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Cauchy Problem for Matrix Factorizations of the Helmholtz Equation in the Space $\mathbb{R}^m$

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ABSTRACT: In this paper, we consider the problem of recovering solutions for matrix factorizations of the Helmholtz equation in a multidimensional bounded domain from their values on a part of the boundary of this domain, i.e., the Cauchy problem. An approximate solution to this problem is constructed based on the Carleman matrix method.

Key Words: The Cauchy problem, regularization, factorization, regular solution, fundamental solution.

Contents

1. Introduction

It is known that the Cauchy problem for systems of elliptic equations with constant coefficients belongs to the family of ill-posed problems: the solution of the problem is unique, but unstable. In unstable problems, the image of the operator is not closed, so the solvability condition cannot be written in terms of continuous linear functionals. There is a sizable literature on the subject (see, e.g. [29], [42], [31]- [32], [35], [1]). Boundary problems, as well as numerical solutions of some problems, are considered in works [25]- [26], [28], [5], [33], [44]- [46], [2]- [4]. In this work, based on the results of works [31]- [32], [38]- [41], based on the Cauchy problem for the Laplace and Helmholtz equations, an explicit Carleman matrix was constructed and, on its basis, a regularized solution of the Cauchy problem for the matrix factorization of the Helmholtz equation. In many well-posed problems of the system of equations of the first order elliptic type with constant coefficients that factorize the Helmholtz operator, calculating the values of the vector function on the entire boundary is not possible. Therefore, the problem of reconstructing the solution of system of equations of the first order elliptic type with constant coefficients and factorizing the Helmholtz operator are among the more challenging problems in the theory of differential equations.

Problems in which any of the three conditions for the correct formulation of the problem (existence, uniqueness, stability) is not fulfilled belong to the class of ill-posed problems. In this case, the condition of continuous dependence of the solution on the input data plays a decisive role. In conditionally correct problems, problems correct in the sense of A. N. Tikhonov, we are no longer talking just about a solution, but about a solution that belongs to a certain class. Narrowing the class of admissible solutions allows in some cases to pass to the well-posed problem. We will say that the problem is correctly posed according to Tikhonov if:

1) it is known a priori that the solution of the problem exists in some class;
2) in this class the solution is unique;
3) the solution of the problem depends continuously on the input data.

The fundamental difference lies precisely in the selection of the class of admissible solutions. The class of a priori constraints on the solution can be different. The very statement of the problem when considering ill-posed problems changes significantly - the statement of the problem includes the condition that the solution belongs to some set (see [1]).

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Many scientific and applied problems, studied at the world level, in many cases are reduced to the study of ill-posed boundary value problems for partial differential equations. Applied research on conditional correctness and construction of an approximate solution for given values on a part of the boundary of the region, for equations of elliptical type, are especially important in hydrodynamics, geophysics and electrodynamics. The study of a family of regularizing solutions to ill-posed problems served as an impetus for the beginning of studies of the well-posedness class when narrowed to a compact set. Therefore, the study of ill-posed problems for linear elliptic systems of the first order is one of the topical problems in the theory of partial differential equations. At present, in the world, in the study of ill-posed boundary value problems for linear elliptic systems of the first order, the construction of a regularized solution plays a special role. The Cauchy problem for elliptic equations is ill-posed (example Hadamard, see for instance [27], p. 39).

At present, special attention is paid to topical aspects of differential equations and mathematical physics, which have scientific and practical applications in the fundamental sciences. In particular, special attention is paid to the study of various ill-posed boundary value problems for partial differential equations of elliptic type, which have practical application in applied sciences. As a result, significant results were obtained in studies of ill-posed boundary value problems for partial differential equations, that is, approximate solutions were constructed using Carleman matrices in explicit form from approximate data in special domains, estimates of conditional stability and solvability criteria were established. The first results, from the point of view of practical importance, for ill-posed problems and for reducing the class of possible solutions to a compact set and reducing problems to stable ones were obtained in the works of A.N. Tikhonov (see [1]). In the works of M.M. Lavrent'ev, estimates were obtained that characterize the stability of the spatial problem in the class of bounded solutions of the Cauchy problem for the Laplace equation and some other ill-posed problems of mathematical physics in a straight cylinder, as well as for an arbitrary spatial domain with a sufficiently smooth boundary (see, for instance [31]-[32]).

In this work, based on the results of works [31]-[32], [38]-[41], based on the Cauchy problem for the Laplace and Helmholtz equations, an explicit Carleman matrix was constructed and, on its basis, a regularized solution of the Cauchy problem for the matrix factorization of the Helmholtz equation. In work [44], the calculation of double integrals with the help of some connection between wave equation and ODE system was considered. Novel symmetric numerical methods for solving symmetric mathematical problems are considered in paper [45]. In work [5] it is considered integral representation and explicit formula at rational arguments for Apostol - Tangent polynomials. Optimal Control Problem for Systems of Hyperbolic Equations is considered in works [3]-[4].

The problem of reconstructing the solution for matrix factorization of the Helmholtz equation (see, for instance [6]-[21]), is one of the topical problems in the theory of differential equations.

At present, there is still interest in classical ill-posed problems of mathematical physics. This direction in the study of the properties of solutions of Cauchy problem for Laplace equation was started in [42], [31]-[32], [38]-[41] and subsequently developed in [6]-[24], [28]-[29], [34]-[35].

This article is divided into four sections. The first section provides historical information about ill-posed problems of equations of mathematical physics. And also we are talking about the Cauchy problem for matrix factorizations of the Helmholtz equation and the technique for solving it. The second section presents the basic concepts and notation, which is required in the further study of this problem. We briefly present some properties of the Mittag-Leffler function and prove the lemma. In the third section, the theorems that constitute the main meaning of this study are fully proved. A regularized solution of the Cauchy problem for matrix factorizations of the Helmholtz equation in a multidimensional bounded domain is found in explicit form. In the last section, a conclusion is made about the results obtained on the basis of the proved theorems.

2. Basic information and statement problem

Let $\mathbb{R}^m$, $(m = 2k + 1,\ k \geq 1)$ be a $m-$dimensional real Euclidean space,

$$x = (x_1, \ldots, x_m) \in \mathbb{R}^m, \quad y = (y_1, \ldots, y_m) \in \mathbb{R}^m,$$

$$x' = (x_1, \ldots, x_{m-1}) \in \mathbb{R}^{m-1}, \quad y' = (y_1, \ldots, y_{m-1}) \in \mathbb{R}^{m-1}.$$

We introduce the following notation:

$$r = |y - x|, \quad \alpha = |y' - x'|, \quad w = i\tau\sqrt{u^2 + \alpha^2} + \beta, \quad w_0 = i\tau\alpha + \beta,$$

$$\beta = \tau y_m, \quad \tau = tg\frac{\pi}{2\rho}, \quad \rho > 1, \quad u \geq 0,\ s = \alpha^2,$$

$$G_\rho = \{y : |y'| < \tau y_m,\, y_m > 0\}, \quad \partial G_\rho = \{y : |y'| = \tau y_m,\, y_m > 0\},$$

$$\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m}\right)^T, \quad \frac{\partial}{\partial x} = \xi^T, \quad \xi^T = \begin{pmatrix} \xi_1 \\ \ldots \\ \xi_m \end{pmatrix} \text{- transposed vector } \xi,$$

$$U(x) = (U_1(x), \ldots, U_n(x))^T, \quad u^0 = (1, \ldots, 1) \in \mathbb{R}^n, \quad n = 2^m, \quad m \geq 3,$$

$$E(z) = \begin{Vmatrix} z_1 & 0 & \cdots & 0 \\ 0 & z_2 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & 0 & z_n \end{Vmatrix} \text{- diagonal matrix, } z = (z_1, \ldots, z_n) \in \mathbb{R}^n.$$

$G_\rho \subset \mathbb{R}^m$, $(m = 2k + 1,\ k \geq 1)$ be a bounded simply-connected domain, the boundary of which consists of the surface of the cone ∂G_ρ, and a smooth piece of the surface S, lying in the cone G_ρ, i.e., $\partial G_\rho = S \bigcup T,\ T = \partial G_\rho \backslash S$. Let $(0, 0, \ldots, x_m) \in G_\rho$, $x_m > 0$.

Let $D(\xi^T)$ be a $(n \times n)-$ dimensional matrix with elements consisting of a set of linear functions with constant coefficients of the complex plane for which the following condition is satisfied:

$$D^*(\xi^T)D(\xi^T) = E((|\xi|^2 + \lambda^2)u^0),$$

where $D^*(\xi^T)$ is the Hermitian conjugate matrix $D(\xi^T)$, $\lambda-$ is a real number.

We consider a system of linear partial differential equations with constant coefficients in the region G_ρ

$$D\left(\frac{\partial}{\partial x}\right)U(x) = 0, \tag{2.1}$$

where $D\left(\frac{\partial}{\partial x}\right)$ is the matrix of first-order differential operators.

We denote by $A(G_\rho)$ the class of vector functions in the domain G_ρ continuous on $\overline{G}_\rho = G_\rho \bigcup \partial G_\rho$ and satisfying system (2.1).

Cauchy problem. Suppose $U(y) \in A(G_\rho)$ and

$$U(y)|_S = f(y), \quad y \in S. \tag{2.2}$$

Here, $f(y)$ a given continuous vector-function on a smooth piece of the surface S. It is required to restore the vector function $U(y)$ in the domain G_ρ, based on it's values $f(y)$ given on the surface S.

If $U(y) \in A(G_\rho)$, then the following integral formula of Cauchy type is valid

$$U(x) = \int\limits_{\partial G_\rho} N(y, x; \lambda)U(y)ds_y, \quad x \in G, \tag{2.3}$$

where

$$N(y, x; \lambda) = \left(E\left(\varphi_m(\lambda r)u^0\right) D^*\left(\frac{\partial}{\partial x}\right)\right) D(t^T).$$

Here $t = (t_1, \ldots, t_m)-$ is the unit exterior normal, drawn at a point y, the surface ∂G_ρ, $\varphi_m(\lambda r)-$ is the fundamental solution of the Helmholtz equation in $\mathbb{R}^m$, $(m = 2k + 1,\, k \geq 1)$, where $\varphi_m(\lambda r)$ defined by the following formula:

$$\varphi_m(\lambda r) = P_m \lambda^{(m-2)/2} \frac{H^{(1)}_{(m-2)/2}(\lambda r)}{r^{(m-2)/2}},$$
$$P_m = \frac{1}{2i(2\pi)^{(m-2)/2}}, \quad m = 2k+1, \quad k \geq 1. \tag{2.4}$$

Here $H^{(1)}_{(m-2)/2}(\lambda r)-$ is the Hankel function of the first kind of $(m-2)/2-$ th order (see, for instance [37]).

We denote by $K(w)$ is an entire function taking real values for real w, ($w = u+iv$, $u, v-$real numbers) and satisfying the following conditions:

$$K(u) \neq 0, \ \sup_{v\geq 1} \left| v^p K^{(p)}(w) \right| = B(u, p) < \infty,$$
$$-\infty < u < \infty, \quad p = 0, 1, \ldots, m. \tag{2.5}$$

We define the function $\Phi(y, x; \lambda)$ at $y \neq x$ by the following equality

$$\Phi(y, x; \lambda) = \frac{1}{c_m K(x_m)} \frac{\partial^{k-1}}{\partial s^{k-1}} \int\limits_0^\infty \operatorname{Im}\left[\frac{K(w)}{w - x_m}\right] \frac{\cos(\lambda u)}{\sqrt{u^2+\alpha^2}} du,$$
$$m = 2k+1, \quad k \geq 1, \tag{2.6}$$

where $c_m = (-1)^k 2^{-k}(2k-1)!(m-2)\pi\omega_m$; ω_m- area of a unit sphere in space $\mathbb{R}^m$.

In the formula (2.6), choosing

$$K(w) = E_\rho(\sigma^{1/\rho} w), \quad K(x_m) = E_\rho(\sigma^{1/\rho}\gamma), \quad \gamma = \tau x_m, \quad \sigma > 0, \tag{2.7}$$

we get

$$\Phi_\sigma(y, x; \lambda) = \frac{E_\rho(\sigma^{1/\rho}\gamma)}{c_m} \frac{\partial^{k-1}}{\partial s^{k-1}} \int\limits_0^\infty \operatorname{Im}\left[\frac{E_\rho(\sigma^{1/\rho} w)}{w - x_m}\right] \frac{\cos(\lambda u)}{\sqrt{u^2+\alpha^2}} du. \tag{2.8}$$

Here $E_\rho(\sigma^{1/\rho} w)-$ is the entire Mittag-Leffler function (see [30]). In [36], using the S-generalized beta function, a new generalization of the Mittag-Leffler function and its properties is presented.

The formula (2.3) is true if instead $\varphi_m(\lambda r)$ of substituting the function

$$\Phi_\sigma(y, x; \lambda) = \varphi_m(\lambda r) + g_\sigma(y, x; \lambda), \tag{2.9}$$

where $g_\sigma(y, x)-$ is the regular solution of the Helmholtz equation with respect to the variable y, including the point $y = x$.

Then the integral formula has the form:

$$U(x) = \int\limits_{\partial G_\rho} N_\sigma(y, x; \lambda) U(y) ds_y, \quad x \in G, \tag{2.10}$$

where

$$N_\sigma(y, x; \lambda) = \left(E\left(\Phi_\sigma(y, x; \lambda) u^0\right) D^*\left(\frac{\partial}{\partial x}\right)\right) D(t^T).$$

Recall the basic properties of the Mittag-Leffler function. The entire function of Mittag-Leffler is defined by a series

$$\sum_{n=1}^\infty \frac{w^n}{\Gamma(1+\rho^{-1}n)} = E_\rho(w), \quad w = u+iv,$$

where $\Gamma(s)-$ is the Euler gamma function.

We denote by $\gamma_\varepsilon(\beta_0)$ ($\varepsilon > 0$, $0 < \beta_0 < \pi$) the contour in the complex plane ζ, run in the direction of non-decreasing $\arg\zeta$ and consisting of the following parts:

1. The beam $\arg\zeta = -\beta_0, \quad |\zeta| \geq \varepsilon$;
2. The arc $-\beta_0 < \arg\zeta < \beta_0$ of circle $|\zeta| = \varepsilon$;
3. The beam $\arg\zeta = \beta_0, \quad |\zeta| \geq \varepsilon$.

The contour $\gamma_\varepsilon(\beta_0)$ divides the plane ζ into two unbounded simply connected domains G_ρ^- and G_ρ^+ lying to the left and to the right of $\gamma_\varepsilon(\beta_0)$, respectively.

Let $\rho > 1, \quad \dfrac{\pi}{2\rho} < \beta_0 < \dfrac{\pi}{\rho}$.

Denote

$$\psi_\rho(w) = \frac{1}{2\pi i} \int\limits_{\gamma_\varepsilon(\beta_0)} \frac{\exp(\zeta^\rho)}{\zeta - w} d\zeta, \tag{2.11}$$

Then the following integral representations are valid:

$$E_\rho(w) = \psi_\rho(w), \quad z \in G_\rho^-, \tag{2.12}$$

$$E_\rho(w) = \rho\exp(w^\rho) + \psi_\rho(w), \quad z \in G_\rho^+. \tag{2.13}$$

From these formulas we find

$$\left.\begin{aligned} &|E_\rho(w)| \leq \rho\exp(\operatorname{Re} w^\rho) + \left|\psi_\rho(w)\right|, \quad |\arg w| \leq \frac{\pi}{2\rho} + \eta_0, \\ &|E_\rho(w)| \leq \left|\psi_\rho(w)\right|, \quad \frac{\pi}{2\rho} + \eta_0 \leq |\arg w| \leq \pi, \quad \eta_0 > 0 \end{aligned}\right\} \tag{2.14}$$

$$\left|\psi_\rho(w)\right| \leq \frac{M}{1 + |w|}, \quad M = const, \tag{2.15}$$

$$E_\rho(w) \approx \rho\exp(w^\rho), \quad w > 0, \quad w \to \infty. \tag{2.16}$$

Further, since $E_\rho(w)$ is real with real w, then

$$\operatorname{Re}\psi_\rho(w) = \frac{\rho}{2\pi i} \int\limits_{\gamma_\varepsilon(\beta_0)} \frac{2\zeta - \operatorname{Re} w}{(\zeta - w)\zeta - \overline{w})} \exp(\zeta^\rho) d\zeta,$$

$$\operatorname{Im}\psi_\rho(w) = \frac{\rho \operatorname{Im}(w)}{2\pi i} \int\limits_{\gamma_\varepsilon(\beta_0)} \frac{\exp(\zeta^\rho)}{(\zeta - w)\zeta - \overline{w})} d\zeta.$$

The information given here concerning the function $E_\rho(w)$ is taken from (see, [12], [15]).

In what follows, to prove the main theorems, we need the following estimates for the function $\Phi_\sigma(y, x; \lambda$.

Lemma 2.1. *Let $x = (x_1, \ldots, x_m) \in G_\rho$, $y \neq x$, $\sigma \geq \lambda + \sigma_0$, $\sigma_0 > 0$, then*

1) at $\beta \leq \alpha$ inequalities are satisfied

$$|\Phi_\sigma(y, x; \lambda)| \leq C(\rho, \lambda) \frac{\sigma^{m-2}}{r^{m-2}} \exp(-\sigma\gamma^\rho), \quad x \in G_\rho, \tag{2.17}$$

$$\left|\frac{\partial\Phi_\sigma(y, x; \lambda)}{\partial y_j}\right| \leq C(\rho, \lambda) \frac{\sigma^m}{r^{m-1}} \exp(-\sigma\gamma^\rho), \quad \sigma > 1, \quad j = 1, \ldots, m. \tag{2.18}$$

2) at $\beta > \alpha$ inequalities are satisfied

$$|\Phi_\sigma(y, x; \lambda)| \leq C(\rho, \lambda) \frac{\sigma^{m-2}}{r^{m-2}} \exp(-\sigma\gamma^\rho + \sigma \operatorname{Re} w_0^\rho), \quad \sigma > 1, \quad x \in G_\rho, \tag{2.19}$$

$$\left|\frac{\partial\Phi_\sigma(y, x; \lambda)}{\partial y_j}\right| \leq C(\rho, \lambda) \frac{\sigma^m}{r^{m-1}} \exp(-\sigma\gamma^\rho + \sigma \operatorname{Re} w_0^\rho), \quad \sigma > 1, \quad x \in G_\rho, \quad j = 1, \ldots, m. \tag{2.20}$$

Here $C(\rho, \lambda)$ is the function depending on ρ and λ.

For a fixed $x \in G_\rho$ we denote by S^* the part of S on which $\beta \geq \alpha$. If $x \in G_\rho$, then $S = S^*$ (in this case, $\beta = \tau y_m$ and the inequality $\beta \geq \alpha$ means that y lies inside or on the surface cone).

3. Estimation of the stability of the solution to the Cauchy problem

Theorem 3.1. *Let $U(y) \in A(G_\rho)$ it satisfy the inequality*

$$|U(y)| \leq M, \quad y \in T = \partial G_\rho \backslash S^*. \tag{3.1}$$

If

$$U_\sigma(x) = \int\limits_{S^*} N_\sigma(y, x; \lambda) U(y) ds_y, \quad x \in G_\rho, \tag{3.2}$$

then the following estimate is true

$$|U(x) - U_\sigma(x)| \leq M C_\rho(\lambda, x) \sigma^{k+1} \exp(-\sigma \gamma^\rho), \quad \sigma > 1, \quad x \in G_\rho. \tag{3.3}$$

Here and below functions bounded on compact subsets of the domain G_ρ, we denote by $C_\rho(\lambda, x)$.

Proof. Using the integral formula (2.10) and the equality (3.2), we obtain

$$U(x) = \int\limits_{S^*} N_\sigma(y, x; \lambda) U(y) ds_y + \int\limits_{\partial G_\rho \backslash S^*} N_\sigma(y, x; \lambda) U(y) ds_y =$$

$$= U_\sigma(x) + \int\limits_{\partial G_\rho \backslash S^*} N_\sigma(y, x; \lambda) U(y) ds_y, \quad x \in G_\rho.$$

Taking into account the inequality (3.1), we estimate the following

$$\begin{gathered} |U(x) - U_\sigma(x)| \leq \left| \int\limits_{\partial G_\rho \backslash S^*} N_\sigma(y, x; \lambda) U(y) ds_y \right| \leq \\ \leq \int\limits_{\partial G_\rho \backslash S^*} |N_\sigma(y, x; \lambda)| \, |U(y)| \, ds_y \leq M \int\limits_{\partial G_\rho \backslash S^*} |N_\sigma(y, x; \lambda)| \, ds_y, \quad x \in G_\rho. \end{gathered} \tag{3.4}$$

To prove this, we estimate the following integrals $\int\limits_{\partial G_\rho \backslash S^*} |\Phi_\sigma(y, x; \lambda)| \, ds_y$, $\int\limits_{\partial G_\rho \backslash S^*} \left| \frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial y_j} \right| ds_y$, $(j = 1, 2, \dots, m-1)$ and $\int\limits_{\partial G_\rho \backslash S^*} \left| \frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial y_m} \right| ds_y$ on the part $\partial G_\rho \backslash S^*$ of the plane $y_m = 0$.

Separating the imaginary part of (2.8), we obtain

$$\begin{gathered} \Phi_\sigma(y, x; \lambda) = \frac{E_\rho(\sigma^{1/\rho} \gamma)}{c_m} \left[\frac{\partial^{k-1}}{\partial s^{k-1}} \int\limits_0^\infty \frac{(y_m - x_m) \mathrm{Im} E_\rho(\sigma^{1/\rho} w)}{u^2 + r^2} \frac{\cos(\lambda u)}{\sqrt{u^2 + \alpha^2}} du - \right. \\ \left. - \frac{\partial^{k-1}}{\partial s^{k-1}} \int\limits_0^\infty \frac{\mathrm{Re}\, E_\rho(\sigma^{1/\rho} w)}{u^2 + r^2} \cos(\lambda u) \, du \right], \quad y \neq x, \quad x_m > 0. \end{gathered} \tag{3.5}$$

Given equality (3.5), we have

$$\int\limits_{\partial G_\rho \backslash S^*} |\Phi_\sigma(y, x; \lambda)| \, ds_y \leq C_\rho(\lambda, x) \sigma^{k+1} \exp(-\sigma \gamma^\rho), \quad \sigma > 1, \quad x \in G_\rho. \tag{3.6}$$

To estimate the second integral, we use the equality

$$\frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial y_j} = \frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial s}\frac{\partial s}{\partial y_j} = 2(y_j - x_j)\frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial s}, \quad s = \alpha^2, \quad j = 1,2,\dots,m-1. \tag{3.7}$$

Given equality (3.5) and equality (3.7), we obtain

$$\int\limits_{\partial G_\rho \setminus S^*} \left|\frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial y_j}\right| ds_y \le C_\rho(\lambda,x)\sigma^{k+1}\exp(-\sigma\gamma^\rho), \quad \sigma > 1, \quad x \in G_\rho, \quad j = 1,2,\dots,m-1. \tag{3.8}$$

Now, we estimate the integral $\int\limits_{\partial G_\rho \setminus S^*} \left|\frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial y_m}\right| ds_y$.

Taking into account equality (3.5), we obtain

$$\int\limits_{\partial G_\rho \setminus S^*} \left|\frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial y_m}\right| ds_y \le C_\rho(\lambda,x)\sigma^{k+1}\exp(-\sigma\gamma^\rho), \quad \sigma > 1, \quad x \in G_\rho. \tag{3.9}$$

From inequalities (3.4), (3.6), (3.8) and (3.9), we obtain an estimate (3.3). □

Corollary 3.2. *The limiting equality*

$$\lim_{\sigma\to\infty} U_\sigma(x) = U(x),$$

holds uniformly on each compact set from the domain $x \in G_\rho$.

Suppose that the surface S is given by the equation

$$y_m = \psi(y'), \quad y' \in \mathbb{R}^{m-1},$$

where $\psi(y')$ is a single-valued function satisfying the Lyapunov conditions.

Theorem 3.3. *Let $U(y) \in A(G_\rho)$ satisfy condition (3.1), and on a smooth surface S the inequality*

$$|U(y)| \le \delta, \ 0 < \delta < M. \tag{3.10}$$

Then the following estimate is true

$$|U(x)| \le C_\rho(\lambda,x)\sigma^{k+1}M^{1-\left(\frac{\gamma}{R}\right)^\rho}\delta^{\left(\frac{\gamma}{R}\right)^\rho}, \quad \sigma > 1, \quad x \in G_\rho. \tag{3.11}$$

Here is $R^\rho = \max\limits_{y\in S} \operatorname{Re} w_0^\rho$.

Proof. Using the integral formula (2.10), we have

$$U(x) = \int\limits_{S^*} N_\sigma(y,x;\lambda)U(y)ds_y + \int\limits_{\partial G_\rho \setminus S^*} N_\sigma(y,x;\lambda))U(y)ds_y, \quad x \in G_\rho. \tag{3.12}$$

We estimate the following

$$|U(x)| \le \left|\int\limits_{S^*} N_\sigma(y,x;\lambda)U(y)ds_y\right| + \left|\int\limits_{\partial G_\rho \setminus S^*} N_\sigma(y,x;\lambda)U(y)ds_y\right|, \quad x \in G_\rho. \tag{3.13}$$

Given inequality (2.10), we estimate the first integral of inequality (3.13).

$$\left|\int\limits_{S^*} N_\sigma(y,x;\lambda)U(y)ds_y\right| \le \int\limits_{S^*} |N_\sigma(y,x;\lambda)|\,|U(y)|\,ds_y \le$$
$$\le \delta \int\limits_{S^*} |N_\sigma(y,x;\lambda)|\,ds_y, \quad x\in G_\rho. \tag{3.14}$$

To do this, we estimate the integrals $\int\limits_{S^*} |\Phi_\sigma(y,x;\lambda)|\,ds_y$, $\int\limits_{S^*} \left|\frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial y_j}\right| ds_y$, $(j=1,2,\dots,m-1)$ and $\int\limits_{S^*} \left|\frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial y_m}\right| ds_y$ on a smooth surface S.

Given equality (3.5), we have

$$\int\limits_{S^*} |\Phi_\sigma(y,x;\lambda)|\,ds_y \le C_\rho(\lambda,x)\sigma^{k+1}\exp\sigma(\tau^\rho R^\rho-\gamma^\rho), \quad \sigma>1, \quad x\in G_\rho. \tag{3.15}$$

To estimate the second integral, using equalities (3.5) and (3.7), we obtain

$$\int\limits_{S^*} \left|\frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial y_j}\right| ds_y \le C_\rho(\lambda,x)\sigma^{k+1}\exp\sigma(\tau^\rho R^\rho-\gamma^\rho), \quad \sigma>1, \quad x\in G_\rho,$$
$$j=1,\dots,m-1. \tag{3.16}$$

To estimate the integral $\int\limits_{S^*} \left|\frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial y_m}\right| ds_y$, using equality (3.5), we obtain

$$\int\limits_{S^*} \left|\frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial y_m}\right| ds_y \le C_\rho(\lambda,x)\sigma^{k+1}\exp\sigma(\tau^\rho R^\rho-\gamma^\rho), \quad \sigma>1, \quad x\in G_\rho. \tag{3.17}$$

From (3.14), (3.15) - (3.17), we obtain

$$\left|\int\limits_{S^*} N_\sigma(y,x;\lambda)U(y)ds_y\right| \le C_\rho(\lambda,x)\sigma^{k+1}\delta\,\exp\sigma(\tau^\rho R^\rho-\gamma^\rho), \quad \sigma>1, \quad x\in G_\rho. \tag{3.18}$$

The following is known

$$\left|\int\limits_{\partial G_\rho\setminus S^*} N_\sigma(y,x;\lambda)U(y)ds_y\right| \le C_\rho(\lambda,x)\sigma^{k+1}M\exp(-\sigma\gamma^\rho), \quad \sigma>1, \quad x\in G_\rho. \tag{3.19}$$

Now taking into account (3.18) - (3.19), we have

$$|U(x)| \le \frac{C_\rho(\lambda,x)\sigma^{k+1}}{2}(\delta\,\exp(\sigma\tau^\rho R^\rho)+M)\exp(-\sigma\gamma^\rho), \quad \sigma>1, \quad x\in G_\rho. \tag{3.20}$$

Choosing σ from the equality

$$\sigma = \frac{1}{\tau^\rho R^\rho}\ln\frac{M}{\delta}, \tag{3.21}$$

we obtain an estimate (3.11). □

Let $U(y) \in A(G_\rho)$ and instead $U(y)$ on S with its approximation $f_\delta(y)$, respectively, with an error $0 < \delta < M$,

$$\max_S |U(y) - f_\delta(y)| \le \delta. \tag{3.22}$$

We put

$$U_{\sigma(\delta)}(x) = \int_{S^*} N_\sigma(y, x; \lambda) f_\delta(y) ds_y, \quad x \in G_\rho. \tag{3.23}$$

Theorem 3.4. *Let $U(y) \in A(G_\rho)$ on the part of the plane $y_m = 0$ satisfy condition (3.1). Then the following estimate is true*

$$\left|U(x) - U_{\sigma(\delta)}(x)\right| \le C_\rho(\lambda, x)\sigma^{k+1} M^{1-\left(\frac{\gamma}{a}\right)^\rho} \delta^{\left(\frac{\gamma}{R}\right)^\rho}, \quad \sigma > 1, \quad x \in G_\rho. \tag{3.24}$$

Proof. From the integral formulas (2.10) and (3.23), we have

$$U(x) - U_{\sigma(\delta)}(x) = \int_{\partial G_\rho} N_\sigma(y, x; \lambda) U(y) ds_y -$$

$$- \int_{S^*} N_\sigma(y, x; \lambda) f_\delta(y) ds_y = \int_{S^*} N_\sigma(y, x; \lambda) U(y) ds_y +$$

$$+ \int_{\partial G_\rho \backslash S^*} N_\sigma(y, x; \lambda) U(y) ds_y - \int_S N_\sigma(y, x; \lambda) f_\delta(y) ds_y =$$

$$= \int_{S^*} N_\sigma(y, x; \lambda) \left\{U(y) - f_\delta(y)\right\} ds_y + \int_{\partial G_\rho \backslash S^*} N_\sigma(y, x; \lambda) U(y) ds_y.$$

Using conditions (3.1) and (3.22), we estimate the following:

$$\left|U(x) - U_{\sigma(\delta)}(x)\right| = \left|\int_{S^*} N_\sigma(y, x; \lambda) \left\{U(y) - f_\delta(y)\right\} ds_y\right| +$$

$$+ \left|\int_{\partial G_\rho \backslash S^*} N_\sigma(y, x; \lambda) U(y) ds_y\right| \le \int_{S^*} |N_\sigma(y, x; \lambda)| \left|\left\{U(y) - f_\delta(y)\right\}\right| ds_y +$$

$$+ \int_{\partial G_\rho \backslash S^*} |N_\sigma(y, x; \lambda)| \, |U(y)| \, ds_y \le \delta \int_{S^*} |N_\sigma(y, x; \lambda)| \, ds_y +$$

$$+ M \int_{\partial G_\rho \backslash S^*} |N_\sigma(y, x; \lambda)| \, ds_y.$$

Now, repeating the proof of Theorems 3.1 and 3.3, we obtain

$$\left|U(x) - U_{\sigma(\delta)}(x)\right| \le \frac{C_\rho(\lambda, x)\sigma^{k+1}}{2} (\delta \exp(\sigma \tau^\rho R^\rho) + M) \exp(-\sigma \gamma^\rho).$$

From here, choosing σ from equality (3.21), we obtain an estimate (3.24). □

Corollary 3.5. *The following limit equality*

$$\lim_{\delta \to 0} U_{\sigma(\delta)}(x) = U(x),$$

holds uniformly on every compact set from the domain G_ρ.

4. Conclusion

This article obtained the following results:

Using the Carleman function, a formula is obtained for the continuation of the solution of linear elliptic systems of the first order with constant coefficients in a spatial bounded domain $\mathbb{R}^m$, $(m = 2k+1,\ k \geq 1)$. The resulting formula is an analogue of the classical formula of B. Riemann, W. Voltaire and J. Hadamard, which they constructed to solve the Cauchy problem in the theory of hyperbolic equations. An estimate of the stability of the solution of the Cauchy problem in the classical sense for matrix factorizations of the Helmholtz equation is given. The problem is considered in which instead of the exact data of the Cauchy problem; their approximations with a given deviation in the uniform metric are given and under the assumption that the solution of the Cauchy problem is bounded on part T of the boundary of the domain G_ρ; an explicit regularization formula is obtained.

We note that when solving applied problems, one should find the approximate values of $U(x),\ x \in G_\rho$.

In this paper, we construct a family of vector-functions $U(x, f_\delta) = U_{\sigma(\delta)}(x)$ depending on a parameter σ, and prove that under certain conditions and a special choice of the parameter $\sigma = \sigma(\delta)$, at $\delta \to 0$, the family $U_{\sigma(\delta)}(x)$ converges in the usual sense to a solution $U(x)$ at a point $x \in G_\rho$.

Following A.N. Tikhonov (see [1]), a family of vector-valued functions $U_{\sigma(\delta)}(x)$ is called a regularized solution of the problem. A regularized solution determines a stable method of approximate solution of the problem.

Thus, functional $U_{\sigma(\delta)}(x)$ determines the regularization of the solution of problem (2.1) - (2.2).

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