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Results on Various Derivations and Posner's Theorem in Prime Ideals of Rings

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ABSTRACT: Let R be a ring and P be a prime ideal of R. In this work, we study the structure of the quotient ring R/P in a new and more general way by discussing various algebraic identities on appropriate subsets of R involving multiplicative (generalized)-(α,β)-derivations, multiplicative generalized (α,β)-derivations, multiplicative generalized derivations and generalized derivations. In addition, we give examples exhibiting the crucially of the hypothesis of our results.

Key Words: Multiplicative generalized (α,β)-derivation, SCP map, prime ring, commutativity.

Contents

1. Introduction

In all that follows, unless specially stated, R always denotes an associative ring with centre $Z(R)$, extended centroid C, central closure R_C and $Q_r(R)$ is the right Matindale ring of quotients of R. As usual, the symbols $s\circ t$ and $[s,t]$ will denote the anti-commutator $st+ts$ and commutator $st-ts$, respectively. The usual commutator and anti-commutator identities, which will be used extensively in the forthcoming sections, are given as follows:

(i) $[x,yz]=y[x,z]+[x,y]z$.

(ii) $[xy,z]=[x,z]y+x[y,z]$.

(iii) $xy\circ z=(x\circ z)y+x[y,z]=x(y\circ z)-[x,z]y$.

(vi) $x\circ yz=y(x\circ z)+[x,y]z=(x\circ y)z+y[z,x]$.

Recall that an ideal P of R is said to be prime if $P\neq R$ and for $x,y\in R$, $xRy\subseteq P$ implies that $x\in P$ or $y\in P$. A prime ideal P of R is minimal if P does not properly include any prime ideals of R. The ring R is a prime ring if and only if $\{0\}$ is the only minimal prime ideal of R. A ring R is semiprime if $xRx=\{0\}$ implies $x=0$. If α and β are automorphisms of R, then an additive mapping d from R to R is called an (α,β)-derivation of R if $d(xy)=d(x)\beta(y)+\alpha(x)d(y)$ holds for all $x,y\in R$. For example, let $R=\left\{\begin{pmatrix} a & b\\ 0 & c\end{pmatrix} \mid a,b,c\in\mathbb{R}\right\}$, be a ring and

$$d\begin{pmatrix} a & b\\ 0 & c\end{pmatrix}=\begin{pmatrix} 0 & b\\ 0 & 0\end{pmatrix},\ \alpha\begin{pmatrix} a & b\\ 0 & c\end{pmatrix}=\begin{pmatrix} a & a+2b-c\\ 0 & c\end{pmatrix},$$

$$\beta\begin{pmatrix} a & b\\ 0 & c\end{pmatrix}=\begin{pmatrix} a & 2a+b-2c\\ 0 & c\end{pmatrix}.$$

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Then it can be verified that d is an (α, β)-derivation of R, with associated automorphisms α and β of R. For $\alpha = \beta = I_R$, d is an ordinary derivation of R. The notion of (α, β)-derivation has been extended to generalized (α, β)-derivation and many interesting results has been established with generalized (α, β)-derivations such as [2], [25]. Generalized (α, β)-derivation is defined as follows: An additive mapping $F : R \to R$ is said to be a (right) generalized (α, β)-derivation of R if there exists a (α, β)-derivation d of R with associated automorphisms α, β such that $F(xy) = F(x)\beta(y) + \alpha(x)d(y)$ holds for all $x, y \in R$, d is called an associated (α, β)-derivation of F. For $\alpha = \beta = I_R$, F is called a generalized derivation of R. According to the definition of [10], an additive mapping $F : R \to R$ satisfying $F(xy) = F(x)\alpha(y)$ for all $x, y \in R$ is called a multiplicative α-left centralizer of R. For $\alpha = I_R$, F is called a left multiplier (or centralizer) of R.
For a ring R with identity, a multiplicative α-left centralizer is just a map $F(x) = k\alpha(x)$, where $k \in R$ is a fixed element. The same it is with generalized (α, β)-derivation, which is just a map of the form $d(x) + k\beta(x)$, where d is a derivation. By the way, if the ring R does not contain identity, then with the help of a regular representation (left multiplications of right regular modules R_R), one can embed a semiprime ring in a semiprime ring with identity and then extend derivations and automorphisms. A mapping $d : R \to R$ (not necessarily additive) is called a multiplicative (α, β)-derivation of R, if $d(xy) = d(x)\beta(y) + \alpha(x)d(y)$ holds for all $x, y \in R$, where $\alpha, \beta : R \to R$ are automorphisms. A mapping $G : R \to R$ (not necessarily additive) is called a multiplicative right (generalized)-(α, β)-derivation (resp. multiplicative left (generalized)-(α, β)-derivation) of R, if there exists a multiplicative (α, β)-derivation $d : R \to R$ such that $G(xy) = G(x)\beta(y) + \alpha(x)d(y)$ (resp. $G(xy) = \alpha(x)G(y) + d(x)\beta(y)$) holds for all $x, y \in R$, where $\alpha, \beta : R \to R$ are automorphisms. Now G is said to be a multiplicative (generalized)-(α, β)-derivation with associated multiplicative (α, β)-derivation d if it is both a multiplicative left and right (generalized)-(α, β)-derivation. In case d is additive, G is called multiplicative generalized (α, β)-derivation.

2. SCP multiplicative (generalized)-(α, β)-derivations

We say that a map $F : R \to R$ preserves commutativity if $[F(x), F(y)] = 0$ whenever $[x, y] = 0$ for all $x, y \in R$. The study of commutativity preserving mappings has been an active research area in matrix theory, operator theory and ring theory (see [11], [30] for references). According to [4], let S be a subset of R, a map $F : R \to R$ is said to be strong commutativity preserving (SCP) on S if $[F(x), F(y)] = [x, y]$ for all $x, y \in S$. In [5], Bell and Daif investigated the commutativity in rings admitting a derivation which is SCP on a nonzero right ideal. Precisely, they proved that if a semiprime ring R admits a derivation d satisfying $[d(x), d(y)] = [x, y]$ for all x, y in a right ideal I of R, then $I \subseteq Z(R)$. In particular, R is commutative if $I = R$. Later, Deng and Ashraf [12] proved that if there exists a derivation d of a semiprime ring R and a map $F : I \to R$ defined on a nonzero ideal I of R such that $[F(x), d(y)] = [x, y]$ for all $x, y \in I$, then R contains a nonzero central ideal. In particular, they showed that R is commutative if $I = R$. Recently, this result was extended to Lie ideals and symmetric elements of prime rings by Lin and Liu in [17] and [18], respectively, and to the case of generalized derivations by Ma, Xu and Niu in [21]. There is also a growing literature on strong commutativity preserving (SCP) maps and derivations (for reference see [5], [11], [22], etc.) In [3], Ali et al. showed that if R is a semiprime ring and f is an endomorphism which is a strong commutativity preserving (simply, SCP) map on a nonzero ideal U of R, then f is commuting on U. In [28], Samman proved that an epimorphism of a semiprime ring is strong commutativity preserving if and only if it is centralizing. Derivations as well as SCP mappings have been extensively studied by researchers in the context of operator algebras, prime rings and semiprime rings too. Many related generalizations of these results can be found in the literature (see for instance [11], [16], [19], [20], [26]).

In this section, we study the concept of multiplicative (generalized)-(α, β)-derivation, and for more information about multiplicative (generalized)-(α, β)-derivations see [27]. More specifically, we investigate the following identities:
(i) $[G(x), G(y)] = [x, y]$ for all $x, y \in U$;
(ii) $G(x) \circ G(y) = x \circ y$ for all $x, y \in U$;
(iii) $[G_1(x), y] + [x, G_2(y)] \in P$ for all $x, y \in R$;
(iv) $G_1(x) \circ y + x \circ G_2(y) \in P$ for all $x, y \in R$;

involving multiplicative (generalized)-(α,β)-derivations in prime rings. Finally, an example is given to demonstrate that the restrictions imposed on the hypothesis of our result are not superfluous.

Lemma 2.1. *Let R be a semiprime ring and $\alpha,\beta : R \to R$ are automorphisms. Suppose that R admits a multiplicative (generalized)-(α,β)-derivation G associated with a nonzero map d and automorphisms α and β, then $G \neq 0$.*

Proof: Assume on the contrary that $G(x) = 0$ for all $x \in R$, then $0 = G(xy) = G(x)\beta(y) + \alpha(x)d(y)$ for all $x, y \in R$. So by using hypothesis $0 = \alpha(x)d(y)$ for all $x, y \in R$. Left-multiplying the last relation by $d(y)$ and using semiprimenes of R, we obtain $d = 0$ which is a contradiction. □

Note that the following result is a generalization of Lemma 2.3 of [15].

Theorem 2.2. *Let R be a semiprime ring with automorphisms $\alpha,\beta : R \to R$. Suppose that R admits a multiplicative (generalized)-(α,β)-derivation G associated with a map d. Then d is a multiplicative (α,β)-derivation.*

Proof: First let us assume that G is a multiplicative right (generalized)-(α,β)-derivation, for all $x, y, z \in R$

$$\begin{aligned} G((xy)z) &= G(xy)\beta(z) + \alpha(xy)d(z) \\ &= G(x)\beta(yz) + \alpha(x)d(y)\beta(z) + \alpha(xy)d(z). \end{aligned}$$

But $G(x(yz)) = G(x)\beta(yz) + \alpha(x)d(yz)$ for all $x, y, z \in R$. So by subtracting two last equations, we get

$$\alpha(x)(d(yz) - d(y)\beta(z) - \alpha(y)d(z)) = 0 \text{ for all } x, y, z \in R. \tag{2.1}$$

Left-multiplying Eq. (2.1) by $(d(yz) - d(y)\beta(z) - \alpha(y)d(z))$, we obtain

$$(d(yz) - d(y)\beta(z) - \alpha(y)d(z))\alpha(x)(d(yz) - d(y)\beta(z) - \alpha(y)d(z)) = 0 \text{ for all } x, y, z \in R.$$

But α is an automorphism of R, so the last equation means that

$$(d(yz) - d(y)\beta(z) - \alpha(y)d(z))R(d(yz) - d(y)\beta(z) - \alpha(y)d(z)) = \{0\}. \tag{2.2}$$

Using semi-primeness of R, we find that $d(yz) = d(y)\beta(z) + \alpha(y)d(z)$ for all $y, z \in R$. Which shows that d is a multiplicative (α,β)-derivation associated with the automorphisms α and β.

In the same way, we see that the associated mapping of a multiplicative left (generalized) -(α,β)-derivation is a multiplicative (α,β)-derivation. □

Letting $\alpha = \beta = I_R$ in Theorem 2.2, we obtain [15, Lemma 2.3].

Theorem 2.3. *Let R be a prime ring, $\alpha,\beta : R \to R$ be automorphisms and U a nonzero ideal of R. Suppose that R admits a multiplicative right (generalized)-(α,β)-derivation G associated with a multiplicative (α,β)-derivation d. If G is SCP on U and $d(Z(R)) \neq \{0\}$, then R is commutative.*

Proof: Suppose that

$$[G(x), G(y)] = [x, y] \quad \text{for all } x, y \in U. \tag{2.3}$$

Replacing x by xh in (2.3), where $h \in Z(R)$, and using it with definition of G, it follows immediately that

$$[x, y](\beta(h) - h) + [\alpha(x), G(y)]d(h) + \alpha(x)[d(h), G(y)] = 0. \tag{2.4}$$

Letting xh' in place of x in (2.4), where $h' \in Z(R)$ and using it again, we arrive at

$$[x, y](\beta(h) - h)(\alpha(h') - h') = 0 \quad \text{for all } x, y \in U, h, h' \in Z(R). \tag{2.5}$$

It implies that

$$(\beta(h)-h)R[x,y]R(\alpha(h')-h')=\{0\} \quad \text{for all } \ x,y\in U, h,h'\in Z(R). \tag{2.6}$$

By primeness of R, we obtain $[x,y]=0$ for all $x,y\in U$ or $\beta=I_{Z(R)}$ or $\alpha=I_{Z(R)}$. $[x,y]=0$ for all $x,y\in U$ means that I is a commutative ideal, by primeness of R and $U\neq\{0\}$, we get commutativity of R.

Let us assume that $\beta=I_{Z(R)}$. It follows from (2.4) that U satisfies $[\alpha(x),G(y)]d(h)+\alpha(x)[d(h),G(y)]=0$. Taking tx instead of x in the last expression, we get

$$[\alpha(t),G(y)]\alpha(x)d(h)=0 \text{ for all } t,x,y\in U.$$

Since $d(Z(R))\neq\{0\}$, it forces that $[\alpha(t),G(y)]=0$ for all $t,y\in U$. For any $r\in R$, replace t by tr in the last equation in order to obtain $\alpha(U)[R,G(U)]=\{0\}$. It yields that $G(U)\subset Z(R)$. In view of our initial hypothesis, we get $[U,U]=\{0\}$, i.e., U is commutative, and hence R is commutative.

Finally, we consider the case $\alpha=I_{Z(R)}$. Substituting hx for x in (2.3), where $h\in Z(R)$, we get

$$d(h)[\beta(x),G(y)]+[d(h),G(y)]\beta(x)+\alpha(h)[G(x),G(y)]=h[x,y]$$

Using (2.3), we get $d(h)[\beta(x),G(y)]+[d(h),G(y)]\beta(x)=0$ for all $x,y\in U$. Applying a similar technique as given above, we get the conclusion. □

Theorem 2.4. *Let R be a prime ring with $\alpha,\beta:R\to R$ be automorphisms and U a nonzero ideal of R. Suppose that R admits a multiplicative (generalized)-(α,β)-derivation G associated with a multiplicative (α,β)-derivation d. If $G(x)\circ G(y)=x\circ y$ for all $x,y\in R$ and $d(Z(R))\neq\{0\}$, then R is commutative and one of the following holds true:*

(i) *$char(R)=2$,*

(ii) *there exists $\lambda\in C$ such that $G(x)=\lambda x$ for all $x\in R$ with $\lambda^2=1$.*

Proof: To avoid repetition, we rely on proof of Theorem 2.3 with some slight changes, we find that R is commutative. Therefore our initial hypothesis yields that

$$2G(x)G(y)=2xy, \text{ for all } x,y\in R. \tag{2.7}$$

Replacing y by yt in (2.7) to get

$$2G(x)G(yt)=2xyt, \text{ for all } x,y,t\in R. \tag{2.8}$$

Also we have

$$2G(x)G(y)t=2xyt, \text{ for all } x,y,t\in R. \tag{2.9}$$

Comparing (2.8) and (2.9), we obtain $2G(x)(G(yt)-G(y)t)=0$ for all $x,y,t\in R$. Taking ux instead of x in the last relation and using it, we find $2d(u)\beta(x)(G(yt)-G(y)t)=0$ for all $u,x,y,t\in R$. That is $2d(u)R(G(yt)-G(y)t)=\{0\}$ for all $u,y,t\in R$. It implies that either $2d(u)=0$ or $G(yt)=G(y)t$ for all $u,y,t\in R$. Now, we have the following two cases:

Case 1. Let $2d(u)=0$ for all $u\in R$. Replacing u by ur, we get $d(u)(2\beta(r))+\alpha(u)(2d(r))=0$ for all $u,r\in R$. It gives that $d(u)\beta(2r)=0$ for all $u,r\in R$. Taking sr in place of r, we find that $d(u)\beta(s)\beta(2r)=0$ for all $u,r,s\in R$. Since β is an automorphism of R and R is prime, we conclude that either $d=0$ or $2r=0$ for all $r\in R$. But $0\neq d$, thus R is of characteristic 2.

Case 2. Now, we assume that

$$G(yt)=G(y)t, \text{ for all } y,t\in R. \tag{2.10}$$

Polarizing (2.10) in t, we get

$$\begin{aligned} G(yt+yx) &= G(y(t+x)) \\ &= G(y)(t+x) \\ &= G(y)t+G(y)x, \end{aligned}$$

which implies that

$$G(yt + yx) \;=\; G(yt) + G(yx), \text{ for all } x, y, t \in R. \tag{2.11}$$

Polarizing (2.10) in y and using 2.11, we get

$$\begin{aligned} G(y+x)t &= G((y+x)t) \\ &= G(yt+xt) \\ &= G(yt) + G(xt) \end{aligned}$$

which gives

$$G(y)t + G(x)t \;=\; G(y+x)t, \text{ for all } x, y, t \in R.$$

Therefore, we find $(G(y+x)-G(y)-G(x))R = \{0\}$ for all $x, y \in R$. It forces that G is additive. Invoking a classical result of Hvala [14, Lemma 2], there exists some $\lambda \in Q_r(R_C)$ such that $G(x) = \lambda x$ for all $x \in R$. Since R is already commutative, so $\lambda \in C$. In this view Eq. (2.7) implies that $\lambda^2 xy = xy$ for all $x, y \in R$. Thus we conclude that $\lambda^2 = 1$. It completes the proof. □

The following example shows that the restrictions imposed in the hypotheses of Theorems 2.3, Theorem 2.4 cannot be omitted.

Example 2.5. *Let us defined R, U and $G, d, \alpha, \beta : R \to R$ as follow:*

$$R = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}, \; U = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid x, y \in \mathbb{R} \right\},$$

$$G\begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & xz \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; d\begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & yz \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\alpha\begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2x & y \\ 0 & 0 & \frac{1}{2}z \\ 0 & 0 & 0 \end{pmatrix}, \text{ and } \beta = I_R.$$

It is easy to see that R is not prime and $d(Z(R)) = \{0\}$, U is a nonzero ideal of R, G is a multiplicative (generalized)-(α, β)-derivation such that

(i) $[G(x), G(y)] = [x, y]$ *for all* $x, y \in U$,

(ii) $G(x) \circ G(y) = x \circ y$ *for all* $x, y \in U$,

but R not commutative.

3. Identities in prime ideal of a ring

It is a common fact of ring theory that the rings in which the only prime ideal is $\{0\}$ is called a prime ring. Now, if P is a prime ideal of an arbitrary ring R, then obviously the prime ideal of the quotient ring R/P is $\{0\}$ only, and hence a prime ring. By keeping this fact in mind, recently, Almahdi et al. [1] gave a generalization of well known Posner's Second theorem, and proved that: Let R be an arbitrary ring, P be a prime ideal of R, and d be a derivation of R. If $[[x, d(x)], y] \in P$ (i.e., $[x, d(x)] \in Z(R/P)$) for all $x, y \in R$, then $d(R) \subseteq P$ or R/P is a commutative integral domain. Moreover, in two consecutive paper [23] and [24], Mamouni et al. proved many results involving derivations and generalized derivations in this direction. More specifically, they established the connection between the structure of the quotient ring R/P and behaviour of derivations and generalized derivations that satisfy certain algebraic identities in prime ideals.

In this section, we first prove some more general results in this line of investigation and then we extend some results of [23] and [24]. In the end of this section, we proved Posner's second theorem with generalized derivation and consequently extend [1, Theorem 2.2].

3.1. Results on multiplicative generalized $(\alpha,\beta)-$derivations

Theorem 3.1. *Let R be a ring, P be a prime ideal of R, and G be a multiplicative left generalized (α,β)-derivation of R associated with (α,β)-derivation d, where α,β are the automorphisms of R. If $[\alpha(x),G(y)]\in P$ for all $x,y\in R$, then $d(R)\subseteq P$ or R/P is a commutative integral domain.*

Proof: Let us consider

$$[\alpha(x),G(y)]\in P \quad \text{for all } x,y\in R. \tag{3.1}$$

Replacing y by xy in (3.1) and using it to find

$$[\alpha(x),d(x)]\beta(y)+d(x)[\alpha(x),\beta(y)]\in P \text{ for all } x,y\in R. \tag{3.2}$$

Changing y by yt in (3.2) and using it, we obtain $d(x)\beta(y)[\alpha(x),\beta(t)]\in P$ for all $x,y,t\in R$. Since β is an automorphism of R, it may be rewritten as

$$d(x)R[\alpha(x),\beta(R)]\subseteq P \text{ for all } x\in R.$$

It implies that for each $x\in R$ either $d(x)\in P$ or $[\alpha(x),\beta(R)]\subseteq P$. Therefore, R is a set-theoretic union of the additive subgroups $A=\{x\in R:[\alpha(x),\beta(R)]\subseteq P\}$ and $B=\{x\in R:d(x)\in P\}$. Using a well-known fact (*Brauer's trick*) that a group cannot be written as union of two of its proper subgroups, we are forced to conclude that either $R=A$ or $R=B$. That means, either $[\alpha(R),\beta(R)]\subseteq P$ implying $[R,R]\subseteq P$ or $d(R)\subseteq P$. Consequently, we have either R/P is a commutative integral domain or $d(R)\subseteq P$. Thus the proof is completed. □

Corollary 3.2. *Let R be a prime ring and G be a multiplicative left generalized (α,β)-derivation of R associated with (α,β)-derivation d, where α,β are the automorphisms of R. If $[\alpha(x),G(y)]=0$ for all $x,y\in R$, then R is commutative.*

Proof: Let us consider $[\alpha(x),G(y)]=0$ for all $x,y\in R$. Since R is a prime ring, there exits a family $\mathcal{P}$ of prime ideal of R such that $\cap_{P\in\mathcal{P}}P=\{0\}$. In view of Theorem 3.1 it follows that $[R,R]\subseteq P$ or $d(R)\subseteq P$ for all $P\in\mathcal{P}$. Therefore, we conclude that either $[R,R]=\{0\}$ or $d(R)=\{0\}$, but our assumption $0\neq d$ forces that R is commutative. □

Theorem 3.3. *Let R be a ring with $char(R)\neq 2$, P be a prime ideal of R, and G be a multiplicative left generalized (α,β)-derivation of R associated with (α,β)-derivation d, where α,β are the automorphisms of R. If $\alpha(x)\circ G(y)\in P$ for all $x\in R$, then $G(R)\subseteq P$.*

Proof: Let us consider $\alpha(x)\circ G(y)\in P$ for all $x,y\in R$. Replacing x by rx, where $r\in R$, we find $(\alpha(r)\circ G(y))\alpha(x)+[\alpha(r),G(y)]\alpha(x)\in P$. In view of the given condition, it reduces to $[\alpha(r),G(y)]\alpha(x)\in P$ for all $x,y,r\in R$. It implies that either $\alpha(r)\subseteq P$ for all $r\in R$ or $[\alpha(x),G(y)]\in P$ for all $x,y\in R$. Comparing the last relation with our initial hypothesis, we find $2\alpha(x)G(y)\in P$ for all $x,y\in R$. It forces $G(R)\subseteq P$. □

The following corollary is an immediate consequence of Theorem 3.3.

Corollary 3.4. *Let R be a prime ring and G be a multiplicative left generalized (α,β)-derivation of R associated with (α,β)-derivation d where α,β are the automorphisms of R. If $\alpha(x)\circ G(y)=0$ for all $x,y\in R$, then one of the following holds true:*

(i) *R is commutative of characteristic 2,*

(ii) $G=0$.

Theorem 3.5. *Let R be a ring, P be a prime ideal of R and where α_2,β_2 are the automorphisms of R. If G_2 is a multiplicative generalized (α_2,β_2)-derivation of R associated with (α_2,β_2)-derivation d_2 and G_1 is any mapping of R such that $[G_1(x),y]+[x,G_2(y)]\in P$ for all $x,y\in R$, then R/P is a commutative integral domain provided $d_2(Z(R))\nsubseteq P$.*

Proof: Let us consider

$$[G_1(x), y] + [x, G_2(y)] \in P \quad \text{for all} \;\; x, y \in R. \tag{3.3}$$

Taking yz in place of y in (3.3), where $z \in Z(R)$, we get

$$[G_1(x), y]z + [x, G_2(y)]\beta_2(z) + \alpha_2(y)[x, d_2(z)] + [x, \alpha_2(y)]d_2(z) \in P \text{ for all } x, y \in R. \tag{3.4}$$

Right multiplying (3.3) by z, we get

$$[G_1(x), y]z + [x, G_2(y)]z \in P \text{ for all } x, y \in R. \tag{3.5}$$

Combining (3.4) and (3.5) to find

$$[x, G_2(y)](\beta_2(z) - z) + \alpha_2(y)[x, d_2(z)] + [x, \alpha_2(y)]d_2(z) \in P \text{ for all } x, y \in R.$$

For $x = G_2(y)$, we can easily arrive at

$$\alpha_2(y)[G_2(y), d_2(z)] + [G_2(y), \alpha_2(y)]d_2(z) \in P \text{ for all } y \in R.$$

Polarizing this equation in y, we find

$$\begin{aligned} \alpha_2(y)[G_2(u), d_2(z)] + \alpha_2(u)[G_2(y), d_2(z)] + [G_2(y), \alpha_2(u)]d_2(z) \\ +[G_2(u), \alpha_2(y)]d_2(z) \in P \text{ for all } u, y \in R. \end{aligned} \tag{3.6}$$

Replacing y by $yz = zy$ in (3.6), where $z \in Z(R)$, and using it to get

$$\begin{aligned} \alpha_2(z)\alpha_2(y)[G_2(u), d_2(z)] + \alpha_2(z)\alpha_2(u)[G_2(y), d_2(z)] + \alpha_2(u)[d_2(z)\beta_2(y), d_2(z)] \\ +\alpha_2(z)[G_2(y), \alpha_2(u)]d_2(z) + [d_2(z)\beta_2(y), \alpha_2(u)]d_2(z) + \alpha_2(z)[G_2(u), \alpha_2(y)]d_2(z) \in P \end{aligned}$$

Combining (3.6) with the last relation, we find

$$\alpha_2(u)[d_2(z)\beta_2(y), d_2(z)] + [d_2(z)\beta_2(y), \alpha_2(u)]d_2(z) \in P \text{ for all } u, y \in R. \tag{3.7}$$

Taking tu in place of u in (3.7) and using it, we obtain

$$[d_2(z)\beta_2(y), \alpha_2(t)]\alpha_2(u)d_2(z) \in P \text{ for all } t, u, y \in R.$$

It implies that

$$[d_2(z)\beta_2(y), \alpha_2(t)]\alpha_2(u)[d_2(z)\beta_2(y), \alpha_2(t)] \in P.$$

In other words, we have

$$[d_2(z)\beta_2(y), \alpha_2(t)]R[d_2(z)\beta_2(y), \alpha_2(t)] \subseteq P.$$

Since P is a prime ideal, it forces that $[d_2(z)\beta_2(y), \alpha_2(t)] \in P$ for all $y, t \in R$. Replacing y by yr, where $r \in R$, we get $d_2(z)\beta_2(y)[\beta_2(r), \alpha_2(t)] \in P$. Since α_2 and β_2 are automorphisms of R, it implies that $d_2(Z(R))R[R, R] \subseteq P$. In view of our assumption, it follows that $[R, R] \subseteq P$. Hence R/P is a commutative integral domain. □

3.2. Results on multiplicative generalized derivations

Lemma 3.6. *Let R be a ring and P be a prime ideal of R. If $char(R/P) \neq 2$ and for some fixed $a, b \in R$, $axb + bxa \in P$ for all $x \in R$, then either $axb \in P$ or $bxa \in P$ for all $x \in R$.*

Proof: Our assumption gives that

$$\overline{axb} + \overline{bxa} = \overline{0}, \text{ for all } \overline{x} \in R/P. \tag{3.8}$$

Now by applying the same computations of Bell [8, Lemma 3.1], we find that either $\overline{axb} = \overline{0}$ or $\overline{bxa} = \overline{0}$. Hence, we have $axb \in P$ and $bxa \in P$ for all $x \in R$. □

The following theorem extends Theorem 1 of Mamouni et al. [23].

Theorem 3.7. *Let R be a ring, P be a prime ideal of R, and G be a multiplicative right generalized derivation with associated derivation d such that $[G(x), y] \in P$ for all $x, y \in R$, then R/P is commutative or $G(R) \subseteq P$.*

Proof: Let us consider that

$$[G(x), y] \in P, \text{ for all } x, y \in R. \tag{3.9}$$

Taking xu in place of x in (3.9) and utilizing it, we get

$$G(x)[u, y] + [x, y]d(u) + x[d(u), y] \in P, \text{ for all } x, y, u \in R. \tag{3.10}$$

Putting $u = y$ in the above expression, we get $[x, u]d(u) + x[d(u), u] \in P$ for all $x, u \in R$. Replacing x by rx in the last relation and using it, we get

$$[r, u]xd(u) \in P, \text{ for all } x, u, r \in R.$$

It forces that for each $u \in R$, either $d(u) \in P$ or $[R, u] \subseteq P$. Applying Brauer's trick we find that either $d(R) \subseteq P$ or $[R, R] \subseteq P$. In the second case, we are done. We assume that $d(R) \subseteq P$. In this view, it follows from (3.10) that $G(R)[R, R] \subseteq P$. It assures that $G(R) \subseteq P$. It completes the proof. □

In case R is a prime ring, then we know that $P = \{0\}$. With these settings, the above theorem gives a commutativity criterion, which is a refinement of [23, Corollary 1] and stated as follows:

Corollary 3.8. *Let R be a prime ring and G be a multiplicative right generalized derivation associated with a nonzero derivation d. Then the following assertions are equivalent:*

(i) $G(R) \subseteq Z(R)$,

(ii) *R is commutative.*

The following result is a complete generalization of [24, Theorem 1] to the class of generalized derivations.

Theorem 3.9. *Let R be a ring, P be a prime ideal of R, and G_1, G_2 are multiplicative generalized derivations with associated derivations d_1, d_2 respectively. If $[G_1(x), G_2(y)] \in P$ for all $x, y \in R$, then one of the following assertion holds true:*

(i) *char*$(R/P) = 2$,

(ii) *R/P is commutative integral domain,*

(iii) $G_1(R) \subseteq P$,

(iv) $G_2(R) \subseteq P$.

Proof: Let us consider

$$[G_1(x), G_2(y)] \in P, \text{ for all } x, y \in R. \tag{3.11}$$

Replacing y by yr in (3.11) and using it, we find

$$G_2(y)[G_1(x), r] + y[G_1(x), d_2(r)] + [G_1(x), y]d_2(r) \in P, \text{ for all } x, y, r \in R. \tag{3.12}$$

Replacing y by ry in (3.12) and using it to get

$$d_2(r)y[G_1(x), r] + [G_1(x), r]yd_2(r) \in P, \text{ for all } x, y, r \in R. \tag{3.13}$$

Now if char$(R/P) = 2$, then we are done. Therefore now onwards we suppose that char$(R/P) \neq 2$. From Lemma 3.6, we obtain $d_2(r)y[G_1(x), r] \subseteq P$ or $[G_1(x), r]yd_2(r) \in P$ for all $x, y, r \in R$. Primeness of P yields that for each $r \in R$, either $d_2(r) \subseteq P$ or $[G_1(R), r] \subseteq P$. An application of Brauer's trick implies that either $d_2(r) \in P$ for all $r \in R$ or $[G_1(x), t] \in P$ for all $x, r, t \in R$.

First, we consider $d_2(r) \in P$ for all $r \in R$. Then from (3.12), we get $G_2(y)[G_1(x), r] \in P$ for all $x, y, r \in R$. Replacing y by ys, we get $G_2(R)R[G_1(R), R] \subseteq P$. In view of primeness of P and our assumption, it follows that $G_2(R) \subseteq P$.

On the other hand, we have $[G_1(x), r] \in P$ for all $x, r \in R$. In light of Theorem 3.7, we get the conclusion. □

Consequently, we have obtained the following generalization of a classical theorem of Herstein [13, Theorem 2].

Corollary 3.10. *Let R be a $2-$torsion free prime ring. If R admits generalized derivations G_1 and G_2 associated with nonzero derivations d_1 and d_2 respectively such that $[G_1(x), G_2(y)] = 0$ for all $x, y \in R$, then R is a commutative integral domain.*

Theorem 3.11. *Let R be a ring, P be a prime ideal of R, and G_1, G_2 are multiplicative generalized derivations with associated derivations d_1, d_2 respectively. If $G_1(x) \circ G_2(y) \in P$ for all $x, y \in R$, then one of the following assertion holds true:*

(i) *char*$(R/P) = 2$,

(ii) $G_1(R) \subseteq P$,

(iii) $G_2(R) \subseteq P$.

Proof: By hypothesis, we have

$$G_1(x) \circ G_2(y) \in P, \text{ for all } x, y \in R. \tag{3.14}$$

Substituting yr for y in (3.14) and utilizing it, we get

$$-G_2(y)[G_1(x), r] + y(G_1(x) \circ d_2(r)) + [G_1(x), y]d_2(r) \in P, \text{ for all } x, y \in R. \tag{3.15}$$

Replacing y by ry in (3.15) and using it to find

$$-d_2(r)y[G_1(x), r] + [G_1(x), r]yd_2(r) \in P, \text{ for all } x, y, r \in R. \tag{3.16}$$

It implies that R satisfies

$$[[G_1(x), r], yd_2(r)] \in P. \tag{3.17}$$

Substituting wy in place of y in (3.17) and using it to find $[[G_1(x), r], w]Rd_2(r) \subseteq P$ for all $x, w, r \in R$. In view of primeness of P, it follows that for each $r \in R$, either $[[G_1(R), r], R] \subseteq P$ or $d_2(r) \in P$. Recalling Brauer's trick, we obtain that either $[[G_1(R), R], R] \subseteq P$ or $d_2(R) \subseteq P$. In the former case, we have

$$[[G_1(u), v], t] \in P, \text{ for all } u, v, t \in R.$$

Replacing v by tv in the above expression, we arrive at $[G_1(u), t][v, t] \in P$ for all $u, v, t \in R$. A simple calculation gives that $[G_1(u), t]R[G_1(u), t] \subseteq P$ for all $u, t \in R$. It implies that $[G_1(R), R] \subseteq P$. In the light of Theorem 3.7, we get either R/P is a commutative integral domain or $G_1(R) \subseteq P$. Let us assume that R/P is commutative. By our initial hypothesis, we find that $2G_1(x)G_2(y) \in P$ for all $x, y \in R$. Our assumption on characteristic of R forces that

$$G_1(x)G_2(y) \in P, \text{ for all } x, y \in R. \tag{3.18}$$

Replacing x by rx in (3.18) and using it, we find $d_1(R)RG_2(R) \in P$. It implies that either $d_1(R) \subseteq P$ or $G_2(R) \subseteq P$. In the latter case, we are done. Assume that $d_1(R) \subseteq P$. Replacing x by xr in (3.18), where $r \in R$, we get $G_1(R)RG_2(R) \subseteq P$ for all $x, y \in R$. It forces that $G_1(R) \subseteq P$.

On the other hand if $d_2(R) \subseteq P$, then from (3.15), we find that $G_2(y)[G_1(x), r] \in P$ for all $x, y, r \in R$. It implies that $G_2(R)R[G_1(R), R] \subseteq P$. By primeness of P, we have either $G_2(R) \subseteq P$ or $[G_1(R), R] \subseteq P$. The latter case has already discussed, so we get $G_2(R) \subseteq P$, as required. It completes the proof. □

Corollary 3.12. [24, Theorem 2] Let R be a ring, P be a prime ideal of R, and d_1, d_2 be derivations of R. If $d_1(x) \circ d_2(y) \in P$ for all $x, y \in R$, then one of the following assertion holds true:

(i) $\text{char}(R/P) = 2$,

(ii) $d_1(R) \subseteq P$,

(iii) $d_2(R) \subseteq P$.

Corollary 3.13. *Let R be a 2−torsion free prime ring. If R admits generalized derivations G_1 and G_2 associated with derivations d_1 and d_2 respectively such that $G_1(x) \circ G_2(y) = 0$ for all $x, y \in R$, then either $G_1 = 0$ or $G_2 = 0$.*

The following example shows that the condition "primeness of P" in Theorems 3.9, 3.11 cannot be omitted.

Example 3.14. *Let S be any ring, and R and P are defined as follows:* $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & a & c \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c \in S \right\}$, $P = \left\{ \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix} \mid a \in S \right\}$. *One can easily check that P is not a prime ideal of R. Now we define $G_1, G_2 : R \to R$ as* $G_1 \begin{pmatrix} 0 & a & b \\ 0 & a & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & b^2 - c^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $G_2 \begin{pmatrix} 0 & a & b \\ 0 & a & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & bc \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. *Then it is verified that G_1, G_2 are multiplicative generalized derivations of R associated with $d_1 = d_2 = 0$ and satisfying the following identities:*

a) *$[G_1(x), G_2(y)] \in P$ for all $x, y \in R$,*

b) *$G_1(x) \circ G_2(y) \in P$ for all $x, y \in R$.*

But none of the following assertions hold:

(i) *$char(R) = 2$,*

(ii) *R/P is a commutative integral domain,*

(iii) *$G_1(R) \subseteq P$,*

(iv) *$G_2(R) \subseteq P$.*

It shows that the assumption of "primeness of P" in Theorem 3.9 and Theorem 3.11 is not superfluous.

3.3. A more general version of Posner's second theorem

Theorem 3.15. *Let R be a ring and P a prime ideal of R. If $\underline{char(R/P)} \neq 2$ and R admits a right generalized derivation F with associated a derivation d such that $\overline{[F(x), x]} \in Z(R/P)$ for all $x \in R$, then either R/P is a commutative integral domain or there exists $\lambda \in C$ such that $\overline{F}(\overline{x}) = \overline{\lambda}\overline{x}$.*

Proof: Let us consider

$$\overline{[F(x), x]} \in Z(R/P), \text{ for all } x \in R. \tag{3.19}$$

Polarizing (3.19), we get

$$\overline{[F(x), y]} + \overline{[F(y), x]} \in Z(R/P), \text{ for all } x, y \in R. \tag{3.20}$$

In particular, putting $y = x^2$ in (3.20), we find that

$$\overline{[F(x), x^2]} + \overline{[F(x^2), x]} \in Z(R/P), \text{ for all } x \in R.$$

In view of (3.19), it implies that

$$3\overline{x}\overline{[F(x),x]}+\overline{x}\overline{[d(x),x]}\in Z(R/P),\text{ for all } x\in R.$$

Commuting with $\overline{x}$ to find

$$[3\overline{x}\overline{[F(x),x]},\overline{x}]+[\overline{x}\overline{[d(x),x]},\overline{x}]=\overline{0}.$$

Simplifying it, we conclude that

$$\overline{x}\overline{[d(x),x]}\overline{x}=\overline{x}^2\overline{[d(x),x]},\text{ for all } x\in R. \tag{3.21}$$

Again replacing x by x^2 in (3.19), we get

$$\begin{aligned}\overline{[F(x^2),x^2]} &= \overline{[F(x)x+xd(x),x]}\overline{x}+\overline{x}\overline{[F(x)x+xd(x),x]}\\ &= \overline{[F(x),x]}\overline{x}^2+\overline{x}\overline{[d(x),x]}\overline{x}+\overline{x}\overline{[F(x),x]}\overline{x}+\overline{x}^2\overline{[d(x),x]}\\ &= 2\overline{[F(x),x]}\overline{x}^2+\overline{x}\overline{[d(x),x]}\overline{x}+\overline{x}^2\overline{[d(x),x]}\in Z(R/P).\end{aligned}$$

In view of (3.21), it follows that

$$2\overline{[F(x),x]}\overline{x}^2+2\overline{x}^2\overline{[d(x),x]}=\overline{[F(x^2),x^2]}\in Z(R/P). \tag{3.22}$$

Once again we go back to our initial hypothesis (3.20) and substitute xy for y and utilize it in order to obtain

$$2\overline{[F(x),x]}\overline{y}+\overline{x}\overline{[F(x),y]}+\overline{F(x)}\overline{[y,x]}+\overline{x}\overline{[d(y),x]}\in Z(R/P),\text{ for all } x,y\in R.$$

In particular, taking $\overline{y}=\overline{x}^2$ in the above expression and solving, we get

$$4\overline{[F(x),x]}\overline{x}^2+\overline{x}\overline{[d(x),x]}\overline{x}+\overline{x}^2\overline{[d(x),x]}\in Z(R/P).$$

Using (3.21), we obtain

$$4\overline{[F(x),x]}\overline{x}^2+2\overline{x}^2\overline{[d(x),x]}\in Z(R/P).$$

With the aid of (3.22), we conclude that $2\overline{[F(x),x]}\overline{x}^2\in Z(R/P)$ for all $x\in R$. Since $\text{char}(R/P)\neq 2$, it gives $\overline{[F(x),x]}\overline{x}^2\in Z(R/P)$ for all $x\in R$. Commuting this expression with $\overline{F(x)}$, we find that

$$\begin{aligned}\overline{0} &= \overline{[[F(x),x]\overline{x}^2,\overline{F(x)}]}\\ &= \overline{[F(x),x^2][F(x),x]}\\ &= 2\overline{x}\overline{[F(x),x]}^2,\end{aligned}$$

which implies that

$$\overline{0} = \overline{x}\overline{[F(x),x]}^2. \tag{3.23}$$

Left multiplying (3.23) with $\overline{F(x)}$, we get

$$\overline{F(x)}\overline{x}\overline{[F(x),x]}^2=\overline{0}. \tag{3.24}$$

Right multiplying (3.23) with $\overline{F(x)}$ and using (3.19), we obtain

$$\overline{x}\overline{F(x)[F(x),x]}^2=\overline{0}. \tag{3.25}$$

Subtracting (3.25) form (3.24) to conclude that $\overline{[F(x),x]}^3=\overline{0}$ for all $x\in R$. Since R/P is a prime ring and center of a prime ring contains no nilpotent elements, thus we conclude that $\overline{[F(x),x]}=\overline{0}$ for all $x\in R$. For convention purpose, we write it as

$$[F(x),x]\equiv 0\ (modP),\text{ for all } x\in R.$$

It gives that

$$[F(x), y] + [F(y), x] \equiv 0 \ (modP), \ \text{for all } x, y \in R. \tag{3.26}$$

Replacing y by yx in the above relation in order to obtain

$$[y, x]d(x) + y[d(x), x] \equiv 0 \ (modP), \ \text{for all } x, y \in R. \tag{3.27}$$

Replacing y by ty in (3.27), we get $[t, x]yd(x) \equiv 0 \ (modP)$ for all $x, y, t \in R$. It forces that for each $x \in R$, either $[t, x] \equiv 0 \ (modP)$ or $d(x) \equiv 0 \ (modP)$. Invoking Brauer's trick, we find that either $[R, R] \equiv \{0\} \ (modP)$ or $d(R) \equiv \{0\} \ (modP)$. In other words, we have either R/P commutative or $d(R) \subseteq P$.

Now, let us assume that $d(x) \equiv 0 \ (modP)$ for all $x \in R$. In the light of this fact, replace y by yt in (3.26), we find

$$y[F(x), t] + F(y)[t, x] \equiv 0 \ (modP), \ \text{for all } x, y, t \in R.$$

Replacing y by sy in the above relation, it implies

$$(F(s)y - sF(y))[t, x] \equiv 0 \ (modP), \ \text{for all } x, y, t, s \in R.$$

Now, replacing s by su and x by xr in order to find

$$(F(s)uI_d(y) - I_d(s)uF(y))x[t, r] \equiv 0 \ (modP), \ \text{for all } x, y, t, s, u, r \in R.$$

It forces that $F(s)uI_d(y) - I_d(s)uF(y) \equiv 0 \ (modP)$ for all $x, y, t, s, u, r \in R$. It can also be written as $\overline{F}(\overline{s})\overline{u}\overline{I_d}(\overline{y}) = \overline{I_d}(\overline{s})\overline{u}\overline{F}((y))$ for all $\overline{s}, \overline{u}, \overline{y} \in R/P$, where R/P is a prime ring. In view of a result of Brešar [9, Lemma], there exists $\lambda \in C$ such that $\overline{F}(\overline{x}) = \overline{\lambda}\overline{x}$ for all $x \in R$. It completes the proof. □

Corollary 3.16. [1, Theorem 2.2] Let R be a ring and P a prime ideal of R. If $\text{char}(R/P) \neq 2$ and R admits a derivation d such that $\overline{[d(x), x]} \in Z(R/P)$ for all $x \in R$, then either R/P is a commutative integral domain or $d(R) \subseteq P$.

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