, we get

$$[u, v]\, h(u) + (u \circ v)\, h(u) = 0, \ \text{for all } u, v \in I. \tag{2.7}$$

Replacing v by $rv, r \in R$ in the last equation, we see that

$$r\,[u, v]\, h(u) + [u, r]\, vh(u) + r(u \circ v)h(u) + [u, r]vh(u) = 0.$$

Using equation (2.6), we get

$$2[u, r]vh(u) = 0, \ \text{for all } u, v \in I, \ r \in R.$$

Since R is $2-$torsion free ring, we obtain that

$$[u, r]vh(u) = 0, \ \text{for all } u, v \in I, \ r \in R.$$

The rest of the proof is the same as equation (1.2). This completes proof. □

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