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Flexible Modules and Graded Rings

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ABSTRACT: A $G-$graded $R-$module is called flexible if $M_g = R_g M_e$ for every $g \in G$. In this paper, we study the relationship between a flexible module and the graded ring R through different aspects. On one hand, we distinguish the flexible modules from other graded modules by characterizing the influence of the e-component of a flexible module on the graded module itself. On the other hand, we extend the class covered by flexible graded modules to include free and protective modules in a comparatively simple manner.

Key Words: Graded rings, graded modules, gr-faithful, flexible modules, first strongly graded rings, graded ideals, free modules, projective modules, injective modules.

Contents

1. Introduction

Graded Ring and Module Theory is very connected to Homology and Cohomology Theory. Actually, it was born there. It drew the attention of many mathematicians who engaged serious research on the application of this branch of mathematics to other branches, especially Homology and Cohomology Theory, or on the extension of the results of the ordinary abstract algebra to the graded type of Algebra, which is the approach adopted mostly in this paper. We tickle the first approach in the final part of this article.

Let G be a group with identity e. A $G-$graded ring R is a ring R such that $R = \bigoplus_{g\in G} R_g$ where R_g is an additive abelian subgroup of R (called the g-component of R) for every $g \in G$ such that $R_g R_h \subseteq R_{gh}$, for every $g, h \in G$. A left module M over a $G-$graded ring R is said to be a $G-$graded left $R-$module if $M = \bigoplus_{g\in G} M_g$ where M_g is an additive abelian subgroup of M (called the g-component of M) for every $g \in G$ such that $R_g M_h \subseteq M_{gh}$ for every $g, h \in G$. If we replace "$\subseteq$" with "$=$" we obtain strongly graded rings and modules. The identity component R_e is a subring of R, and if R has unity 1, then $1 \in R_e$. Notice that we can consider R as a $G-$graded $R-$module. An element belonging to a g-component is called a homogeneous element of degree g. Each element $x \in M$ can be written uniquely in the form $x = \sum_{g\in G} x_g$, where $x_g \in M_g$, and all x_g's are zeros except finitely many. An $R-$submodule N of M is described to be $G-$graded if $N = \bigoplus_{g\in G}(N \cap M_g)$, or equivalently, If $x \in N$, then $x_g \in N$, for every $g \in G$. In the case where $M = R$, N becomes a graded left ideal.

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A $G-$graded $R-$module is said to be flexible if $M_g = R_g M_e$, for all $g \in G$ or equivalently if $M = RM_e$. The $e-$component M_e is expected to have a great impact on M when it is flexible. Actually, it does. Which means, we can connect the graded modules properties of a flexible module to similar properties known in ordinary abstract algebra. This fact is what makes flexible modules important. It was proved in [5], [11], and other articles that the class of flexible modules includes many types of graded modules such as strongly graded modules, graded modules over crossed product rings, first strongly graded modules with equal supports (the support of R equals the support of M), augmented graded modules. We show in the final part of this paper that free modules, graded free modules, projective modules, and graded projective modules are also parts of the class of flexible graded modules, when equipped with a particular gradation by G. Further, flexible injective graded modules are contained only in flexible modules. Also, we prove that any graded module, if not flexible, includes a maximal flexible submodule and itself is included in a flexible module as its $e-$component.

In another part of the paper, we investigate several properties of flexible graded modules. Some of these properties are related to the graded ring R and its graded ideals. For example we prove that "For M being a flexible $G-$graded $R-$module such that R is a crossed product on the support, and N a $G-$graded $R-$submodule of M, then N is gr-prime if and only if N_e is a prime R_e-submodule of M_e". Also, we prove that " For R being a commutative graded ring, A a gr-maximal ideal, and M a flexible $R-$module, if M is not a flexible $A-$module, then AM is a gr-prime $R-$module. Moreover, if M is gr-multiplication $R-$module, then M is a gr-local $R-$module with AM being the unique gr-maximal $R-$submodule of M". Besides, other properties in this paper explain the relationship between the flexible modules and other graded modules.

In the first part of the paper, we introduce and study a new but necessary class of graded modules under the name of "gr-faithful modules on the supports". The motivation behind this introductory is that the property of "faithfulness" from abstract algebra is not enough to serve our results, and we need a stronger condition which is more concerned with the gradation by the group G and has the flavor of faithful modules.

The necessary background that helps understand our argument in the paper is given in the second section. While the new results exist in the third section.

2. Preliminaries

This section presents some background of graded rings and graded modules necessary to this paper. More details can be found in the references. Throughout this article, unless otherwise stated:

- G and G' are groups with identity e and e', respectively;
- $R = \bigoplus_{g\in G} R_g$ and $R' = \bigoplus_{g'\in G'} R'_{g'}$ are $G-$graded and $G'-$graded rings with unities 1 and $1'$, respectively;
- $M = \bigoplus_{g\in G} M_g$ and $M' = \bigoplus_{g'\in G'} M'_{g'}$ are $G-$graded left $R-$module and $G'-$graded left $R'-$module, respectively;
- The set $supp(R, G) = \{g \in G : \ R_g \neq 0\}$ is called the support of R. The support of M, $supp(M, G)$, is defined similarly.
- The set $h(R) = \bigcup_{g\in G} R_g$ is the set of homogeneous elements of R. Similar definition stands for $h(M)$.
- The subring $C_R(R_e) = \{r \in R : \ rx = xr, \text{ for all } x \in R_e\}$ is the commutant of R_e in R.

To avoid repetition, we assume that all underlying rings and modules are non-trivial, and all modules are left modules.

Definition 2.1. [12] A $G-$graded ring R is first strong, if $1 \in R_g R_{g^{-1}}$, for all $g \in supp(R, G)$, or equivalently, if $supp(R, G)$ is a subgroup of G, and $R_g R_h = R_{gh}$, for all $g, h \in supp(R, G)$
If $supp(R, G) = G$ and R is first strong, then R is strong [7].

Definition 2.2. [9] A $G-$graded ring R is said to be augmented if the following conditions hold:

1. $R_e = \bigoplus\limits_{g \in G} R_{e-g}$ is a $G-$graded ring.
2. For each $g \in G$, there exists $r_g \in R_g$ such that $R_g = R_e r_g$. We assume $r_e = 1$.
3. If $r_g \neq 0$ and $r_h \neq 0$ are as in (2), then $r_g r_h = r_{gh}$.
4. If $r_g \neq 0$ and $r_h \neq 0$ are as in (2), and $x, y \in R_e$, then $(x r_g)(y r_h) = x y r_{gh}$.

An r_g that appears in condition (2) of Definition 2.2 is called a $g-$representative. The set of all selected nonzero homogeneous representatives is denoted by $\Lambda(R, G)$ (or simply Λ). Thus, the set $\Lambda(R, G)$ varies as the representatives vary. However, once an augmented graded ring is under consideration, we fix Λ. We denote an augmented graded ring with a selected homogeneous-representatives set Λ by (R, G, Λ).

Proposition 2.3. *[5] If (R, G, Λ) is an augmented graded ring, then R is first strong. In particular, $supp(R, G)$ is a subgroup of G.*

Proposition 2.4. *[5] If (R, G, Λ) is an augmented graded ring, then*

1. *Λ is a multiplicative subgroup of R, with identity 1.*
2. *Λ is group isomorphic to $supp(R, G)$.*
3. *$\Lambda \subseteq C_R(R_e)$.*

Definition 2.5. [8] Let (R, G, Λ) be an augmented $G-$graded ring. A $G-$graded $R-$module M is said to be augmented if the following conditions hold:

1. $M_g = \bigoplus\limits_{h \in G} M_{g-h}$ is a $G-$graded R_e-module.
2. $R_{g-h} M_{g'-h'} \subseteq M_{gg'-hh'}$, for every $g, h, g', h' \in G$.

Definition 2.6. [1] A $G-$graded ring R is called a crossed product over the support if $R_g \cap U(R) \neq \emptyset$, for every $g \in supp(R, G)$, where $U(R)$ is the set of units of R.

Lemma 2.7. *[1] If R is a crossed product over the support, then R is first strong. In particular, $supp(R, G)$ is a subgroup of G.*

Definition 2.8. [12] Let R be a $G-$graded ring and R' a $G'-$graded ring. A ring homomorphism $\eta : R \to R'$ is said to be an almost equivalence between R and R' if for every $g' \in G'$ there exists $g \in G$ such that $\eta(R_g) = R'_{g'}$.

Lemma 2.9. *[12] Let R be a $G-$graded ring, R' a $G'-$graded ring, and $\eta : R \to R'$ an almost equivalence between R and R'. Then*

1. *For each $g \in G$ and $g' \in G'$, if $r_g \in h(R)$ then $\eta(r_g) \in h(R')$, and if $r'_{g'} \in h(R')$ then $\eta^{-1}(r'_{g'}) \in h(R)$.*
2. *If $0 \neq r_g \in R_g$ and $\eta(r_g) \in R'_{g'}$, then $\eta(R_g) = R'_{g'}$.*
3. *if $\eta(R_g) = R'_{g'}$ and either $R_g R_{g^{-1}} \neq 0$ or $R_{g^{-1}} R_g \neq 0$, then $\eta(R_{g^{-1}}) = R'_{g'^{-1}}$.*

4. supp(R, G) and supp(R′, G′) have the same cardinality.

Remark 2.1. The original version of Lemma 2.9 that exists in [12] assumes $R = R'$. In addition, item 3 assumes only that $R_g R_{g^{-1}} \neq 0$ and omits the case $R_{g^{-1}} R_g \neq 0$. We rewrote the lemma in the current form - the proof is the same - to fit our notations and to be harmonic with our sequel work.

Remark 2.2. [11] Let M be a flexible $G-$graded $R-$module. Then

1. $M \neq 0$ iff $M_e \neq 0$ iff $e \in spp(M, G)$.
2. $supp(M, G) \subseteq supp(R, G)$.

Theorem 2.10. *[11] Let R be an augmented $G-$graded ring, and M a $G-$graded $R-$module such that M_e is a $G-$graded R_e-module. Then M is flexible if and only if M is an augmented $G-$graded $R-$module and $supp(R, G) = supp(M, G)$.*

Definition 2.11. [10] Let R be a $G-$graded ring. A graded $R-$module M is called first strong if $supp(R, G)$ is a subgroup of G and $R_g\, M_h = M_{gh}$, for every $g \in supp(R, G)$ and $h \in G$.

Theorem 2.12. *[10] Let R be a $G-$graded ring. R is first strong if and only if every $G-$graded $R-$module is first strong.*

Lemma 2.13. *[10] Suppose R is a $G-$graded ring, and M a first strongly graded $R-$module. If $supp(R, G) \cap supp(M, G) \neq \emptyset$, then $e \in supp(R, G) \cap supp(M, G)$.*

Theorem 2.14. *[10] If R is a first strongly graded ring and M a graded $R-$module such that $supp(R, G) \cap supp(M, G) \neq \emptyset$. Then $supp(R, G) \subseteq supp(M, G)$.*

Definition 2.15. [11] Let R be a $G-$graded ring, and M a flexible $R-$module. A $G-$graded $R-$submodule N of M is said to be flexible, if N is itself a flexible R–module.

Theorem 2.16. *[11] Let R be a $G-$graded ring, and M a graded $R-$module. If X is an R_e-submodule of M_e, then RX is a flexible R-submodule of M.*

The following theorem informs that on first strongly graded rings, flexibility of a graded module is completely determined by the behavior of the supports of the ring and the module.

Theorem 2.17. *[10] Let R be a first strongly graded ring, and M a graded $R-$module. Then, M is flexible if and only if $supp(M, G) = supp(R, G)$.*

Theorem 2.18. *[11] If R is a first strongly graded ring, and M a flexible $R-$module, then every $G-$graded $R-$submodule of M is also flexible.*

Theorem 2.19. *[11] Let R be a $G-$graded ring and M a graded $R-$module. Then any two of the following conditions together imply the third condition:*

1. *$supp(R, G) = supp(M, G)$.*
2. *M is flexible.*
3. *M is first strong.*

Definition 2.20. [3] Let M be a $G-$graded $R-$module and N a $G-$graded $R-$submodule of M. We call N a graded direct summand (or gr-direct summand) of M if there exists a $G-$graded $R-$submodule K of M such that $M = N \oplus K$.

Theorem 2.21. *[5] Let R be a $G-$graded ring, and M a free $R-$module. Then there exists a graduation of M by G that transforms M into a flexible $G-$graded $R-$module.*

The following definitions are well known in the literature. Each definition exists in non-graded sense.

Definition 2.22. Let R be a $G-$graded ring and M a $G-$graded $R-$module. Then

- If N is a $G-$graded $R-$submodule of M, then $(N :_R M) = \{r \in R : rM \subseteq N\}$ is a $G-$graded ideal of R.
- Assume R is commutative. A proper $G-$graded ideal I is gr-prime, if $ab \in I$ implies either $a \in I$ or $b \in I$, where $a, b \in h(R)$.
- A proper $G-$graded $R-$submodule N of M is gr-prime, if $rm \in N$ implies either $m \in N$ or $r \in (N :_R M)$, where $r \in h(R)$ and $m \in h(M)$.
- M is a gr-multiplication $R-$module if for every graded $R-$submodule N of M, there exists a graded ideal I of R such that $N = IM$.
- A proper $G-$graded $R-$submodule N of M is gr-maximal, if it is not a subset of another proper graded $R-$submodule of M.
- M is gr-local (or sometimes gr-quasi-local) if it has unique graded maximal $R-$submodule.
- M is gr-free if it is a free $R-$module with homogeneous basis.
- M is a gr-projective $R-$module if for every exact sequence $N \xrightarrow{f} L \to 0$ of $G-$graded $R-$modules and gr-homomorphisms, and a gr-homomorphism $g : M \to L$ there exists a gr-homomorphism $h : M \to N$ such that $f = gh$.
- M is a gr-injective $R-$module if for every exact sequence $0 \to N \xrightarrow{f} L$ of $G-$graded $R-$modules and gr-homomorphisms, and a gr-homomorphism $g : N \to M$ there exists a gr-homomorphism $h : L \to M$ such that $g = hf$.

Theorem 2.23. *[4] A $G-$graded $R-$module is gr-projective if and only if it is a gr-direct summand of a gr-free $R-$module. While, A $G-$graded $R-$module is gr-injective if and only if it is a gr-direct summand of a $G-$graded $R-$module containing it.*

3. Main Results

In this section, we exhibit the results of this article. We start by introducing the concept of a gr-faithful module on the supports which is a special case of faithful modules. Their properties and nexus to other graded modules are investigated. We shall see that the "gr-faithful on the supports" property combined with other properties produce a well-structured graded modules such as augmented graded modules, first strong modules, flexible modules, and so on. Furthermore, we prove that "gr-faithful on the supports" property is preserved under certain isomorphisms between graded modules, called fittings (see Definition 3.3). Afterward, we give different results of flexible modules. Mainly, we show that we can provide free modules and projective modules (graded or not) with a suitable gradation that turns them into flexible modules.

3.1. The gr-Faithful Modules on The Supports

Definition 3.1. A $G-$graded $R-$module is said to be gr-faithful on the supports if $supp(R, G)$ is a subgroup of G and $r_g M_h \neq 0$, for every $r_g \in h(R) - 0$, $g \in supp(R, G)$, and $h \in supp(M, G)$.

If $M = R$ in Definition 3.1, we obtain the definition of a gr-faithful graded ring on the support.

Example 3.2. Let $G = \mathbb{Z}_4$, $R = \mathbb{Z}$, and $M = \mathbb{Z} \oplus i\mathbb{Z}$. Then, R has the trivial gradation by G and M is a $G-$graded $R-$module with a gradation $M_0 = \mathbb{Z}$, $M_2 = i\mathbb{Z}$ and $M_1 = M_3 = 0$. We have M is a gr-faithful on the supports graded module.

It is clear that a left (right) gr-faithful module on the supports is left (right) gr-faithful if and only if supp(R,G)=G.

Analogous to many properties in graded ring and graded module theories, a question rises up about the "gr-faithful on the supports" property whether it can be preserved by a suitable isomorphism between graded modules that preserves the gradation of modules or fits them together. Moving in this route we start with a definition that generalizes the notion of almost equivalence between graded rings (Definition 2.8) to graded modules.

Definition 3.3. Let M be a $G-$graded $R-$module and M' a $G'-$graded $R'-$module. A 4-tuple (η, f, M, M') is said to be fitting M in M' (or fitting for short) if

1. the mapping $\eta : R \to R'$ is a ring isomorphism such that for each $g' \in G'$ there exists $g \in G$ such that $\eta(R_g) = R_{g'}$. That is, R is almost equivalent to R' by η [12].

2. the mapping $f : M \to M'$ is a group isomorphism such that $f(rm) = \eta(r)f(m)$, for every $r \in R$ and $m \in M$; and for each $g' \in G'$, there exists $g \in G$ such that $f(M_g) = M_{g'}$.

We say that M is gr-equivalent to M' if there exists a fitting (η, f, M, M'). Notice that $(\eta^{-1}, f^{-1}, M', M)$ is a fitting.

It is not difficult to prove that the gr-equivalence is an equivalence relation on the category of graded modules. In Definition 3.3, the group isomorphism $f : M \to M'$ exists in the literature and is described by an $\eta-$isomorphism [6].

The proof of the next lemma partially follows the similar argument of the proof of Lemma 2.9.

Lemma 3.4. *Let M be a $G-$graded $R-$module, M' a $G'-$graded $R'-$module, and (η, f, M, M') a fitting. Then*

1. *If $f(M_g) \cap M'_{g'} \neq 0$, then $f(M_g) = M'_{g'}$, where $g \in G$ and $g' \in G'$.*

2. *Let m' and x' be homogeneous elements of M'. Then $f^{-1}(m')$ and $f^{-1}(x')$ are homogeneous elements of M of the same degree if and only if m' and x' have the same degree in M'.*

3. *$f(m_g) \in h(M')$ and $f^{-1}(m_{g'}) \in h(M)$, for every $m_g \in M_g$, $m'_{g'} \in M'_{g'}$, $g \in G$, and $g' \in G'$. Moreover, for every $g \in G$, there exists $g' \in G'$ such that $f(M_g) = M'_{g'}$.*

4. *Suppose for some $g \in G$ that*

   (a) either $R_g R_{g^{-1}} \neq 0$ or $R_{g^{-1}} R_g \neq 0$;

   (b) $R_g M_{g^{-1}} \neq 0$ and $R_{g^{-1}} M_g \neq 0$; and

   (c) $f(M_e) = M'_{e'}$,

   then the equality $f(M_g) = M'_{g'}$ implies the equality $f(M_{g^{-1}}) = M'_{g'^{-1}}$.

5. *supp(M, G) and supp(M', G') have the same cardinality.*

Proof. 1. Suppose $f(M_g) \cap M'_{g'} \neq 0$, for some $g \in G$ and $g' \in G'$. There exists $h \in G$ such that $f(M_h) = M'_{g'}$. We get that $f(M_g \cap M_h) \neq 0$. Since f is an isomorphism, $M_g \cap M_h \neq 0$. Thus, $g = h$ and $f(M_g) = M'_{g'}$.

2. Assume $f^{-1}(m')$ and $f^{-1}(x')$ are homogeneous elements of M of the same degree. Let $m' \in M'_{g'}$ and $x' \in M'_{h'}$, where $g', h' \in G'$. There exist $g \in G$ and $m, x \in M_g$ such that $m' = f(m)$ and $x' = f(x)$. By part 1, $f(M_g) = M'_{g'} = M'_{h'}$ which yields $g' = h'$. The converse is an immediate consequence of the definition of f.

3. Firstly, it follows by the definition of f that if $m'_{g'} \in h(M')$, then $f^{-1}(m'_{g'}) \in h(M)$. Secondly, if $m_g \in M_g$, then $f(m_g) = \sum_{g' \in G'} m'_{g'}$. Thus $m_g = \sum_{g' \in G'} f^{-1}(m'_{g'})$. Since $f^{-1}(m'_{g'}) \in h(M)$, for all $g' \in G'$, part 2 implies $m'_{g'} = 0$ for all $g' \in G'$ except one element $g' \in G'$. This means that $f(m_g) = m'_{g'} \in h(M')$. By part (1), for every $g \in G$, there exists $g' \in G'$ such that $f(M_g) = M'_{g'}$.
4. Assume $f(M_g) = M'_{g'}$. Let $f(M_{g^{-1}}) = M'_{h'}$. we have
$$0 \neq f(R_g M_{g^{-1}}) = \eta(R_g) f(M_{g^{-1}}) = R'_{k'} M'_{h'} \subseteq M'_{k'h'},$$
where $\eta(R_g) = R'_{k'}$ for some $k' \in G'$ by Lemma 2.9. From the fact that $R_g M_{g^{-1}} \subseteq M_e$ and the assumption $f(M_e) = M'_{e'}$ we deduce that $M'_{e'} = M'_{k'h'}$ or $k'h' = e'$. On the other hand,
$$0 \neq f(R_{g^{-1}} M_g) = \eta(R_{g^{-1}}) f(M_g) = R'_{k'^{-1}} M'_{g'} \subseteq M'_{k'^{-1}g'},$$
where $\eta(R_{g^{-1}}) = R'_{k'^{-1}}$ by Lemma 2.9. Again, the fact that $R_{g^{-1}} M_g \subseteq M_e$ and the assumption $f(M_e) = M'_{e'}$ yield $M'_{k'^{-1}g'} = M'_{e'}$ or $k'^{-1}g' = e'$.

   Now, from the equalities $k'h' = e'$ and $k'^{-1}g' = e'$ we obtain that $h' = g'^{-1}$, and the proof is complete.
5. Define the function $\varphi : supp(M, G) \to supp(M', G')$ by $\varphi(g) = g'$, where $f(M_g) = M'_{g'}$. The definition of f and parts 1, 2 and 3 ensure that φ is a well-defined bijection.

□

Consider the function $\psi : supp(R, G) \to supp(R', G')$ defined by $\psi(g)$ is the unique element in G' such that $\eta(R_g) = R_{\psi(g)}$. This function is the one used in Lemma 2.9 to show that the cardinality of $supp(R, G)$ is equal to the cardinality of $supp(R', G')$. We now study the functions φ and ψ and discover the relationship between them. In the next work, R, R', G, G', φ, ψ, η and f are as mentioned above.

Lemma 3.5. *If R is a gr-faithful graded ring on the support, then $\psi : supp(R, G) \to supp(R', G')$ is a group isomorphism. Further, $supp(R', G')$ is a subgroup of G'.*

Proof. We have $supp(R, G)$ is a subgroup of G. In the definition of ψ, replace $supp(R', G')$ by G' to obtain the function $\psi : supp(R, G) \to G'$ between two groups defined in the same way as above. To show that the new ψ is a group homomorphism, let $g, h \in supp(R, G)$. By assumptions
$$0 \neq \eta(R_g R_h) \subseteq \eta(R_{gh}) = R'_{\psi(gh)}. \tag{3.1}$$
On the other hand,
$$0 \neq \eta(R_g R_h) = \eta(R_g)\eta(R_h) = R'_{\psi(g)} R'_{\psi(h)} \subseteq R'_{\psi(g)\psi(h)}. \tag{3.2}$$
From Equations 3.1 and 3.2, we obtain $\psi(gh) = \psi(g)\psi(h)$.
Since ψ is one to one and $Im\psi = supp(R', G')$, we conclude that the original ψ is a group isomorphism. Consequently, $Im\psi = supp(R', G')$ is a subgroup of G'. □

Lemma 3.6. *Let $g \in G$ such that $R_g M_e \neq 0$. Then $\varphi(g) = \psi(g)\varphi(e)$. Moreover, if $f(M_e) = M_{e'}$, then $\varphi(g) = \psi(g)$.*

Proof. Since $R_g M_e \neq 0$, $g \in supp(R, G) \cap supp(M, G)$. Hence
$$0 \neq f(R_g M_e) \subseteq f(M_g) = M'_{\varphi(g)}. \tag{3.3}$$
Also,
$$0 \neq f(R_g M_e) = \eta(R_g) f(M_e) \subseteq R'_{\psi(g)} M'_{\varphi(e)} \subseteq M'_{\psi(g)\varphi(e)}. \tag{3.4}$$
Equations 3.3 and 3.4 yield $\varphi(g) = \psi(g)\varphi(e)$.
In addition, if $f(M_e) = M'_{e'}$, we get that $\varphi(e) = e'$ and hence $\varphi(g) = \psi(g)$. □

Corollary 3.7. *If both R and M are gr-faithful on the supports and $f(M_e) = M'_{e'} \neq 0$, then*

1. *$supp(R,G) \subseteq supp(M,G)$.*

2. *$\varphi | supp(R,G) = \psi$, where $\varphi | supp(R,G)$ is the restriction of φ on $supp(R,G)$.*

Proof. The proof is easy by applying Lemmas 3.4, 3.5, and 3.6. □

The next theorem asserts that the gr-equivalence between graded modules preserves the "gr-faithful on the supports" property under certain conditions.

Theorem 3.8. *Let R be a gr-faithful $G-$graded ring on the support, M a gr-faithful module on the supports, M' a $G'-$graded $R'-$module, and (η, f, M, M') a fitting. Then M' is gr-faithful on the supports.*

Proof. By Lemma 3.5, $supp(R', G')$ is a subgroup of G'. Let $r'_{g'} \in h(R) - 0$, and $h' \in supp(M', G')$. Set $g' = \psi(g)$ with $g \in supp(R,G)$ and $h' = \varphi(h)$ with $h \in supp(M,G)$. Then $r'_{g'} = \eta(r_g)$ for some $r_g \in R_g - 0$ and $M'_{h'} = f(M_h)$. Then, $r'_{g'} M'_{h'} = \eta(r_g) f(M_h) = f(r_g M_h) \neq 0$. Consequently, M' is gr-faithful on the supports. □

Theorem 3.9. *If M is gr-faithful on the supports such that $M_e \neq 0$, then $Ann_R(M) = (0 :_R M) = 0$, and hence M is faithful $R-$module.*

Proof. Assume M is gr-faithful on the supports. Let $g \in G$ and $r_g \in (0 :_R M)_g$. Then $r_g M = 0$ and hence $r_g M_e = 0$. Since $e \in supp(M,G)$ and M is gr-faithful on the supports, we obtain $r_g = 0$. Thus $(0 :_R M)_g = 0$ for every $g \in G$. So, $(0 :_R M) = 0$. □

Theorem 3.10. *Let R be a first strongly $G-$graded ring and M a $G-$graded $R-$module such that $supp(R,G) = supp(M,G)$. Then M is gr-faithful on the supports if and only if $(0 :_R M_e) = 0$.*

Proof. Suppose $(0 :_R M_e) = 0$. Let $r_g \in h(R)$ and $h \in supp(M,G)$ such that $r_g M_h = 0$. Then

$$R_{h^{-1}} r_g M_h = 0 \Rightarrow r_g M_e = 0 \Rightarrow r_g = 0.$$

We deduce that M is gr-faithful on the supports.
Conversely, Suppose that M is gr-faithful on the supports. Let $r_g \in h(R)$ such that $r_g M_e = 0$. Since $M_e \neq 0$, $r_g = 0$. This implies $(0 :_R M_e) = 0$. □

Proposition 3.11. *If M is a gr-faithful module on the supports and J a $G-$graded ideal of R such that $JM = 0$, then $J = 0$.*

Proof. Suppose that $J \neq 0$. Then there exists $r \in J$ such that $r \neq 0$, and then $r_g \neq 0$ for some $g \in G$. Since J is a graded ideal, $r_g \in J$, and then $r_g M = 0$. Let $h \in supp(M,G)$. Then $r_g M_h = 0$, and hence $r_g = 0$ as M is graded faithful on the supports, which is a contradiction. Hence, $J = 0$. □

Theorem 3.12. *If M is a gr-faithful module on the supports such that $supp(R,G) = supp(M,G)$ and M_g is simple R_e-module for every g in G, then M is first strongly graded module.*

Proof. By the definition of gr-faithful modules on the supports, $supp(R,G)$ is a subgroup of G. Let $g \in supp(R,G)$ and $h \in G$. If $h \in supp(R,G)$, then $0 \neq R_g M_h$ is an R_e-submodule of M_{gh}. Thus $R_g M_h = M_{gh}$. If $h \notin supp(M,G)$, then $gh \notin supp(R,G)$ and hence $R_g M_h = M_{gh} = 0$. This proves that M is first strong. □

3.2. Flexible Modules

From this point, we assume R is a commutative G-graded ring with unity, unless otherwise stated. The following theorem gives the first characterization of flexible modules through the graded ring.

Theorem 3.13. *Let M be a $G-$graded $R-$module. Then M is flexible if and only if $\bigcap_{g\in G}(N_g :_{R_e} M_g) = (N_e :_{R_e} M_e)$, for every graded submodule N of M.*

Proof. Suppose M is a flexible module. Let $a \in (N_e :_{R_e} M_e)$ and $g \in G$. Then

$$aM_e \subseteq N_e \Rightarrow R_g(aM_e) \subseteq R_gN_e \subseteq N_g \Rightarrow a(R_gM_e) \subseteq N_g \Rightarrow aM_g \subseteq N_g.$$

Thus, $a \in (N_g :_{R_e} M_g)$, for each $g \in G$. Therefore, $(N_e :_{R_e} M_e) \subseteq \bigcap_{g\in G}(N_g :_{R_e} M_g)$, which implies $\bigcap_{g\in G}(N_g :_{R_e} M_g) = (N_e :_{R_e} M_e)$.

For the converse, Suppose $\bigcap_{g\in G}(N_g :_{R_e} M_g) = (N_e :_{R_e} M_e)$ for every graded $R-$submodule N of M, or equivalently $(N_e :_{R_e} M_e) \subseteq (N_g :_{R_e} M_g)$, for every $g \in G$ and every graded $R-$submodule N of M. Let $N = RM_e$ and fix $g \in G$. Then the inclusion $(N_e :_{R_e} M_e) \subseteq (N_g :_{R_e} M_g)$ induces $R_e = (M_e :_{R_e} M_e) \subseteq (R_gM_e :_{R_e} M_g) \subseteq R_e$. That is $(R_gM_e :_{R_e} M_g) = R_e$ which gives $M_g = R_gM_e$, for each $g \in G$. We conclude that M is flexible. □

Theorem 3.14. *Let R be a first strongly $G-$graded ring and M a $G-$graded $R-$module. Then M is flexible if and only if $(N :_R M)$ is a flexible ideal, for every $G-$graded $R-$submodule N of M.*

Proof. Suppose M is flexible. By Theorem 2.12, $(N :_R M)_g = R_g(N :_R M)_e$, for every $g \in G$, which means that $(N :_R M)$ is a flexible $R-$module.
For the converse, assume $(N :_R M)$ is a flexible ideal of R, for every $G-$graded $R-$submodule N of M. Set $N = RM_e$. Then $(RM_e :_R M) = R(RM_e :_R M)_e = R(M_e :_{R_e} M_e) = RR_e = R$. Hence, $M = RM_e$ which means that M is a flexible $R-$module. □

Theorem 3.15. *The sum of flexible $R-$submodules is a flexible $R-$submodule.*

Proof. Let $N = \sum_{i\in I} N_i$ where N_i is a flexible $R-$submodule of a graded $R-$module M with $e-$component N_i^e, for each $i \in I$. Obviously, N is a graded $R-$submodule of M. Further,

$$N = \sum_{i\in I} RN_i^e \subseteq R\sum_{i\in I} N_i^e = R\left(\sum_{i\in I} N_i\right)_e = RN_e.$$

This yields N is flexible. □

Definition 3.16. [2] Let R be a $G-$graded ring. The ideal $\theta_{gr}(M)$ is defined by $\theta_{gr}(M) = \sum_{x\in h(M)} (Rx :_R M)$.

It is easy to prove that $\theta_{gr}(M)$ is a graded ideal.

Theorem 3.17. *Let R be a first strongly graded ring and M a flexible graded $R-$module. Then the graded ideal $\theta_{gr}(M)$ is a flexible ideal.*

Proof. The proof follows from Theorems 3.14 and 3.15. □

Theorem 3.18. *Let R be a crossed product on the support, M a flexible $G-$graded $R-$module, and N a $G-$graded $R-$submodule of M. Then N is gr-prime if and only if N_e is a prime R_e-submodule of M_e.*

Proof. Assume N is gr-prime. Let $a \in R_e$ and $m \in M_e$ such that $am \in N_e$. Then $am \in N$. Hence, either $m \in N$ or $a \in (N :_R M)$. It follows that either $m \in N \cap M_e = N_e$ or $a \in (N :_R M) \cap R_e = (N :_R M)_e = (N_e :_{R_e} M_e)$. Thus, N_e is a prime R_e-submodule of M_e.
For the converse, assume N_e is a prime R_e-submodule of M_e. Let $r_g \in R_g$ and $m_h \in M_h$ such that $r_g m_h \in N$. Let $u_g \in U(R) \cap R_g$ for each $g \in supp(R, G)$. Since M is flexible and R is a crossed product on the support, $R_g = R_e u_g$ for every $g \in supp(R, G)$, and hence $M_h = u_h M_e$. Set $r_g = tu_g$ and $m_h = u_h m_e$ for some $t \in R_e$ and $m_e \in M_e$. We have

$$tu_g u_h m_e \in N \Rightarrow u_g u_h t m_e \in N_{gh} \Rightarrow t m_e \in u_h^{-1} u_g^{-1} N_{gh} \subseteq N_e.$$

Since N_e is prime, either $m_e \in N_e$ or $t \in (N_e :_{R_e} M_e)$. Therefore, either $u_h m_e \in u_h N_e = N_h \subseteq N$ or $tu_g = (N_e :_{R_e} M_e) u_g = (N :_R M)_g \subseteq (N :_R M)$. This proves that N is gr-prime. $\square$

Theorem 3.19. *Let R be a first strongly $G-$graded ring and M a flexible $R-$module. Then M is gr-faithful on the supports if and only if M_e is faithful R_e-module.*

Proof. Suppose M_e is a faithful R_e-module. Let $r_g \neq 0$ and $M_h \neq 0$, where $g \in supp(R, G)$ and $h \in supp(M, G)$. By Theorem 3.10, R is a gr-faithful ring on the supports. Thus $r_g R_f \neq 0$ for every $g, f \in supp(R, G)$. If we let $f = g^{-1}$, we obtain that $r_g R_{g^{-1}} M_e \neq 0$ which implies $r_g M_{g^{-1}} \neq 0$. Now

$$\begin{aligned} r_g M_{g^{-1}} \neq 0 \quad &\Rightarrow \quad r_g R_e M_{g^{-1}} \neq 0 \Rightarrow r_g R_h R_{h^{-1}} M_{g^{-1}} \neq 0 \Rightarrow r_g R_h R_{h^{-1}} R_{g^{-1}} M_e \neq 0 \\ &\Rightarrow \quad r_g R_{h^{-1}} R_{g^{-1}} R_h M_e \neq 0 \Rightarrow r_g R_{h^{-1}} R_{g^{-1}} M_h \neq 0 \Rightarrow R_{h^{-1}g^{-1}}(r_g M_h) \neq 0 \\ &\Rightarrow \quad r_g M_h \neq 0. \end{aligned}$$

Consequently, M is gr-faithful on the supports.
The converse is straightforward from the definition of the gr-faithful module on the supports. $\square$

Recall that if R is a $G-$graded ring, then the group ring $R[G]$ is $G-$graded by $(R[G])_\sigma = \bigoplus_{g \in G} R_{\sigma g}\, g^{-1}$ with $e-$component $(R[G])_e = R$ (up to gr-isomorphism). Furthermore, the group ring $R[G]$ is strongly graded. Similarly, if M is a $G-$graded $R-$module, then the $R[G]-$module $M[G]$, defined like $R[G]$, is $G-$graded by $(M[G])_\sigma = \bigoplus_{g \in G} M_{\sigma g}\, g^{-1}$ with $e-$component $(M[G])_e = M$ (up to isomorphism). More details can be found in [6].

Theorem 3.20. *Every graded $R-$module contains a flexible $R-$submodule, and is included in a flexible module as its $e-$component.*

Proof. Assume M is a $G-$graded $R-$module such that $M_e \neq \emptyset$. Then RM_e is a flexible $R-$submodule of M. On the other hand, since $R[G]$ is strongly graded, then the $R[G]-$module $M[G]$ is also a strongly graded $R[G]-$module and therefore $M[G]$ is a flexible graded $R[G]-$module. $\square$

Theorem 3.21. *Let $f : M \longrightarrow N$ be a gr-epimorphism of degree e between $G-$graded $R-$modules such that M is flexible. Then N is flexible.*

Proof. Assume M is a flexible $R-$module. Then $N = f(M) = f(RM_e) = Rf(M_e) = RN_e$. Thus N is flexible. $\square$

The previous theorem will yield many results. We list some below.

Theorem 3.22. *Let M be a $G-$graded $R-$module. The following statements are true:*

1. *If $M_e = R$, then M is flexible if and only if the gradation of M by G is trivial.*
2. *If M is flexible and N is an $R-$submodule, then M/N is a flexible $R-$module.*
3. *If M is flexible then, every graded direct summand of M is a flexible $R-$module.*
4. *If $\bigoplus_{\lambda\in\Lambda} M_\lambda$ is a flexible $R-$module, where $M_\lambda = M$ for each $\lambda \in \Lambda$, then M is a flexible R-module, where Λ is an indexing set.*
5. *Assume $M = N \oplus K$, where N and K are $G-$graded $R-$submodules of M, if either of M, N, or K is flexible, then the other two are flexible.*

Proof. 1. Assume M is flexible with $M_e = R$. We have $M = RM_e = RR = R = M_e$, which implies $supp(M, G) = \{e\}$ and hence M has the trivial gradation by G. Conversely, if M has the trivial gradation with $M_e = R$, then $M = R$ which is obviously flexible.

2. Apply Theorem 3.21 to the gr-epimorphism of degree e, $f : M \to M/N$ defined by $f(m) = m + N$.

3. Assume N is a gr-direct summand $R-$submodule of M, then $M = N \oplus K$, where K is a graded $R-$submodule of M. The function $f : M \to N$ defined by $f(n + k) = n$ is a gr-epimorphism of degree e. By Theorem 3.21, N is flexible.

4. Apply Theorem 3.21 on the projection gr-epimorphism$f : \bigoplus_{\lambda\in\Lambda} M_\lambda \to M$.

5. Firstly, assume M is flexible. Then both N and K are flexible modules since the projection maps $\pi_N : M \to N$ defined by $\pi_N(n + k) = n$ and $\pi_K : M \to K$ defined by $\pi_K(n + k) = k$ are gr-epimorphisms of degree e. Secondly, without loss of generality, assume N is flexible. Since M/N is gr-isomorphic to K of degree e, by (2), we get K is flexible. Now, by Theorem 3.15 , M is flexible. □

Lemma 3.23. *Let R be a $G-$graded ring, I a $G-$graded right ideal, and M a $G-$graded $R-$module, and N a flexible $R-$submodule of M. Then IN is a flexible $I-$module.*

Proof. If $N = RN_e$, then $IN = IRN_e = IN_e$. □

Proposition 3.24. *[3] Let R be a $G-$ graded ring, M a graded $R-$module, and N a graded $R-$submodule of M. The following hold:*

1. *If $(N :_R M)$ is a graded maximal ideal of R, then N is a graded prime submodule of M.*
2. *If Q is a graded maximal ideal of R with $QM \neq M$, then QM is graded prime.*

Theorem 3.25. *Let R be a graded ring, A a gr-maximal ideal, and M a flexible $R-$module. If M is not flexible a $A-$module, then $AM = AM_e$ is a gr-prime $R-$module.*

Proof. We have $AM = ARM_e = AM_e$. Also, $M \neq AM_e$. Thus $AM \neq M$. By Proposition 3.24, $AM = AM_e$ is a graded prime submodule. □

Proposition 3.26. *Let R be a $G-$graded ring, A a maximal graded ideal, and M a flexible $R-$submodule. Then each of the conditions below makes M a non-flexible $A-$module.*

1. *$A_g M_e \neq M_g$ for some $g \in G$,*
2. *M_e is a finitely generated faithful R_e-module.*
3. *M is gr-faithful on the supports and M_e is finitely generated R_e-module.*

4. *There exists a graded $A-$submodule N of M and $g \in supp(R,G)$ such that $(N_e :_{A_e} M_e) \subsetneq (N_g :_{A_e} M_g)$.*

Proof. 1. Follows from the definition of flexible modules.
2. By Nakayama's Lemma, we get $A_e M_e \neq M_e$. Then apply (1).
3. Follows by Theorem 3.9 and (2).
4. Apply Theorem 3.13. □

Corollary 3.27. *Let R be a gr-local ring with A being the unique gr-maximal ideal of R, and M be a gr-multiplication flexible $R-$module. If M is not a flexible $A-$module, then AM_e is the unique gr-maximal $R-$submodule of M which yields M is gr-local.*

Proof. We have $AM = AM_e$. Let N be a $G-$graded $R-$submodule of M such that $AM \subsetneq N \subseteq M$. . Assume $N = JM$ where J is a graded ideal of R. Then, there exists $r_g \in J$ and $m_e \in M_e$ such that $r_g m_e \notin AM_e$. Thus, $r_g \notin A$ and hence r_g is a unit. This implies $J = R$ and $N = M$, which means that AM_e is a maximal graded R-submodule of M. To prove that AM is the unique gr-maximal $R-$submodule of M, assume $N \neq M$ is a graded $R-$submodule of M. Then $N = (N :_R M)M$ and $(N :_R M) \neq R$. So, $(N :_R M) \subseteq A$. Thus, $N \subseteq AM$. We deduce that all the proper graded $R-$submodules of M are contained in the gr-maximal submodule AM. Therefore, AM is the unique gr-maximal $R-$submodule of M and hence M is gr-local. □

3.3. Flexible Modules and Other Graded Modules

In this section, we investigate the relationship between flexible graded modules and other types of graded modules. The main result in this section states that the protective and free modules can be accommodated with a gradation by G that transforms them into flexible graded module, and this gradation is trivial if and only if the gradation of the ring R is trivial.

Theorem 3.28. *Let M be a flexible gr-faithful $R-$module on the supports. Then*

$$supp(R,G) = supp(M,G)$$

and M is first strong.

Proof. Assume M is gr-faithful on the supports. Let $g \in supp(R,G)$, we have $R_g M_e = M_g$. Since M is gr-faithful on the supports, and $e \in supp(M,G)$, then $g \in supp(M,G)$. So $supp(R,G) \subseteq supp(M,G)$. By Remark 2.2 $supp(M,G) \subseteq supp(R,G)$. Thus, $supp(R,G) = supp(M,G)$. □

The following theorem states that the flexible gr-faithful modules on the supports are augmented graded modules.

Theorem 3.29. *Let R be an augmented $G-$graded ring, and M a flexible gr-faithful $R-$module on the supports such that M_e is a $G-$graded R_e-module. Then M is an augmented $G-$graded $R-$module.*

Proof. Combine Theorem 2.10 and Theorem 3.28. □

A partial converse of Theorem 3.29 is given next.

Theorem 3.30. *Let M be an augmented $G-$graded $R-$module such that $supp(R,G) = supp(M,G)$ and $Ann_{R_e}(M_e) = 0$. Then, M is gr-faithful on the supports.*

Proof. We have $supp(R,G)$ is a subgroup of G by Proposition 2.3. Suppose for contrary that $r_g M_h = 0$ for some $0 \neq r_g \in h(R)$ and $g,h \in supp(R,G)$. Let $t_g \in \Lambda(R,G)$. Then $r_g = r_e t_g = t_g r_e$, for some $r_e \in R_e$. So, $t_g r_e M_h = 0$. Since M is flexible, $M_h = t_h M_e$, where $t_h \in \Lambda(R,G)$. Thus, $t_g t_h r_e M_e = 0$ or $t_{gh} r_e M_e = 0$. Since t_{gh} is invertible, we have $r_e M_e = 0$. By assumptions, $r_e = 0$ which yields $r_g = 0$, a contradiction. Consequently, $r_g M_h \neq 0$ for every $0 \neq r_g \in h(R)$ and $g,h \in supp(R,G) = supp(M,G)$. That is, M is gr-faithful on the supports. □

The final part of the article is related to Homology and Cohomology Theory.

Theorem 3.31. *Given a $G-$graded free $R-$module M, the gradation of R by G induces a gradation by G on M that turns it into a flexible $R-$module. This gradation is trivial if and only if R is trivially graded.*

Proof. By applying Theorem 2.21, the proof is straightforward since every $G-$graded free module is a free module. According to Theorem 2.21, the desired gradation is defined by $M_g = R_gT$, for all $g \in G$, where T is a basis of M. It is obvious that this gradation is trivial if and only if the gradation of R is trivial. □

Since every graded module is the gr-homomorphic image of a gr-free module, the reader may think by combining Theorem 3.21 and Theorem 3.31 that every graded module is flexible, which is of course not true. We should be careful that not every gr-free module is flexible, but we can change the gradation of the module to make it flexible. Even with the new gradation, we are not sure that the graded module is still a gr-homomorphic image of degree e of the new gr-free module. For example, If $G = \mathbb{Z}$ and K is a graded field with the trivial gradation and $M = K[1, x, x^2, \ldots]$ is graded by $M_n = Kx^n$ for $n \geq 0$ and $M_n = 0$ for $n < 0$. Then M is a gr-free module but not flexible. However, if we change the gradation of M to the trivial gradation, i.e., $M_0 = K[1, x, x^2, \ldots]$ and $M_n = 0$ for $n \neq 0$, the resulting gr-free module is flexible.

Proposition 3.32. *[4] Let R be a $G-$graded ring and P a $G-$graded $R-$module. The following statements are equivalent:*

1. *P is gr- projective.*
2. *P is graded and projective.*
3. *P is gr-isomorphic of a direct summand of gr-free module.*

Theorem 3.33. *Let R be a $G-$graded ring. Every $G-$graded projective (or projective) $R-$module has a gradation by G that turns it into a flexible $R-$module.*

Proof. Let P be a projective or $G-$graded projective $R-$module. By Proposition 3.32, P is a direct summand of a free $R-$module M. By Theorem 3.31, M possesses a gradation by G that makes it a graded flexible $R-$module. This gradation is induced to P from the fact that P is isomorphic to M/K, where $M = P \oplus K$. Equipped with this gradation by G, this isomorphism becomes a gr-isomorphism of degree e. Since the graded module P is a direct summand of the flexible graded module M, by Theorem 3.22 P is flexible. □

Theorem 3.34. *Let R be a $G-$graded ring, M a $G-$graded $R-$module, and N a gr-injective $R-$submodule of M. Then M is flexible if and only if N is flexible.*

Proof. The proof follows easily by Theorems 2.23 and 3.22. □

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