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Amenable Quasi-lattice Ordered Groups and True Representations

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ABSTRACT: Let (G, P) be a quasi-lattice ordered group. In previous work, the author constructed a universal covariant representation (A, U) for (G, P) in a way that avoids some of the intricacies of the other approaches in [11] and [8]. Then showed if (G, P) is amenable, true representations of (G, P) generate C^*-algebras which are canonically isomorphic to the C^*-algebra generated by the universal covariant representation. In this paper, we discuss characterizations of amenability in a comparatively simple and natural way to introduce this formidable property.

Key Words: Quasi-lattice ordered groups, C^*-algebra, true representation, amenable groups.

Contents

1. Introduction

Nica introduced a class of groups termed quasi-lattice ordered groups. To each quasi-lattice ordered group (G, P) there corresponds representations of P by isometries called covariant representations. There is also a unique covariant representation with the universal property. Nica used this universal object to define amenability, which is an interesting property of some quasi-lattice ordered groups. The term 'amenability' is already used in group representation theory, but an amenable quasi-lattice ordered group (G, P) is not necessarily amenable in the usual sense. However, a quasi-lattice ordered group (G, P) is necessarily amenable in Nica's sense if G is amenable in the usual sense. In this paper we follow Nica's sense.

Recall that, for a quasi-lattice ordered group (G, P), a representation of (G, P) by isometries is a pair (A, V) consisting of a unital C^*-algebra A and a map V from P to A that satisfies the following three conditions:

(i) $V_e = 1_A$;

(ii) $V_p^* V_p = 1_A$ for all $p \in P$;

(iii) $V_p V_q = V_{pq}$ for all $p, q \in P$.

If in addition V satisfies

$$V_p V_p^* V_q V_q^* = \begin{cases} V_{p\vee q} V_{p\vee q}^*, & \text{if } p, q \text{ have a common upper bound in } P; \\ 0, & \text{otherwise.} \end{cases}$$

then V is a *covariant isometric representation*.

To set up our notions, we denote the C^*-algebra generated by the set $\{V_p : p \in P\}$ by $C^*(V)$. We also write $A_V = \{V_p V_p^* : p \in P\}$ and $B_V = \{V_p V_q^* : p, q \in P\}$. Furthermore, recall from [2] that a covariant representation (A, V) of a quasi-lattice ordered group (G, P) is called a true representation if $\prod_{p\in F}(1 - V_p V_p^*) \neq 0$ for all finite subsets F of $P\backslash\{e\}$.

In [2, Proposition 2.4] we showed that a true representation (A, V) of a quasi-lattice ordered group (G, P) has four properties and we mention here

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1. The set B_V is linearly independent with $\mathrm{span}(B_V)$ dense in $C^*(V)$.

2. There is a continuous linear map Φ_V of $C^*(V)$ onto $\overline{\mathrm{span}}(A_V)$ such that

$$\Phi_V(V_pV_q^*) = \begin{cases} V_pV_p^*, & \text{if } p = q \\ 0, & \text{otherwise.} \end{cases}$$

Then in [2, Theorem 3.2] we showed, every quasi-lattice ordered group (G, P) has a universal covariant representation (A, U). So, $(C^*(U), U)$ will be referred to as the universal covariant representation and $C^*(U)$ will be given the symbol $C^*(G, P)$. In fact, [2, Theorem 3.2] tells that the universal covariant representation (A, U) in true. Moreover, for a covariant representation (A, V) of the lattice ordered group (G, P), there is a $*$-homomorphism $\phi : C^*(G, P) \to C^*(V)$ such that $\phi(U_p) = V_p$. Later on we showed in [2, Theorem 3.6] that the C^*-algebra generated by a true representation of an amenable quasi-lattice ordered group (G, P) is canonically isomorphic to the C^*-algebra generated by the universal covariant representation.

We would like to point out that amenability of the quasi-lattice ordered group (G, P) is in some sense a topological restriction on the $\mathrm{span}(B_V)$. In this paper, we discuss characterizations of amenability in a comparatively simple way, which can be established by investigating the behavior of Φ_U on the range of a positive, faithful, linear map rather than the whole algebra $C^*(G, P)$.

In section 2, we give the important background material about quasi-lattice ordered groups and covariant representations. We also refer to important theorems from [2]. In section 3, we develop some techniques for determining whether a quasi-lattice ordered group is amenable and show our main theorem. In section 4, we discuss amenability of a quasi-lattice ordered group and show that the amenability of a given quasi-lattice ordered group (G, P) can be established by investigating the behavior of Φ_U on the range of a positive, faithful, linear map rather than the whole algebra. The idea of using an action to reduce the amenability question in this way is drawn from [8, Remark 3.6, Proposition 4.2]. In our paper, care has been taken to minimize the amount of vector valued integration theory used.

2. Preliminaries

Let P be a subsemigroup of a group G with identity e such that $P \bigcap P^{-1} = \{e\}$. There is a relation '$\leq$' on G with respect to P where $x \leq y$ if $x^{-1}y \in P$. This relation is a partial order on G which is left invariant in the sense that $x \leq y$ implies $zx \leq zy$ for any $x, y, z \in G$. This partial order is known as the natural partial order determined by P.

Convention 2.1. *We now use (G, P) to refer to the group G with the natural partial order $\leq$ on G determined by P.*

Definition 2.2. *The partially ordered group (G, P) is* quasi-lattice ordered *if every finite subset of G with an upper bound in P has a least upper bound in P [3, Section 2].*

Equivalently, (G, P) is quasi-lattice ordered if and only if every element of G with an upper bound in P has a least upper bound in P, and every two elements in P with a common upper bound in P have a least upper bound in P [11, Section 2.1].

Notation 2.3. *The least upper bound or* sup *of the elements x and y will be denoted by $x \vee y$.*

The following property about quasi-lattice ordered groups can be found in [3].

Lemma 2.4. *Let (G, P) be a quasi-lattice ordered group. If $x, y \in G$ have a common upper bound in P and $z \in G$ satisfies $z(x \vee y) \in P$ then zx and zy have a common upper bound in P. If, in addition, $z \leq zx \vee zy$, then $zx \vee zy = z(x \vee y)$.*

Definition 2.5. *Let (G, P) be a quasi-lattice ordered group. A representation of (G, P) by isometries is a pair (A, V) consisting of a unital C^*-algebra A and a map V from P to A that satisfies the following three conditions:*

(i) $V_e = 1_A$;
(ii) $V_p^* V_p = 1_A$ *for all* $p \in P$;
(iii) $V_p V_q = V_{pq}$ *for all* $p, q \in P$.
If in addition V *satisfies*

$$V_p V_p^* V_q V_q^* = \begin{cases} V_{p\vee q} V_{p\vee q}^*, & \text{if } p,q \text{ have a common upper bound in } P; \\ 0, & \text{otherwise.} \end{cases}$$

then V *is a* covariant isometric representation.

Notation 2.6. *The* C^**-algebra generated by the set* $\{V_p : p \in P\}$ *will be denoted by* $C^*(V)$. *We write* $A_V = \{V_p V_p^* : p \in P\}$ *and* $B_V = \{V_p V_q^* : p, q \in P\}$.

Remark 2.7. *A covariant isomeric representation of the quasi-lattice ordered group* (G, P) *may be defined as a pair* (A, V) *consisting of a unital* C^**-algebra* A *and a map* V *from* P *to* A *such that*

1. $V_e = 1_A$;
2. $V_p V_q = V_{pq}$ *for all* $p, q \in P$,
3. $V_p^* V_q = \begin{cases} V_{p^{-1}(p\vee q)} V_{q^{-1}(p\vee q)}^*, & \text{when } p,q \text{ have a common upper bound in } P; \\ 0, & \text{otherwise.} \end{cases}$

To see that the first definition implies the second, notice first that if $p, q \in P$ have no common upper bound in P then the covariance condition gives $V_P V_p^* V_q V^* q = 0$ and hence

$$V_p^* V_q = (V_p^* V_p) V_p^* V_q (V_q^* V_q) = 0$$

However if p, q have a common upper bound in P, then

$$V_p^* V_q = (V_p^* V_p) V_p^* V_q (V_q^* V_q) = V_p^* V_{p\vee q} V_{p\vee q}^* V_q.$$

But $p \le p \vee q$, so $p^{-1}(p \vee q) \in P$. Therefore,

$$V_{p\vee q} V_{p\vee q}^* = V_p V_{p^{-1}(p\vee q)} V_{q^{-1}(p\vee q)}^* V_q^*$$

thus the result follows. The reverse implication is easily checked.

Definition 2.8. *A covariant representation* (A, V) *of a quasi-lattice ordered group* (G, P) *is called a true representation if* $\prod_{p\in F}(1 - V_p V_p^*) \neq 0$ *for all finite subsets* F *of* $P \backslash \{e\}$.

Remark 2.9. *The name 'true' reflects that* V_p *is a true isometry (that is,* $V_p V_p^* \neq 1$*) for all* $p \in P$.

Recall that:

1. The C^*-algebra generated by A_V is commutative and hence any product in $\mathrm{span}(A_V)$ may be rearranged as necessary.

2. For a quasi-lattice ordered group (G, P) and a finite subset F of P. A subset I of F is an initial segment if and only if whenever $x, y \in F$, $x \le y$ and $y \in I$ imply that $x \in I$.

Definition 2.10. *A universal covariant representation* (A, U) *of the quasi-lattice ordered group* (G, P) *is a covariant representation such that if* (B, V) *is any other covariant representation of* (G, P)*, there is a unique* $*$*-homomorphism* $\phi : C^*(U) \to C^*(V)$ *such that* $\phi(U_p) = V_p$ *for all* $p \in P$.

Recall the following facts from [2]

- [2, Theorem 3.2] Let (G, P) be a quasi-lattice ordered group. Then there is a universal covariant representation (A, U) of (G, P).

- [2, Theorem 3.6] The C^*-algebra generated by a true representation of an amenable quasi-lattice ordered group (G,P) is canonically isomorphic to the C^*-algebra generated by the universal covariant representation.

- For any covariant representation (B,V) of (G,P), [2, Theorem 3.2] provides a unique $*$-homomorphism $\phi : C^*(G,P) \to C^*(V)$ such that $\phi(U_p) = V_p$. Where $C^*(G,P)$ is the C^*-algebra $C^*(U)$ generated by the universal covariant representation.

Throughout this paper (A,U) denotes the universal covariant representation of the quasi-lattice ordered group (G,P) and Φ_U is the $*$-homomorphism of $C^*(G,P)$ onto $\overline{\text{span}}(A_U)$ given by

$$\Phi_U(U_pU_q^*) = \begin{cases} U_pU_p^*, & \text{if } p = q \\ 0, & \text{otherwise.} \end{cases}$$

in [2, Proposition 2.4].

Definition 2.11. *A quasi-lattice ordered group (G,P) is a amenable if Φ_U is faithful on positive elements, in the sense that if $a \in C^*(G,P)$ then $\Phi_U(a^*a) = 0$ implies $a = 0$.*

3. Characterization of amenability

In this section, we develop some techniques for determining whether a quasi-lattice ordered group is amenable and show our main theorem. We start this section with some technical lemmas, then we show our main theorem in the section.

Throughout this paper (A,U) denotes the universal covariant representation of the quasi-lattice ordered group (G,P) and Φ_U is the $*$-homomorphism of $C^*(G,P)$ onto $\overline{\text{span}}(A_U)$ given by

$$\Phi_U(U_pU_q^*) = \begin{cases} U_pU_p^*, & \text{if } p = q \\ 0, & \text{otherwise.} \end{cases}$$

in [2, Proposition 2.4].

Lemma 3.1. *Let (G,P) be a quasi-lattice ordered group and $(B(\ell_2(P)),T)$ be the Toeplitz representation of (G,P). Then Φ_T is faithful on positive elements.*

Proof. The canonical orthonormal basis for $\ell_2(P)$ consists of the maps $\{\delta_p : p \in P\}$ defined for each $q \in P$ by:

$$\delta_p(q) = \begin{cases} 1 & \text{if } p = q, \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

It follows from the definition of T that

$$\begin{aligned} (T_pT_q^*\delta_r)(s) &= \begin{cases} \delta_r(qp^{-1}s) & \text{if } p^{-1}s \in P, \\ 0 & \text{otherwise.} \end{cases} \\ &= \begin{cases} 1 & \text{if } p^{-1}s \in P \text{ and } qp^{-1}s = r, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Now $p^{-1}s \in P$ and $qp^{-1}s = r$ if and only if $q^{-1}r \in P$ and $s = pq^{-1}r$, and hence

$$T_pT_q^*\delta_r = \begin{cases} \delta_{pq^{-1}r} & \text{if } q^{-1}r \in P, \\ 0 & \text{otherwise.} \end{cases}$$

Now if $a = \sum_{p,q\in F} \alpha_{p,q}T_pT_q^* \in \text{span}(B_T)$, then for all $r \in P$,

$$
\begin{aligned}
(a\delta_r|\delta_r) &= (\sum_{p,q\in F, q\leq r} \alpha_{p,q}\delta_{pq^{-1}r}|\delta_r) \\
&= \sum_{p,q\in F, q\leq r} \alpha_{p,q}(\delta_{pq^{-1}r}|\delta_r).
\end{aligned}
$$

Now, using the orthonormality of $\{\delta_p : p \in P\}$,

$$(\delta_{pq^{-1}r}|\delta_r) \neq 0 \text{ if and only if } p = q.$$

Which implies that

$$
\begin{aligned}
(a\delta_r|\delta_r) &= \sum_{p,q\in F, q\leq r} \alpha_{q,q}(\delta_r|\delta_r) \\
&= (\sum_{p,q\in F, q\leq r} \alpha_{q,q}\delta_r|\delta_r) \\
&= (\Phi_T(a)\delta_r|\delta_r).
\end{aligned}
$$

Hence, by continuity of Φ_T, $(\Phi_T(a)\delta_r|\delta_r) = (a\delta_r|\delta_r)$ for all $a \in C^*(T)$ and $r \in P$. Thus if $\Phi_T(a^*a) = 0$ for some $a \in C^*(T)$, then

$$\|a\delta_r\|^2 = (a^*a\delta_r|\delta_r) = (\Phi_T(a^*a)\delta_r|\delta_r) = 0$$

for all $r \in P$. Hence $a = 0$ as required. □

Lemma 3.2. *Let (A, U) be the universal covariant representation of the quasi-lattice ordered group (G, P), $(B(\ell_2(P)), T)$ the Toeplitz representation of (G, P), $\phi : C^*(G, P) \to C^*(T)$ be the $*$-homomorphism in Equation 3.1 supplied by [2, Theorem 3.2] and $\Phi_U : C^*(G, P) \to \overline{\text{span}}(A_U)$ be the $*$-homomorphism in [2, Proposition 2.4]. Then $\ker(\phi) = \{a \in C^*(G, P) : \Phi_U(a^*a) = 0\}$.*

Proof. Suppose $\phi(a) = 0$. By [2, Remark 2.5], $\|\Phi_U(a^*a)\| = \|\phi \circ \Phi_U(a^*a)\|$ since $\Phi_U(a^*a) \in \overline{\text{span}}(A_U)$. Thus by [2, Lemma 3.5]

$$\|\Phi_U(a^*a)\| = \|\Phi_T \circ \phi(a^*a)\| = 0$$

and hence $\|\Phi_U(a^*a)\| = 0$.

Now suppose $\Phi_U(a^*a) = 0$. Then by [2, Lemma 3.5]

$$\|\Phi_T \circ \phi(a^*a)\| = \|\phi \circ \Phi_U(a^*a)\| = 0.$$

But Φ_T is faithful on positive elements by 3.1, so

$$\phi(a)^*\phi(a) = \phi(a^*a) = 0.$$

Thus

$$\|\phi(a)\|^2 = \|\phi(a)^*\phi(a)\| = 0$$

and the result follows.

□

Now we introduce our main theorem of this section which allows us to determine whether a given quasi-lattice ordered group is amenable.

Theorem 3.3. *Let (G, P) be a quasi-lattice ordered group. Then the following are equivalent*

1. *(G, P) is amenable.*
2. *Any two true representations of (G, P) generate canonically isomorphic C^*-algebras.*

3. *The C^*-algebra generated by the Toeplitz representation of (G, P) is canonically isomorphic to the universal covariant representation.*

The proof requires the following lemma.

Lemma 3.4. *The Toeplitz representation of a quasi-lattice ordered group (G, P) is a true representation.*

Proof. Consider $\delta_e \in \ell^2(P)$, defined by

$$\delta_e(s) = \begin{cases} 1 & \text{if } s = e, \\ 0 & \text{otherwise.} \end{cases}$$

Then for any $p \in P \setminus \{e\}$, $(\mathrm{T}_p^* \delta_e)(s) = \delta_e(ps) = 0$ for all $s \in P$, and hence $(1 - \mathrm{T}_p \mathrm{T}_p^*)\delta_e = \delta_e$. This gives $\big(\prod_{p \in F}(1 - \mathrm{T}_p \mathrm{T}_p^*)\big)\delta_e = \delta_e$ for any $F \subset P$, and the result follows. $\square$

Proof of Theorem 3.3. By [2, Theorem 3.6], 1 implies 2. Since the Toeplitz representation of a quasi-lattice ordered group (G, P) is true by Lemma 3.4, then trivially 2 implies 3. Finally, note that if $\phi : C^*(G, P) \to C^*(T)$ is the $*$-homomorphism supplied by [2, Theorem 3.2] and $\ker(\phi) = \{0\}$, then by Lemma 3.2 the set $\{a \in C^*(G, P) : \Phi_U(a^*a) = 0\} = \ker(\phi) = \{0\}$. Hence (G, P) is amenable, so 3 implies 1. $\square$

4. Faithful representations

The aim of this section is to show that the amenability of a given quasi-lattice ordered group (G, P) can be established by investigating the behavior of Φ_U on the range of a positive, faithful, linear map rather than the whole algebra. We also show that an action of a compact group on $C^*(G, P)$ can be used to construct such a map. The idea of using an action to reduce the amenability question in this way is drawn from [8, Remark 3.6, Proposition 4.2]. In our paper, care has been taken to minimize the amount of vector valued integration theory used.

Definition 4.1. *A $*$-homomorphism $\Phi : A \to A$ on a C^*-algebra A is said to be positive if $\Phi(a^*a)$ is positive for all $a \in A$. The map Φ is said to be faithful if $\Phi(a^*a) = 0$ implies $a = 0$.*

Proposition 4.2. *Let (A, U) be the universal covariant representation of the quasi-lattice ordered group (G, P) and let $\Phi : C^*(G, P) \to C^*(G, P)$ be a positive, faithful, linear map such that $\Phi_U \circ \Phi = \Phi \circ \Phi_U$. Then (G, P) is amenable if and only if Φ_U is faithful on positive elements in the range of Φ.*

Proof. Suppose Φ_U is positive on faithful elements in the range of Φ. Suppose also that $\Phi_U(a^*a)) = 0$ for some $a \in C^*(G, P)$. Then

$$\Phi_U \circ \Phi(a^*a) = \Phi \circ \Phi_U(a^*a) = 0.$$

Now $\Phi(a^*a)$ is positive since Φ is positive. Hence $\Phi(a^*a) = 0$ since Φ_U is faithful on positive elements of $\Phi(C^*(G, P))$. So $a = 0$ by the faithfulness of Φ, and that (G, P) is amenable. The reverse implication is trivial. $\square$

One way to construct a positive, faithful linear map Φ is to integrate over a continuous action. An action of a group K on a C^*-algebra A is a group homomorphism $\alpha : K \to \mathrm{Aut}(A)$ where $\mathrm{Aut}(A)$ is the group of automorphisms on A with composition as a product. An action α will be called continuous if for each $a \in A$, the map defined on K by $\gamma \mapsto \alpha_\gamma(a)$ is continuous.

Theorem 4.3. *Let (G, P) be a quasi-lattice ordered group, K a compact group with a continuous action α on $C^*(G, P)$ and μ be the Haar probability measure on K. Then there is a continuous, positive, faithful, linear map*

$$\Phi_\alpha : C^*(G, P) \to C^*(G, P)$$

such that

$$(\Phi_\alpha(a)\eta \,|\, \xi) = \int_K (\alpha_\gamma(a)\eta \,|\, \xi) d\mu(\gamma)$$

for all $a \in C^*(G, P)$, $\eta, \xi \in H_V$, for some Hilbert space H_V.

The proof of this theorem requires the following proposition.

Proposition 4.4. *Let $f : K \to B(H)$ be a continuous function from a compact group K into the bounded linear operators on a Hilbert space H. Then there is a unique $T \in B(H)$ such that*

$$(T_\eta \,|\, \xi) = \int_K (f(s)\eta \,|\, \xi) d\mu(s)$$

for all $\eta, \xi \in H$, where μ is the Haar probability measure on K. Moreover, $\|T\| \leq \|f\|_\infty$.

Proof. First note that for all $s \in K$ and $\eta, \xi \in H$,

$$|(f(s)\eta \,|\, \xi)| \leq \|f(s)\eta\| \, \|\xi\| \leq \|f(s)\| \, \|\eta\| \, \|\xi\| \leq \|f\|_\infty\| \, \|\eta\| \, \|\xi\|$$

by the Cauchy-Schwartz inequality and boundedness of $f(s)$. Hence, the function $s \mapsto (f(s)\eta \,|\, \xi)$ is integrable.

Define a map $F : H \to \mathbb{C}$ by

$$F(\xi) = \int_K (\xi \,| f(s)\eta) \, d\mu(s).$$

Then one can see that F is linear by the linearity of the integral and the inner product. Moreover,

$$|F(\xi)| = |\int_K (\xi \,| f(s)\eta) \, d\mu(s)| \leq \int_K |(\xi \,| f(s)\eta)| \, d\mu(s) \leq \|f\|_\infty \|\eta\| \, \|\xi\|$$

and hence F is bounded with $\|F\| \leq \|f\|_\infty \|\eta\|$. By the Riesz representation Theorem there is a vector $T_\eta \in H$ such that

$$(\xi \,| T_\eta) = F(\xi) = \int_K (\xi \,| f(s)\eta) \, d\mu(s)$$

and $\|T_\eta\| \leq \|F\| \leq \|f\|_\infty \|\eta\|$.

Now, the map $T : \eta \mapsto T_\eta$ is bounded and linear since given $k \in \mathbb{C}$ and $\eta, \zeta \in H$,

$$\begin{aligned}
(T(\eta + k\zeta) \,|\, \xi) &= \int_K (f(s)(\eta + k\zeta) \,|\xi) \, d\mu(s) \\
&= \int_K (f(s)\eta \,|\, \xi) \, d\mu(s) + k \int_K (f(s)\zeta \,|\, \xi) \, d\mu(s) \\
&= (T_\eta \,|\xi) + k(T_\zeta \,|\, \xi) \\
&= (T_\eta + kT_\zeta \,|\, \xi)
\end{aligned}$$

for all $\xi \in H$. Hence $T(\eta + k\zeta) = T_\eta + kT_\zeta$. Thus we have $T \in B(H)$ and $\|T\| \leq \|f\|_\infty$. $\square$

Proof of Theorem 4.3. Given $a \in C^*(G, P)$, the map $\gamma \mapsto \alpha_\gamma(a)$ is continuous. Proposition 4.4 guarantees the existence of a unique element $\Phi_\alpha(a) \in C^*(G, P)$ such that

$$(\Phi_\alpha(a)\eta \,|\, \xi) = \int_K (\alpha_\gamma(a)\eta \,|\, \xi) \, d\mu(\gamma).$$

The map $\Phi_\alpha : C^*(G, P) \to C^*(G, P)$ is clearly linear, since α_γ is linear for each $\gamma \in K$. Moreover, Proposition 4.4 gives that

$$\|\Phi_\alpha(a)\| \leq \sup_{\gamma \in K} \|\alpha_\gamma(a)\| \leq \|a\|$$

since C^*-homomorphisms are contractive. Hence Φ_α is continuous.

To see that Φ_α is positive, note that for all $a \in C^*(G,P)$ and $\eta \in H_V$ we have

$$\begin{aligned}(\Phi_\alpha(a^*a)\eta \,|\, \eta) &= \int_K (\alpha_\gamma(a^*a)\eta \,|\, \eta)\, d\mu(\gamma) \\ &= \int_K (\alpha_\gamma(a)\eta \,|\, \alpha_\gamma(a)\eta)\, d\mu(\gamma) \\ &= \int_K \|\alpha_\gamma(a)\eta\|^2 \, d\mu(\gamma) \\ &\geq 0.\end{aligned}$$

Note also that if $\Phi_\alpha(a^*a) = 0$, then

$$0 = \int_K \|\alpha_\gamma(a)\eta\|^2 \, d\mu(\gamma).$$

Hence, the map $\gamma \mapsto \|\alpha_\gamma(a)\eta\|^2$ is identically zero since it is continuous and positive valued. In particular, for $\gamma = e$, the identity of K,

$$a\eta = \alpha_\gamma(a) = 0$$

for all $\eta \in H$. Thus $a = 0$ and Φ_α is faithful on positive elements. □

We finish off this section with the following special case of the above techniques.

Proposition 4.5. *Let (G,P) be a quasi-lattice ordered group and $\mathcal{G}$ an abelian group and let $\Phi_U : C^*(G,P) \to \overline{\operatorname{span}}(A_U)$ be the $*$-homomorphism in [2, Proposition 2.4]. Suppose there exists a group homomorphism $\psi : G \to \mathcal{G}$. Then there is a positive, faithful, linear map Φ of $C^*(G,P)$ onto the C^*-subalgebra*

$$K = \overline{\operatorname{span}}\{V_pV_q^* : p,q \in P, \psi(p) = \psi(q)\}$$

such that

$$\Phi(V_pV_q^*) = \begin{cases} V_pV_q^* & \text{if } \psi(p) = \psi(q), \\ 0 & \text{otherwise.}\end{cases}$$

Moreover, $\Phi_U \circ \Phi = \Phi_U = \Phi \circ \Phi_U$.

Proof. I claim

1. There is a continuous action α of the dual group $\widehat{\mathcal{G}}$ on $C^*(G,P)$ such that for each $\gamma \in \widehat{\mathcal{G}}$ and $p \in P$, $\alpha_\gamma(U_p) = \gamma \circ \psi(p)U_p$, and
2. For each $z \in \widehat{\mathcal{G}}$

$$\int_{\widehat{\mathcal{G}}} \gamma(z)\, d\mu(\gamma) = \begin{cases} 1 & \text{if } z = e, \\ 0 & \text{otherwise.}\end{cases}$$

To prove the first claim, note that given $\gamma \in \widehat{\mathcal{G}}$, the map $\gamma U : P \to C^*(G,P)$ defined by $(\gamma U)_p = \gamma \circ \psi(p)U_p$ for all $p \in P$, induces a covariant representation of (G,P). Witness that $(\gamma U_e) = \gamma \circ \psi(e)U_e = 1$, and

$$(\gamma U)_p(\gamma U)_q = \gamma \circ \psi(p)U_p\, \gamma \circ \psi(q)U_q = \gamma \circ \psi(pq)U_{pq} = (\gamma U)_{pq}$$

for all $p,q \in P$. Also, if $p,q \in P$ have a common upper bound in P, then by the covariance condition

$$\begin{aligned}(\gamma U)_p^*(\gamma U)_q &= \overline{\gamma \circ \psi(p)}\, U_p^*\, \gamma \circ \psi(q)\, U_q \\ &= \overline{\gamma \circ \psi(p)}\, \gamma \circ \psi(q)\, U_p^* U_q \\ &= \gamma \circ \psi(p^{-1}(p \vee q))\overline{\gamma \circ \psi(q^{-1}(p \vee q))}\, U_{p^{-1}(p\vee q)}\, U_{q^{-1}(p\vee q)}^* \\ &= (\gamma U)_{p^{-1}(p\vee q)}\, (\gamma U^*)_{q^{-1}(p\vee q)}.\end{aligned}$$

However if p, q have no common upper bound,

$$(\gamma U)_p^*(\gamma U)_q = \overline{\gamma \circ \psi(p)}\, \gamma \circ \psi(q)\, U_p^* U_q = 0$$

by the covariance condition. Thus $(C^*(G,P), \gamma U)$ is a covariant representation of (G,P) as required.

Now, since $(C^*(G,P), U)$ is universal, there is a $*$-homomorphism

$$\alpha_\gamma : C^*(G,P) \to C^*(G,P)$$

such that for each $p \in P$,

$$\alpha_\gamma(U_p) = (\gamma U)_p = \gamma \circ \psi(p) U_p.$$

The map $\alpha : \gamma \mapsto \alpha_\gamma$ is an action since for all $\gamma, \chi \in \widehat{\mathcal{G}}$ and $p \in P$,

$$\alpha_{\gamma\chi}(U_p) = (\gamma\chi)\,(\psi(p))\, U_p = \gamma \circ \psi(p)\, \chi \circ \psi(p)\, U_p = \alpha_\gamma \circ \alpha_\chi(U_p).$$

Thus $\alpha_{\gamma\chi} = \alpha_\gamma \circ \alpha_\chi$ since these maps are $*$-homomorphisms. To see that the map $\gamma \mapsto \alpha_\gamma(a)$ is continuous for each $a \in C^*(G,P)$, fix $\epsilon > 0$ and $a \in C^*(G,P)$. Then there is $a_\epsilon \in \operatorname{span}(B_U)$ such that $\|a - a_\epsilon\| < \epsilon/3$. Consider a sequence $\{\gamma_m\}_{m=1}^\infty$ in $\widehat{\mathcal{G}}$ which converges to some $\gamma \in \widehat{\mathcal{G}}$. Now, since $\widehat{\mathcal{G}}$ has the topology of pointwise convergence, then for all $p, q \in P$:

$$\alpha_{\gamma_m}(U_p\, U_q^*) = \gamma_m \circ \psi(p)\, \overline{\gamma_m \circ \psi(q)}\, U_p\, U_q^* \longrightarrow \gamma \circ \psi(p)\, \overline{\gamma \circ \psi(q)}\, U_p\, U_q^* = \alpha_\gamma(U_p\, U_q^*),$$

and hence by the linearity of these maps there is a positive integer M such that $\|\alpha_{\gamma_m}(a_\epsilon) - \alpha_\gamma(a_\epsilon)\| < \epsilon/3$ whenever $m \geq M$.

Now, since C^*-homomorphisms are norm-reducing, we have

$$\begin{aligned}\|\alpha_{\gamma_m}(a) - \alpha_\gamma(a)\| &\leq \|\alpha_{\gamma_m}(a) - \alpha_{\gamma_m}(a_\epsilon)\| + \|\alpha_{\gamma_m}(a_\epsilon) - \alpha_\gamma(a_\epsilon)\| + \|\alpha_\gamma(a_\epsilon) - \alpha_\gamma(a)\| \\ &\leq 2\|a - a_\epsilon\| + \|\alpha_{\gamma_m}(a_\epsilon) - \alpha_\gamma(a_\epsilon)\| \\ &< \epsilon\end{aligned}$$

whenever $m \geq M$. Thus $\{\alpha_{\gamma_m}(a)\}_{m=1}^\infty$ converges to $\alpha_\gamma(a)$, and hence $\gamma \mapsto \alpha_\gamma(a)$ is continuous.

To prove the second claim, note that if $z = e$ then

$$\int_{\widehat{\mathcal{G}}} \gamma(z)\, d\mu(\gamma) = \int_{\widehat{\mathcal{G}}} d\mu(\gamma) = 1$$

since μ is a probability measure. If $z \neq e$ then there exists $\chi \in \widehat{\mathcal{G}}$ such that $\chi(z) \neq 1$. Hence by the translation invariance of μ,

$$\int_{\widehat{\mathcal{G}}} \gamma(z)\, d\mu(\gamma) = \int_{\widehat{\mathcal{G}}} (\chi\gamma)(z)\, d\mu(\gamma) = \chi(z) \int_{\widehat{\mathcal{G}}} \gamma(z)\, d\mu(\gamma)$$

which implies that $\int_{\widehat{\mathcal{G}}} \gamma(z)\, d\mu(\gamma) = 0$.

Now, suppose $\mathcal{G}$ has the discrete topology. Then the dual group $\widehat{\mathcal{G}}$ is compact in the dual topology, which here is the topology of pointwise convergence. Hence, by the first claim and Theorem 4.3 there is a continuous, positive, faithful, linear map $\Phi : C^*(G,P) \to C^*(G,P)$ such that for all $p, q \in P$ and $\eta, \xi \in H_U$,

$$\begin{aligned}(\Phi(U_p U_q^*)\eta \,|\, \xi) &= \int_{\widehat{\mathcal{G}}} (\alpha_\gamma(U_p U_q^*)\eta \,|\, \xi)\, d\mu(\gamma) \\ &= \int_{\widehat{\mathcal{G}}} (\gamma \circ \psi(p)\, \overline{\gamma \circ \psi(q)}\, U_p U_q^*\, \eta \,|\, \xi)\, d\mu(\gamma) \\ &= \Big(\int_{\widehat{\mathcal{G}}} \gamma \circ \psi(pq^{-1})\, d\mu(\gamma) \Big)\, (U_p U_q^* \eta \,|\, \xi)\end{aligned}$$

where μ is the left invariant Haar probability measure on the compact group $\mathcal{G}$. Then by the second claim

$$(\Phi(U_pU_q^*)\eta \,|\, \xi) = \begin{cases} (U_pU_q^*\eta \,|\, \xi) & \text{if } \psi(pq^{-1}) = e, \\ 0 & \text{otherwise} \end{cases}$$

and thus

$$\Phi(U_pU_q^*) = \begin{cases} U_pU_q^* & \text{if } \psi(p) = \psi(q), \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\Phi(C^*(G,P)) = K$ by the linearity and continuity of Φ. Also note that

$$\begin{aligned} \Phi_U \circ \Phi(U_pU_q^*) &= \begin{cases} \Phi_U(U_pU_q^*) & \text{if } \psi(p) = \psi(q), \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} U_pU_q^* & \text{if } p = q, \\ 0 & \text{otherwise} \end{cases} \\ &= \Phi_U(U_pU_q^*) \\ &= \Phi \circ \Phi_U(U_pU_q^*) \end{aligned}$$

and thus $\Phi_U \circ \Phi = \Phi_U = \Phi \circ \Phi_U$ by linearity and continuity of Φ and Φ_U. □

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