Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 2023 (41)** : 1–16.
ISSN-0037-8712 in press
doi:10.5269/bspm.62948

Nonlinear Elliptic Problems Involving the Generalized $p(u)$-Laplacian Operator with Fourier Boundary Condition

Said Ait Temghart, Chakir Allalou and Khalid Hilal

abstract: This paper considers the existence of entropy solutions for some generalized elliptic $p(u)$-Laplacian problems with Fourier boundary conditions, when the variable exponent p is a real continuous function and we have dependency on the solution u. We get the results by assuming the right-hand side function f to be an integrable function, and by using the regularization approach combined with the theory of Sobolev spaces with variable exponents.

Key Words: Generalized $p(u)$-Laplacian, entropy solutions, Fourier boundary condition, Sobolev space with variable exponents.

Contents

1. Introduction

In this paper, the aim is to study the existence of entropy solutions for some variable exponent problems with exponents p that may depends on the unknown solution u. We consider the case where the dependency of p on u is a local quantity. Namely, we study the following nonlinear Fourier boundary value problem

$$\begin{cases} -\text{div}(|\nabla u - \Theta(u)|^{p(u)-2}(\nabla u - \Theta(u))) + |u|^{p(u)-2}u + \alpha(u) = f \text{ in } \Omega \\ (|\nabla u - \Theta(u)|^{p(u)-2}(\nabla u - \Theta(u))).\eta + \lambda u = g \qquad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω be a bounded domain of $\mathbb{R}^{N\geq 3}$ with Lipschitz boundary $\partial\Omega$, $\lambda > 0$, η is the outer unit normal vector on $\partial\Omega$, α, Θ are real functions defined on $\mathbb{R}$ or $\mathbb{R}^N$, $f \in L^1(\Omega)$, $g \in L^1(\partial\Omega)$ and $p : \mathbb{R} \to [p_-, p_+]$ is a real continuous function such that, $1 < p_- \leq p_+ < +\infty$ and $p'(z) = \frac{p(z)}{p(z)-1}$ is the conjugate exponent of $p(z)$, with

$$p_- := ess \inf_{z\in\mathbb{R}} p(z) \text{ and } p_+ := ess \sup_{z\in\mathbb{R}} p(z).$$

Interest in general forms of differential problems, whose leading operator is of the generalized $p(u)$-Laplacian type, has greatly increased over the last few decades. The main reason is that this kind of nonlinear operator appears naturally in the study of several phenomena which appear in area of oceanography, turbulent fluid flows, induction heating and electrochemical problems. We cite for example the following parabolic model:
- Fluid flow through porous media: this model is governed by the following equation,

$$\frac{\partial\theta}{\partial t} - \text{div}\left(|\nabla\varphi(\theta) - K(\theta)e|^{p-2}(\nabla\varphi(\theta) - K(\theta)e)\right) = 0,$$

where θ is the volumetric content of moisture, $K(\theta)$ the hydraulic conductivity, $\varphi(\theta)$ the hydro-static potential and e is the unit vector in the vertical direction.

2010 *Mathematics Subject Classification:* 35J60, 35D05, 76A05.
Submitted March 18, 2022. Published April 23, 2022

We distinguish the case of constant exponents p (namely, isotropic equations) and the case of variable exponents $p(x)$ (namely, anisotropic equations). The authors developed the existing results in the abstract settings of Lebesgue and Sobolev spaces with and without variable exponents, namely, $L^p(\Omega), W^{1,p}(\Omega), L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$. It is well-known that $L^{p(x)}(\Omega)$ is not invariant with respect to translation (See [12]). This causes many difficulties about convolutions and continuity of functions in the mean. Moreover, the spaces $W^{1,p(x)}(\Omega)$ presents difficulties about the density of smooth functions (See [15]), the Sobolev inequality, and embedding theorems (for further details, we refer to [16] and [14]). This means that the passage from the constant exponent setting to the variable exponent setting needs attention to special cases, and thus, some challenging open problems remain (for further details, we refer to [5] and [12], and the references therein).
Many authors have studied the problem (1.1) when $p(u) = p(x)$ or $p(u) = p$ by proving the existence and the uniqueness of several types of solutions, and by different approaches ([13,6,7]).

The novelty of this work is to study some problems involving the generalized p-Laplacian operator in the case when the variable exponents p depend on the unknown solution u. Here, we consider a Fourier boundary condition which bring some difficulties to treat the term at the boundary. The motivation to study these kind of problems relies in the fact that, in reality the measurements of some physical quantities are not made point-wise but through some local averages. The situation where the variable exponents p depend on the unknown solution u is non-standard as in the classical case (see [1,2,3,4,13,6,7]). This kind of problems appear in the applications of some numerical techniques for the total variation image restoration method that have been used in some restoration problems of mathematical image processing and computer vision [9,10,20]. Türola, J. in [20] have presented several numerical examples suggesting that the consideration of exponents $p = p(u)$ preserves the edges and reduces the noise of the restored images u. A numerical example suggesting a reduction of noise in the restored images u when the exponent of the regularization term is $p = p(|\nabla u|)$ is presented in [9]. Many authors have considered the problem (1.1) in the case when $\Theta = 0$ and especially the study of existence and uniqueness for weak or entropy solutions to the problem (1.1). M. Chipot and H. B. de Oliveira in [12] have proved the existence of weak solutions for some $p(u)-$Laplacian problems, the existence proofs of [12] are based on the Schauder fixed-point theorem. C. Allalou, K. Hilal and S. A. Temghart in [11], extended the results established in [12] by proving some existence results for some local and nonlocal problems. Andreianov et al. [5], have studied the following prototype problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p(u)-2}\nabla u) + u = f & \text{in} \ \ \Omega, \\ \\ u = 0 & \text{on} \ \ \partial\Omega. \end{cases}$$

C. Zhang and X. Zhang in [21] have proved the existence of entropy solutions to problem (1.1) in the case when $\Theta = 0$ and they have provided some positive answers for the two questions proposed by Chipot and de Oliveira in [12]. S. Ouaro and N. Sawadogo in [17] and [18] considered the following nonlinear Fourier boundary value problem

$$\begin{cases} b(u) - \mathrm{div}\, a(x, u, \nabla u) = f & \text{in } \Omega \\ a(x, u, \nabla u) \cdot \eta + \lambda u = g & \text{on } \partial\Omega. \end{cases}$$

The existence and uniqueness results of entropy and weak solutions are established by an approximation method and convergent sequences in terms of Young measure. Recently, C. Vetro in [19] considered a local Dirichlet problem driven by the $(r(u), s(u))$-Laplacian operator, he proved the existence of nontrivial weak solutions via variational methods and the critical point theory.
To study the existence of weak solutions for the nonlinear Fourier boundary value problem (1.1), we first show that the approximate problems admits a sequence of weak solutions by applying the variational method combined with a special type of operators. In the second step, we will prove that the sequence of weak solutions converges to some function u and by using some a priori estimates, we will show that this function u is an entropy solution of elliptic problem (1.1).

This paper is organized us follow. In Sec. 2 we introduce the basic assumptions and we recall some definitions, basic properties of generalized Sobolev spaces that we will use later. The Sec. 3 is devoted to showing the existence of entropy solutions to the local problem (1.1).

2. Preliminaries

The exponent function p depends on the solution u and therefore it depends on the space variable x. This allows us to look for the entropy solutions to the problem (1.1) in a Sobolev space with variable exponents in the following sense,

$$h(x) = p(u(x)).$$

Let Ω be a bounded domain of $\mathbb{R}^N, N \geq 3$, we say that a real-valued continuous function $h(.)$ is log-Hölder continuous in Ω if

$$\exists C > 0 : \ |h(x) - h(y)| \leq \frac{C}{\ln\left(\frac{1}{|x-y|}\right)} \quad \forall x, y \in \Omega, \quad |x-y| < \frac{1}{2}. \tag{2.1}$$

For any Lebesgue-measurable function $h : \Omega \to [1, \infty)$, we define

$$h_- := \operatorname*{ess\,inf}_{x\in\Omega} h(x) \ , \ h_+ := \operatorname*{ess\,sup}_{x\in\Omega} h(x), \tag{2.2}$$

and we introduce the variable exponent Lebesgue space by:

$$L^{h(\cdot)}(\Omega) = \{ \ u : \Omega \to \mathbb{R} \ / \ \rho_{h(\cdot)}(u) \ := \int_\Omega |u(x)|^{h(x)} dx < \infty\}. \tag{2.3}$$

Equipped with the Luxembourg norm

$$\|u\|_{h(\cdot)} \ := \inf\left\{\lambda > 0 \ : \ \rho_{h(\cdot)}\left(\frac{u}{\lambda}\right) \leq 1\right\}, \tag{2.4}$$

$L^{h(\cdot)}(\Omega)$ becomes a Banach space. If

$$1 < h_- \leq h_+ < \infty, \tag{2.5}$$

$L^{h(\cdot)}(\Omega)$ is separable and reflexive. The dual space of $L^{h(\cdot)}(\Omega)$ is $L^{h'(\cdot)}(\Omega)$, where $h'(x)$ is the generalized Hölder conjugate of $h(x)$,

$$\frac{1}{h(x)} + \frac{1}{h'(x)} = 1.$$

From the definitions of the modular $\rho_{h(\cdot)}(u)$ and the norm (2.4), it can be proved that, if (2.5) holds, then

$$\min\left\{\|u\|_{h(\cdot)}^{h_-}, \ \|u\|_{h(\cdot)}^{h_+}\right\} \leq \rho_{h(\cdot)}(u) \leq \max\left\{\|u\|_{h(\cdot)}^{h_-}, \ \|u\|_{h(\cdot)}^{h_+}\right\}. \tag{2.6}$$

One consequence very useful of (2.6) is,

$$\|u\|_{h(\cdot)}^{h_-} - 1 \leq \rho_{h(\cdot)}(u) \leq \|u\|_{h(\cdot)}^{h_+} + 1. \tag{2.7}$$

In particular, if $(u_n)_{n\in\mathbb{N}}$ is a sequence in $L^{h(\cdot)}(\Omega)$, then $\|u_n\|_{L^{h(\cdot)}(\Omega)}$ tends to zero (resp., to infinity) if and only if $\rho_{h(.)}(u_n)$ tends to zero (resp., to infinity), as $n \to +\infty$.

For any functions $u \in L^{h(\cdot)}(\Omega)$ and $v \in L^{h'(\cdot)}(\Omega)$, the generalized Hölder inequality holds:

$$\int_\Omega uv dx \leq (\frac{1}{h_-} + \frac{1}{h'_-})\|u\|_{h(\cdot)}\|v\|_{h'(\cdot)} \leq 2\|u\|_{h(\cdot)}\|v\|_{h'(\cdot)}. \tag{2.8}$$

We define also the generalized Sobolev space by

$$W^{1,h(\cdot)}(\Omega) := \{u \in L^{h(\cdot)}(\Omega) : \nabla u \in L^{h(\cdot)}(\Omega)\},$$

which is a Banach space for the norm

$$\|u\|_{1,h(\cdot)} := \|u\|_{h(\cdot)} + \|\nabla u\|_{h(\cdot)}. \tag{2.9}$$

The space $W^{1,h(\cdot)}(\Omega)$ is separable and is reflexive when (2.5) is satisfied. We have also

$$W^{1,h(\cdot)}(\Omega) \hookrightarrow W^{1,r(\cdot)}(\Omega) \text{ whenever } h(x) \geq r(x) \text{ for a.e. } x \in \Omega. \tag{2.10}$$

For a measurable function $u \in W^{1,h(\cdot)}(\Omega)$ we introduce the following notation:

$$\rho_{1,h(.)}(u) = \int_\Omega |u|^{h(.)} dx + \int_\Omega |\nabla u|^{h(.)} dx.$$

We have the following result that is fundamental in this paper.

Proposition 2.1. *(See [17])*
If $u \in W^{1,h(\cdot)}(\Omega)$, the following properties hold:
i) $\|u\|_{W^{1,h(\cdot)}(\Omega)} > 1 \Rightarrow \|u\|^{h_-}_{W^{1,h(\cdot)}(\Omega)} < \rho_{1,h(.)}(u) < \|u\|^{h_+}_{W^{1,h(\cdot)}(\Omega)}$;
ii) $\|u\|_{W^{1,h(\cdot)}(\Omega)} < 1 \Rightarrow \|u\|^{h_+}_{W^{1,h(\cdot)}(\Omega)} < \rho_{1,h(.)}(u) < \|u\|^{h_-}_{W^{1,h(\cdot)}(\Omega)}$;
iii) $\|u\|_{W^{1,h(\cdot)}(\Omega)} < 1$ *(respectively* $= 1; > 1$*)* $\Leftrightarrow \rho_{1,h(.)}(u) < 1$ *(respectively* $= 1; > 1$*)*.

We give now some embedding results.

Proposition 2.2. *(See [17])*
Assume that $h : \Omega \to [h_-, h_+]$ satisfying the log-Hölder continuity assumption (2.1).
i) Then, $\mathcal{D}(\Omega)$ is dense in $W^{1,h(\cdot)}(\Omega)$.
ii) $W^{1,h(\cdot)}(\Omega)$ is embedded into $L^{h^*(\cdot)}(\Omega)$, where $h^*(.)$ is the Sobolev embedding exponent defined below. If q is a measurable variable exponent such that $\operatorname{ess\,inf}_{x\in\Omega}(h^*(.) - q(.)) > 0$, *then the embedding of $W^{1,h(\cdot)}(\Omega)$ into $L^{q(\cdot)}(\Omega)$ is compact.*
For a given $h(.)$, a function taking values in $[h_-, h_+]$, $h^*(.)$ denotes the optimal Sobolev embedding defined for any $x \in \Omega$ by

$$h^*(x) = \begin{cases} \frac{Nh(x)}{N-h(x)} & \textit{if} \quad h(x) < N \\ \textit{any real value} & \textit{if} \quad h(x) = N \\ +\infty & \textit{if} \quad h(x) > N. \end{cases}$$

Put

$$h^\partial(x) := (h(x))^\partial := \begin{cases} \frac{(N-1)h(x)}{N-h(x)} & \text{if } h(x) < N \\ \infty & \text{if } h(x) \geq N. \end{cases}$$

Proposition 2.3. *(See [17], Proposition 4) Let $h(.) \in C(\bar{\Omega})$ and $h_- > 1$. If $q(x) \in C(\partial\Omega)$ satisfies the condition:*

$$1 \leq q(x) < h^\partial(x), \quad \forall x \in \partial\Omega,$$

then, there is a compact embedding

$$W^{1,h(.)}(\Omega) \hookrightarrow L^{q(.)}(\partial\Omega).$$

In particular there is compact embedding

$$W^{1,h(.)}(\Omega) \hookrightarrow L^{h(.)}(\partial\Omega).$$

Let T_k denote the truncation function at height $k \geqslant 0$:

$$T_k(r) = \min\{k, \max\{r, -k\}\} = \begin{cases} k & \text{if } r \geqslant k \\ r & \text{if } |r| < k, \\ -k & \text{if } r \leqslant -k. \end{cases}$$

For any $u \in W^{1,h(.)}(\Omega)$, we denote by $\tau(u)$ the trace of u on $\partial\Omega$ in the usual sense. We will identify at boundary u and $\tau(u)$. Set

$$\mathcal{T}^{1,h(.)}(\Omega) = \left\{ u : \Omega \to \mathbb{R}, \text{ measurable such that } T_k(u) \in W^{1,h(.)}(\Omega), \text{ for any } k > 0 \right\}.$$

We define $\mathcal{T}^{1,h(.)}_{tr}(\Omega)$ as the set of the functions $u \in \mathcal{T}^{1,h(.)}(\Omega)$ such that there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset W^{1,h_+}(\Omega)$ satisfying the following conditions:

(i) $u_n \to u$ a.e. in Ω.

(ii) $\nabla T_k(u_n) \to \nabla T_k(u)$ in $L^1(\Omega)$.

(iii) There exists a measurable function v on $\partial\Omega$, such that $u_n \to v$ a.e. on $\partial\Omega$.

In the sequel the trace of $u \in \mathcal{T}^{1,h(.)}_{tr}(\Omega)$ on $\partial\Omega$ will be denoted $\operatorname{tr}(u)$. If $u \in W^{1,h(.)}(\Omega), \operatorname{tr}(u)$ coincides with $\tau(u)$ in the usual sense. Moreover, for $u \in \mathcal{T}^{1,h(.)}_{tr}(\Omega)$ and for all $k > 0, \operatorname{tr}(T_k(u)) = T_k(\operatorname{tr}(u))$ and if $\varphi \in W^{1,h(.)}(\Omega)$ then $u - \varphi \in \mathcal{T}^{1,h(.)}_{tr}(\Omega)$ and $\operatorname{tr}(u - \varphi) = \operatorname{tr}(u) - tr(\varphi)$.

Next we define the very weak gradient of a measurable function u with $T_k(u) \in W^{1,h(\cdot)}_0(\Omega)$. The proof follows from Lemma 2.1 of [8] due to the fact that $W^{1,h(\cdot)}_0(\Omega) \subset W^{1,h_-}_0(\Omega)$.

Proposition 2.4. *For every measurable function u with $T_k(u) \in W^{1,h(\cdot)}(\Omega)$, there exists a unique measurable function $v : \Omega \to \mathbb{R}^N$, which we call the very weak gradient of u and denote $v = \nabla u$, such that*

$$\nabla T_k(u) = v\chi_{\{|u|<k\}} \qquad \text{for a.e. } x \in \Omega \text{ and for every } k > 0,$$

where χ_E denotes the characteristic function of a measurable set E.
Moreover, if u belongs to $W^{h(\cdot)}(\Omega)$, then v coincides with the weak gradient of u.

The following lemma prove that the space $W^{1,h(.)}(\Omega)$ is stable by truncation.

Lemma 2.5. *If $u \in W^{1,h(.)}(\Omega)$ then $T_k(u) \in W^{1,h(.)}(\Omega)$.*

Lemma 2.6. *(See [6]) For $\xi, \eta \in \mathbb{R}^N$ and $1 < p < \infty$, we have*

$$\frac{1}{p}|\xi|^p - \frac{1}{p}|\eta|^p \leq |\xi|^{p-2}\xi(\xi - \eta).$$

Lemma 2.7. *For $a \geq 0,\ b \geq 0$ and $1 \leq p < +\infty$, we have*

$$(a + b)^p \leq 2^{p-1}\left(a^p + b^p\right).$$

Lemma 2.8. *(See [6]) Let $(v_n)_{n\in N}$ be a sequence of measurable functions in Ω. If v_n converges in measure to v and is uniformly bounded in $L^{h(\cdot)}(\Omega)$ for some $1 \ll h(.) \in L^\infty(\Omega)$, then v_n strongly converges to v in $L^1(\Omega)$.*

3. Main results

In this section, we prove the existence of entropy solutions of problem (1.1). Firstly, we state the following assumptions:

(H_0) α is continuous function defined on $\mathbb{R}$ such that $\alpha(x).x \geq 0$ for all $x \in \mathbb{R}$.

(H_1) $f \in L^1(\Omega)$ and $g \in L^1(\partial\Omega)$.

(H_2) $\Theta : \mathbb{R} \to \mathbb{R}^N$ is a continuous function such that $\Theta(0) = 0$ and $|\Theta(x) - \Theta(y)| \leq \lambda|x - y|$, for all $x, y \in \mathbb{R}$, where λ is a positive constant such that $\lambda < \dfrac{1}{2C_0}$, and C_0 is the constant given by the Poincaré's inequality.

Now, we give a definition of entropy solutions for the elliptic problem (1.1).

Definition 3.1. *A measurable function u with $u \in \mathcal{T}_{tr}^{1,p(u(\cdot))}(\Omega)$ is said to be an entropy solution for the problem (1.1), if $|u|^{p(u)-2}u \in L^1(\Omega)$, $\alpha(u) \in L^1(\Omega)$, $u \in L^1(\partial\Omega)$ and*

$$\int_\Omega \Phi\left(\nabla u - \Theta(u)\right) \nabla T_k(u-\varphi)dx + \int_\Omega |u|^{p(u)-2}uT_k(u-\varphi)dx + \int_\Omega \alpha(u)T_k(u-\varphi)dx$$
$$+\lambda \int_{\partial\Omega} uT_k(u-\varphi)d\sigma \leqslant \int_\Omega fT_k(u-\varphi)dx + \int_{\partial\Omega} gT_k(u-\varphi)d\sigma, \tag{3.1}$$

for all $\varphi \in W^{1,p(u(\cdot))}(\Omega) \cap L^\infty(\Omega)$ and for every $k>0$, with

$$\Phi\left(\xi\right) = |\xi|^{p(u)-2}\xi \quad \forall \xi \in \mathbb{R}^N.$$

Theorem 3.2. *Let (H_0)-(H_2) be satisfied. Then there exists at least one entropy solution of the problem (1.1) in the sense of the Definition 3.1.*

The proof of Theorem (3.2) is divided into into several steps.
Step 1: The approximate problem.

We consider the sequence of approximate problems

$$(\mathcal{P}_n)\begin{cases} -\text{div}(\Phi\left(\nabla u_n - \Theta(u_n)\right)) + |u_n|^{p(u_n)-2}u_n + T_n(\alpha(u_n)) - \varepsilon\Delta_{p_+} + \varepsilon|u_n|^{p_+-2}u_n = T_n(f) \text{ in } \Omega \\ \left(\Phi\left(\nabla u_n - \Theta(u_n)\right) + \varepsilon|\nabla u_n|^{p_+-2}\nabla u_n\right).\eta + \lambda T_n(u_n) = T_n(g) \qquad \text{on } \partial\Omega, \end{cases}$$

where

$$\Phi\left(\xi\right) = |\xi|^{p(u_n)-2}\xi \quad \forall \xi \in \mathbb{R}^N.$$

We define the following reflexive space

$$E = W^{1,p_+}(\Omega) \times L^{p_+}(\partial\Omega).$$

Let

$$X_0 = \{(u,v) \in E : \quad v = \tau(u)\}.$$

In the sequel, we will identify an element $(u,v) \in X_0$ with its representative $u \in W^{1,p_+}(\Omega)$ (since $W^{1,p_+}(\Omega) \hookrightarrow\hookrightarrow L^{p_+}(\partial\Omega)\big)$.

We define the operator A_n by

$$\langle A_n u, v\rangle = \langle Au, v\rangle + \int_\Omega T_n(\alpha(u))vdx + \lambda \int_{\partial\Omega} T_n(u)vd\sigma + \varepsilon \int_\Omega \left[|\nabla u|^{p_+-2}\nabla u \nabla v + |u|^{p_+-2}uv\right]dx, \quad \text{with} \quad u,v \in X_0,$$

where

$$\langle Au, v\rangle = \int_\Omega \Phi(\nabla u - \Theta(u))\nabla v dx + \int_\Omega |u|^{p(u)-2}uvdx.$$

Assertion 1. The operator A_n is coercive.
From Lemma 2.6, we obtain

$$\begin{aligned}\langle Au, u\rangle &= \int_\Omega \Phi(\nabla u - \Theta(u))\nabla u dx + \int_\Omega |u|^{p(u)}dx \\ &= \int_\Omega |\nabla u - \Theta(u)|^{p(u)-2}(\nabla u - \Theta(u))\nabla u dx + \int_\Omega |u|^{p(u)}dx \\ &\geq \int_\Omega \frac{1}{p(u)}|\nabla u - \Theta(u)|^{p(u)}dx - \int_\Omega \frac{1}{p(u)}|\Theta(u)|^{p(u)}dx + \int_\Omega |u|^{p(u)}dx.\end{aligned}$$

Since

$$(a+b)^p \leq 2^{p-1}\left(|a|^p + |b|^p\right),$$

we have

$$\frac{1}{2^{p_+-1}}|\nabla u|^{p(u)} = \frac{1}{2^{p_+-1}}|\nabla u - \Theta(u) + \Theta(u)|^{p(u)} \\ \leq |\nabla u - \Theta(u)|^{p(u)} + |\Theta(u)|^{p(u)},$$

then

$$\frac{1}{2^{p_+-1}}|\nabla u|^{p(u)} - |\Theta(u)|^{p(u)} \leq |\nabla u - \Theta(u)|^{p(u)}.$$

Therefore, from Poincaré's inequality we get

$$\begin{aligned}
\langle Au, u\rangle &\geq \int_\Omega \frac{1}{p(u)}\left[\frac{1}{2^{p_+-1}}|\nabla u|^{p(u)} - |\Theta(u)|^{p(u)}\right]dx - \int_\Omega \frac{1}{p(u)}|\Theta(u)|^{p(u)}dx + \int_\Omega |u|^{p(u)}dx \\
&\geq \int_\Omega \frac{1}{p(u)}\frac{1}{2^{p_+-1}}|\nabla u|^{p(u)}dx - \int_\Omega \frac{2}{p(u)}|\Theta(u)|^{p(u)}dx \\
&\geq \int_\Omega \frac{1}{p(u)}\frac{1}{2^{p_+-1}}|\nabla u|^{p(u)}dx - \int_\Omega \frac{2}{p(u)}\lambda^{p(u)}|u|^{p(u)}dx \\
&\geq \int_\Omega \frac{1}{p_+}\frac{1}{2^{p_+-1}}|\nabla u|^{p(u)}dx + \int_\Omega \left(1 - \frac{2}{p_-}\lambda^{p(u)}\right)|u|^{p(u)}dx.
\end{aligned}$$

So the choice of the constant λ in (H_2) gives the existence of a positive constant M_0 such that

$$\langle Au, u\rangle \geq \min\{\frac{1}{p_+}\frac{1}{2^{p_+-1}}, M_0\}\left(\int_\Omega |\nabla u|^{p(u)}dx + \int_\Omega |u|^{p(u)}dx\right). \tag{3.2}$$

On the other hand, we have

$$\int_\Omega T_n(\alpha(u))u dx + \lambda \int_{\partial\Omega} T_n(u)u d\sigma \geq 0. \tag{3.3}$$

By (3.2) and (3.3), we get

$$\langle A_n u, u\rangle \geq \varepsilon \int_\Omega \left[|\nabla u|^{p_+} + |u|^{p_+}\right]dx \\ \geq \varepsilon \|\nabla u\|^{p_+}_{W^{1,p_+}}.$$

Consequently

$$\frac{\langle A_n u, u\rangle}{\|u\|_{W^{1,p_+}(\Omega)}} \longrightarrow +\infty \quad \text{as} \quad \|u\|_{W^{1,p_+}(\Omega)} \to +\infty. \tag{3.4}$$

We deduce that the operator A_n is coercive.
Assertion 2. The operator A_n is of type (M).
Let $(u_k)_k$ be a sequence in X_0 such that

$$\begin{cases} u_k \rightharpoonup u \ \ in \ \ X_0 \\ A_n u_k \rightharpoonup \chi \ \ in \ X_0' \\ \limsup\limits_{k\to+\infty}\langle A_n u_k, u_k\rangle \leq \langle \chi, u\rangle. \end{cases}$$

We will prove that $\chi = A_n u$.
As

$$T_n(\alpha(u_k))u_k \geq 0 \ \text{ and } \ \lambda T_n(u_k)u_k \geq 0,$$

by Fatou's Lemma, we deduce that

$$\liminf_{k\to\infty}\left(\int_\Omega T_n(\alpha(u_k))u_k dx + \lambda\int_{\partial\Omega} T_n(u_k)u_k d\sigma\right) \geq \int_\Omega T_n(\alpha(u))u dx + \lambda \int_{\partial\Omega} T_n(u)u d\sigma.$$

On the other hand, thanks to the Lebesgue dominated convergence Theorem, we have

$$\lim_{k\to\infty}\left(\int_\Omega T_n\left(b\left(u_k\right)\right)vdx+\lambda\int_{\partial\Omega}T_n\left(u_k\right)vd\sigma+\varepsilon\int_\Omega\left[\left|u_k\right|^{p_+-2}u_kv+\left|\nabla u_k\right|^{p_+-2}\nabla u_k\nabla v\right]dx\right)$$
$$=\int_\Omega T_n(b(u))vdx+\lambda\int_{\partial\Omega}T_n(u)vd\sigma+\varepsilon\int_\Omega\left[|u|^{p_+-2}uv+|\nabla u|^{p_+-2}\nabla u\nabla v\right]dx,$$

for any $v\in X_0$. Therefore, for k large enough,

$$T_n\left(b\left(u_k\right)\right)+\lambda T_n\left(u_k\right)+\varepsilon\left[\left|u_k\right|^{p_+-2}u_k+\left|\nabla u_k\right|^{p_+-2}\nabla u_k\right]\rightharpoonup T_n(b(u))+\lambda T_n(u)+\varepsilon\left[|u|^{p_+-2}u+|\nabla u|^{p_+-2}\nabla u\right]\text{ in }X_0'.$$

Hence,

$$Au_k\rightharpoonup\chi-\left(T_n(b(u))+\lambda T_n(u)+\varepsilon\left[|u|^{p_+-2}u+|\nabla u|^{p_+-2}\nabla u\right]\right)\text{ in }X_0',\text{ as }k\to+\infty.$$

As the operator A is of type (M), so we have immediately

$$Au=\chi-\left(T_n(b(u))+\lambda T_n(u)+\varepsilon\left[|u|^{p_+-2}u+|\nabla u|^{p_+-2}\nabla u\right]\right).$$

Therefore, we conclude that $A_nu=\chi$.

Besides, the operator A is bounded and hemi-continuous. Therefore, A_n is surjective. Thus, for any $F_n=\langle T_n(f),T_n(g)\rangle\subset E'\subset X_0'$, we can deduce the existence of a solution $u_n\in X_0$ of the problem

$$\langle A_nu,v\rangle=\langle F_nu,v\rangle\text{ for all }v\in X_0.$$

i.e.

$$\int_\Omega|\nabla u_n-\Theta(u_n)|^{p(u_n)-2}\left(\nabla u_n-\Theta(u_n)\right))\nabla vdx+\int_\Omega|u|^{p(u_n)-2}u_nvdx+\int_\Omega T_n\left(\alpha(u_n)\right)vdx$$
$$+\lambda\int_{\partial\Omega}T_n\left(u_n\right)vd\sigma+\varepsilon\int_\Omega\left[\left|u_n\right|^{p_+-2}u_nv+\left|\nabla u_n\right|^{p_+-2}\nabla u_n\nabla v\right]dx=\int_\Omega T_n(f)vdx+\int_{\partial\Omega}T_n(g)vd\sigma. \tag{3.5}$$

Our aim is to prove that a subsequence of these approximate solutions $\{u_n\}$ converges to a measurable function u, which is an entropy solution to (1.1).

Step 2: a priori estimate.

Lemma 3.3. $(\nabla T_k(u_n))_{n\in\mathbb{N}}$ *is bounded in* $L^{p_-}(\Omega)$.

Proof. We take $\varphi=T_k(u_n)$ as a test function in (3.5), we obtain

$$\int_\Omega\Phi\left(\nabla u_n-\Theta(u_n)\right)\nabla T_k(u_n)dx+\int_\Omega|u|^{p(u_n)-2}u_nT_k(u_n)dx$$
$$+\int_\Omega T_n\left(\alpha(u_n)\right)T_k(u_n)dx+\lambda\int_{\partial\Omega}T_n\left(u_n\right)T_k(u_n)d\sigma$$
$$+\varepsilon\int_\Omega\left[|u|^{p_+-2}uT_k(u_n)+|\nabla u|^{p_+-2}\nabla u\nabla T_k(u_n)\right]dx=\int_\Omega T_n(f)T_k(u_n)dx+\int_{\partial\Omega}T_n(g)T_k(u_n)d\sigma.$$

Since the third, the fourth and the fifth terms in the left-hand side of equality above are nonnegative then

$$\int_\Omega\Phi\left(\nabla u_n-\Theta(u_n)\right)\nabla T_k(u_n)dx+\int_\Omega|u|^{p(u_n)-2}u_nT_k(u_n)dx\le k\left(\|f\|_{L^1(\Omega)}+\|g\|_{L^1(\partial\Omega)}\right). \tag{3.6}$$

We have

$$\int_\Omega|u_n|^{p(u_n)-2}u_nT_k(u_n)dx\ge\int_{\{|u_n|\le k\}}|T_k(u_n)|^{p(u_n)}dx+\int_{\{|u_n|>k\}}k^{p(u_n)}dx$$
$$\le\int_\Omega|T_k(u_n)|^{p(u_n)}dx.$$

Then by (3.6), we get

$$\int_{\Omega} \Phi\left(\nabla T_k(u_n) - \Theta(u_n)\right) \nabla T_k(u_n) dx + \int_{\Omega} |T_k(u_n)|^{p(u_n)} dx \leq k\left(\|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)}\right). \tag{3.7}$$

By the same way as in the proof of the coerciveness, we get

$$\rho_{1,p(u_n)}(T_k(u_n)) \leqslant Ck.$$

Therefore,

$$\|T_k(u_n)\|_{1,p(u_n)} \leqslant 1 + (Ck)^{\frac{1}{p_-}},$$

we deduce that for any $k > 0$, the sequence $(T_k(u_n))_{n\in\mathbb{N}}$ is uniformly bounded in $W^{1,p(u_n(\cdot))}(\Omega)$ and also in $W^{1,p_-}(\Omega)$. Then, up to a subsequence still denoted $T_k(u_n)$, we can assume that for any $k > 0$, $T_k(u_n)$ weakly converges to ν_k in $W^{1,p_-}(\Omega)$ and also $T_k(u_n)$ strongly converges to ν_k in $L^{p_-}(\Omega)$.

Lemma 3.4. *$(u_n)_{n\in\mathbb{N}}$ converges in measure to some measurable function u.*

Proof. Firstly, we prove that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in measure. For every fixed $\delta > 0$, and every positive integer $k > 0$, we know that

$$meas\left\{|u_n - u_m| > \delta\right\} \leq meas\left\{|u_n| > k\right\} + meas\left\{|u_m| > k\right\} + meas\left\{|T_k\left(u_n\right) - T_k\left(u_m\right)| > \delta\right\}.$$

Choosing $T_k\left(u_n\right)$ as a test function in (3.5), we get

$$\rho_{1,p(u_n)}(T_k(u_n)) \leqslant k\left(\|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)}\right). \tag{3.8}$$

It follows that

$$\int_{\{|u_n|>k\}} k^{p(u_n)} dx \leqslant k\left(\|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)}\right).$$

Therefore

$$meas\left\{|u_n| > k\right\} \leqslant k^{1-p_-}\left(\|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)}\right).$$

Hence

$$meas\left\{|u_n| > k\right\} \to 0 \text{ as } k \to +\infty.$$

Let $\varepsilon > 0$, we choose $k = k(\varepsilon)$ such that

$$meas\left\{|u_n| > k\right\} \leqslant \frac{\varepsilon}{3} \quad \text{and} \quad meas\left\{|u_m| > k\right\} \leqslant \frac{\varepsilon}{3}.$$

Since $\{T_k\left(u_n\right)\}$ converges strongly in $L^{p_-}(\Omega)$, then it is a Cauchy sequence. Thus

$$meas\left\{|T_k(u_n) - T_k(u_m)| > \delta\right\} \leqslant \frac{\varepsilon}{3},$$

for all $n, m \geqslant n_0(\delta, \varepsilon)$.
Finally, we obtain

$$meas\left\{|u_n - u_m| > \delta\right\} \leqslant \varepsilon,$$

for all $n, m \geqslant n_0(\delta, \varepsilon)$.
Hence

$$\limsup_{n,m\to\infty} \text{meas}\left\{|u_n - u_m| > \delta\right\} = 0,$$

which proves that the sequence $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in measure and then converges almost everywhere to some measurable function u.

$$u_n \to u \quad \text{a.e in} \quad \Omega. \tag{3.9}$$

Therefore

$$T_k(u_n) \rightharpoonup T_k(u) \text{ in } W_0^{1,p_-}(\Omega),$$

$$T_k(u_n) \longrightarrow T_k(u) \text{ in } L^{p_-}(\Omega) \text{ and a.e. in } \Omega.$$

Lemma 3.5. *u_n converges almost everywhere in $\partial\Omega$ to some function v.*

Proof. We have

$$T_k\left(u_n\right) \rightarrow T_k(u) \text{ in } W^{1,p_-}(\Omega) \text{ and } W^{1,p_-}(\Omega) \hookrightarrow L^{p_-}(\partial\Omega),$$

then

$$T_k\left(u_n\right) \rightarrow T_k(u) \text{ in } L^{p_-}(\partial\Omega) \text{ and a.e. on } \partial\Omega,$$

hence

$$T_k\left(u_n\right) \rightarrow T_k(u) \text{ in } L^1(\partial\Omega) \text{ and a.e. in } \partial\Omega.$$

Therefore, there exists $A \subset \partial\Omega$ such that $T_k\left(u_n\right) \rightarrow T_k(u)$ on $\partial\Omega \backslash A$ with $\mu(A)=0$, where μ is area measure on $\partial\Omega$.
For every $k>0$, let $A_k=\{x \in \partial\Omega: \ |T_k(u)|<k\}$ and $B=\partial\Omega \backslash \bigcup_{k>0} A_k$. By using Fatou's Lemma, we have

$$\begin{aligned}\int_{\partial\Omega}\left|T_k(u)\right| d\sigma & \leq \liminf_{n \rightarrow+\infty} \int_{\partial\Omega}\left|T_k\left(u_n\right)\right| d\sigma \\ & \leq \frac{\|f\|_{L^1(\Omega)}+\|g\|_{L^1(\partial\Omega)}}{\lambda}.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\mu(B)=\frac{1}{k} \int_B\left|T_k(u)\right| d\sigma & \leq \frac{1}{k} \int_{\partial\Omega}\left|T_k(u)\right| d\sigma \\ & \leq \frac{\|f\|_{L^1(\Omega)}+\|g\|_{L^1(\partial\Omega)}}{k\lambda}.\end{aligned}$$

We get $\mu(B)=0$, as k goes to ∞. Let' s now define on $\partial\Omega$ the function v by

$$v(x)=T_k(u(x)), \quad x \in A_k.$$

We take $x \in \partial\Omega \backslash(E \cup F)$, then there exists $k>0$ such that $x \in E_k$ and we have

$$u_n(x)-v(x)=\left(u_n(x)-T_k\left(u_n(x)\right)\right)+\left(T_k\left(u_n(x)\right)-T_k(u(x))\right).$$

Since $x \in E_k$, then $|T_k(u(x))|<k$ and so $\left|T_k\left(u_n(x)\right)\right|<k$, from which we deduce that $|u_n(x)|<k$. Therefore,

$$u_n(x)-v(x)=T_k\left(u_n(x)\right)-T_k(u(x)) \rightarrow 0, \text{ as } n \rightarrow+\infty.$$

Which means that u_n converges to v a.e. on $\partial\Omega$, but for all $x \in E_k, T_k(u(x))=u(x)$. Thus, $v=u$ a.e. on $\partial\Omega$. Therefore,

$$u_n \rightarrow u \text{ a.e. on } \partial\Omega.$$

Lemma 3.6. *$\left(\nabla u_n\right)_{n \in \mathbb{N}}$ converges almost everywhere in Ω to ∇u.*

Proof. We first prove that $\{\nabla u_n\}$ is a Cauchy sequence in measure. Let δ, h, ε are positive real numbers, obviously we have

$$\{x \in \Omega:|\nabla u_n-\nabla u_m|>\delta\} \subset\{x \in \Omega:|\nabla u_n|>h\} \cup\{x \in \Omega:|\nabla u_m|>h\} \cup\{x \in \Omega:|u_n-u_m|>1\} \cup E,$$

where

$$E:=\{x \in \Omega:|\nabla u_n| \leqslant h,|\nabla u_m| \leqslant h,|u_n-u_m| \leqslant 1,|\nabla u_n-\nabla u_m|>\delta\}.$$

For $k>0$, we can write

$$\{x \in \Omega:|\nabla u_n| \geqslant h\} \subset\{x \in \Omega:|u_n| \geqslant k\} \cup\left\{x \in \Omega:\left|\nabla T_k\left(u_n\right)\right| \geqslant h\right\},$$

then by using the same method us in Lemma 3.4 we obtain for k sufficiently large,

$$\operatorname{meas}\{\{x \in \Omega:|\nabla u_n|>h\} \cup\{x \in \Omega:|\nabla u_m|>h\} \cup\{x \in \Omega:|u_n-u_m|>1\}\} \leqslant \frac{\varepsilon}{2}.$$

Notice that the application

$$\mathcal{G} : (s, t, \xi_1, \xi_2) \mapsto \left(\Phi\left(\xi_1 - \Theta(s)\right) - \Phi\left(\xi_2 - \Theta(t)\right)\right)\left(\xi_1 - \xi_2\right)$$

is continuous and the set

$$\mathcal{H} := \left\{(s, t, \xi_1, \xi_2) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N, |s| \le h, |t| \le h, |\xi_1| \le h, |\xi_2| \le h, |\xi_1 - \xi_2| > \delta\right\}$$

is compact and

$$\left(\Phi\left(\xi_1 - \Theta(s)\right) - \Phi\left(\xi_2 - \Theta(t)\right)\right)\left(\xi_1 - \xi_2\right) > 0, \quad \forall \xi_1 \neq \xi_2.$$

Then, the application $\mathcal{G}$ has its minimum on $\mathcal{H}$. Therefore, there exists a real valued function $\beta(h,\delta) > 0$ such that

$$\begin{aligned}
\beta(h, \delta) meas(E) \le \int_E \Big[& |\nabla u_n - \Theta(u_n)|^{p(u_n)-2}\left(\nabla u_n - \Theta(u_n)\right) \\
& - |\nabla u_m - \Theta(u_m)|^{p(u_n)-2}\left(\nabla u_m - \Theta(u_m)\right)\Big]\left[\nabla u_n - \nabla u_m\right] dx, \\
= \int_E \Big[& |\nabla u_m - \Theta(u_m)|^{p(u_m)-2}\left(\nabla u_m - \Theta(u_m)\right) \\
& - |\nabla u_m - \Theta(u_m)|^{p(u_n)-2}\left(\nabla u_m - \Theta(u_m)\right)\Big]\left[\nabla u_n - \nabla u_m\right] dx \\
& + \int_E \Big[|\nabla u_n - \Theta(u_n)|^{p(u_n)-2}\left(\nabla u_n - \Theta(u_n)\right) \\
& - |\nabla u_m - \Theta(u_m)|^{p(u_m)-2}\left(\nabla u_m - \Theta(u_m)\right)\Big]\left[\nabla u_n - \nabla u_m\right] dx.
\end{aligned}$$

We take $T_\nu(u_n - u_m)$ as a test function in (3.5) to get

$$\begin{aligned}
& \beta(h, \delta) meas(E) \leqslant \\
& \int_E \left[|\nabla u_m - \Theta(u_m)|^{p(u_m)-2}\left(\nabla u_m - \Theta(u_m)\right) - |\nabla u_m - \Theta(u_m)|^{p(u_n)-2}\left(\nabla u_m - \Theta(u_m)\right)\right]\left[\nabla u_n - \nabla u_m\right] dx \\
& - \int_\Omega \left(|u_n|^{p(u_n)-2} u_n - |u_m|^{p(u_m)-2} u_m\right) T_\nu(u_n - u_m)\, dx - \int_\Omega \left(T_n(\alpha(u_n)) - T_m(\alpha(u_m))\right) T_\nu(u_n - u_m)\, dx \\
& - \lambda \int_{\partial\Omega} \left(T_n(u_n) - T_m(u_m)\right) T_\nu(u_n - u_m)\, dx - \varepsilon \int_\Omega \left(|\nabla u_n|^{p_+-2}\nabla u_n - |\nabla u_m|^{p_+-2}\nabla u_m\right)\nabla T_\nu(u_n - u_m)\, dx \\
& + \int_\Omega \left[T_n(f) - T_m(f)\right] T_\nu(u_n - u_m)\, dx + \int_{\partial\Omega} \left[T_n(g) - T_m(g)\right] T_\nu(u_n - u_m)\, dx,
\end{aligned}$$

from the fact that $\|T_n(\alpha(u_n))\|_{L^1(\Omega)} + \lambda \|T_n(u_n)\|_{L^1(\partial\Omega)} \le \|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)}$, we obtain

$$\begin{aligned}
& \beta(h, \delta) meas(E) \leqslant \int_E \Big[|\nabla u_m - \Theta(u_m)|^{p(u_m)-2}\left(\nabla u_m - \Theta(u_m)\right) \\
& - |\nabla u_m - \Theta(u_m)|^{p(u_n)-2}\left(\nabla u_m - \Theta(u_m)\right)\Big]\left[\nabla u_n - \nabla u_m\right] dx \\
& - \int_\Omega \left(|u_n|^{p(u_n)-2} u_n - |u_m|^{p(u_m)-2} u_m\right) T_\nu(u_n - u_m)\, dx + \nu\left(2\|f\|_{L^1(\Omega)} + 2\|g\|_{L^1(\partial\Omega)}\right) \\
& \qquad + \nu\|T_n(f) - T_m(f)\|_{L^1(\Omega)} + \nu\|T_n(g) - T_m(g)\|_{L^1(\partial\Omega)},
\end{aligned} \tag{3.10}$$

by using the mean value theorem, there exists η taking values between $p(u_n)$ and $p(u_m)$ such that

$$\begin{aligned}
& \int_E \left[|\nabla u_m - \Theta(u_m)|^{p(u_m)-2}\left(\nabla u_m - \Theta(u_m)\right) - |\nabla u_m - \Theta(u_m)|^{p(u_n)-2}\left(\nabla u_m - \Theta(u_m)\right)\right]\left[\nabla u_n - \nabla u_m\right] dx \\
& \qquad \leqslant \int_E |\nabla u_m - \Theta(u_m)|^{\eta-1} \left|\log|\nabla u_m - \Theta(u_m)|\right| \cdot |\nabla u_n - \nabla u_m| \cdot |p(u_m) - p(u_n)|\, dx.
\end{aligned}$$

By using Lemma 2.7, (H_2), the facts that $h \gg 1$ and the definition of E, we get

$$\int_E \left[|\nabla u_m - \Theta(u_m)|^{p(u_m)-2}(\nabla u_m - \Theta(u_m)) - |\nabla u_m - \Theta(u_m)|^{p(u_n)-2}(\nabla u_m - \Theta(u_m))\right][\nabla u_n - \nabla u_m]\,dx$$
$$\leqslant 2^{p^+} h^{p^+}\left(1+\lambda^{\eta-1}\right)\log((1+\lambda)h)\cdot\int_\Omega |p(u_m) - p(u_n)|\,dx.$$

Therefore, from (3.10) and the Lebesgue dominated convergence theorem we obtain

$$\operatorname{meas}(E) \leqslant \frac{\varepsilon}{2},$$

for all $n, m \geqslant N_2(\varepsilon,\delta)$. Consequently, Combining the previous results we get

$$\operatorname{meas}\ \{x \in \Omega : |\nabla u_n - \nabla u_m| > \delta\} \leqslant \varepsilon, \qquad \text{for all } n, m \geqslant \max\{N_1, N_2\},$$

hence $\{\nabla u_n\}$ is a Cauchy sequence in measure. Then we can choose a subsequence (denote it by the original sequence) such that

$$\nabla u_n \to v \quad \text{ a.e. in } \Omega.$$

Thus, using Proposition 2.4 and the fact that $\nabla T_k(u_n) \to \nabla T_k(u)$ in $(L^{p_-}(\Omega))^N$, we deduce that v coincides with the very weak gradient of u almost everywhere. Therefore, we have

$$\nabla u_n \to \nabla u \quad \text{ a.e. in } \Omega. \tag{3.11}$$

Step 3: Passing to the limit.
Since the sequence $(\nabla T_k(u_n))_{n\in\mathbb{N}}$ converges in measure to $\nabla T_k(u)$, then from Lemma 2.8, we get

$$\nabla T_k(u_n) \to \nabla T_k(u) \quad \text{ in } \left(L^1(\Omega)\right)^N. \tag{3.12}$$

Consequently, by using Lemma 3.4, 3.5 and (3.12) we get $u \in \mathcal{T}_{tr}^{1,p(u(\cdot))}(\Omega)$.
Let $\phi \in \mathcal{C}^\infty(\overline{\Omega})$, since $\mathcal{C}^\infty(\overline{\Omega})$ is dense in the space $W^{1,p_+}(\Omega)$ and $T_k(u_n - \phi) \in L^\infty(\partial\Omega)$, then we can choose $T_k(u_n-\phi)$ as a test function in (3.5) to obtain

$$\begin{aligned}
&\int_\Omega \Phi(\nabla u_n - \Theta(u_n))\nabla T_k(u_n-\phi)\,dx + \int_\Omega |u|^{p(u_n)-2}u_n T_k(u_n-\phi)\,dx + \int_\Omega T_n(\alpha(u_n))T_k(u_n-\phi)\,dx\\
&+\lambda\int_{\partial\Omega} T_n(u_n)T_k(u_n-\phi)\,d\sigma + \varepsilon\int_\Omega\left[|u_n|^{p_+-2}u_nT_k(u_n-\phi) + |\nabla u_n|^{p_+-2}\nabla u_n\nabla T_k(u_n-\phi)\right]dx\\
&=\int_\Omega T_n(f)T_k(u_n-\phi)\,dx + \int_{\partial\Omega} T_n(g)T_k(u_n-\phi)\,d\sigma.
\end{aligned}\tag{3.13}$$

We now focus our attention on the first term in left-hand side of (3.13).
We note that, if $L = k + \|\phi\|_{L^\infty(\Omega)}$, we have

$$\begin{aligned}
&\int_\Omega |\nabla u_n - \Theta(u_n)|^{p(u_n)-2}(\nabla u_n - \Theta(u_n))\cdot\nabla T_k(u_n-\phi)\,dx\\
&=\int_\Omega |\nabla T_L(u_n) - \Theta(T_L(u_n))|^{p(u_n)-2}(\nabla T_L(u_n) - \Theta(T_L(u_n)))\cdot\nabla T_k(T_L(u_n)-\phi)\,dx\\
&=\int_\Omega |\nabla T_L(u_n) - \Theta(T_L(u_n))|^{p(u_n)-2}(\nabla T_L(u_n) - \Theta(T_L(u_n)))\cdot\nabla T_L(u_n)\,\chi_{\{|T_L(u_n)-\phi|\leqslant k\}}dx\\
&\quad-\int_\Omega |\nabla T_L(u_n) - \Theta(T_L(u_n))|^{p(u_n)-2}(\nabla T_L(u_n) - \Theta(T_L(u_n)))\cdot\nabla\varphi\chi_{\{|T_L(u_n)-\phi|\leqslant k\}}dx.
\end{aligned}\tag{3.14}$$

From (3.13), we have

$$\begin{aligned}&\int_\Omega\left[|\nabla T_L(u_n)-\Theta(T_L(u_n))|^{p(u_n)-2}(\nabla T_L(u_n)-\Theta(T_L(u_n)))\cdot\nabla T_L(u_n)+\frac{1}{p_-}|\Theta(T_L(u_n)|^\gamma\right]\chi_{\{|T_L(u_n)-\phi|\leqslant k\}}dx\\&-\int_\Omega|\nabla T_L(u_n)-\Theta(T_L(u_n))|^{p(u_n)-2}(\nabla T_L(u_n)-\Theta(T_L(u_n)))\cdot\nabla\varphi\chi_{\{|T_L(u_n)-\phi|\leqslant k\}}dx\\&+\int_\Omega|u|^{p(u_n)-2}u_nT_k(u_n-\phi)\,dx+\int_\Omega T_n(\alpha(u_n))T_k(u_n-\phi)\,dx+\lambda\int_{\partial\Omega}T_n(u_n)T_k(u_n-\phi)\,d\sigma\\&+\varepsilon\int_\Omega\left[|u_n|^{p_+-2}u_nT_k(u_n-\phi)+|\nabla u_n|^{p_+-2}\nabla u_n\nabla T_k(u_n-\phi)\right]dx\\&=\int_\Omega f_nT_k(u_n-\phi)\,dx+\int_{\partial\Omega}T_n(g)T_k(u_n-\phi)\,d\sigma+\int_\Omega\frac{1}{p_-}|\Theta(T_L(u_n)|^\gamma\chi_{\{|T_L(u_n)-\phi|\leqslant k\}}dx,\end{aligned}\tag{3.15}$$

where

$$\gamma=\begin{cases}p_+ & \text{if}\quad|\Theta(T_L(u_n))|\le 1,\\ p_- & \text{if}\quad |\Theta(T_L(u_n))|>1.\end{cases}$$

Since $\{\nabla T_L(u_n)\}$ is bounded in $\left(L^{p'(u_n)}(\Omega)\right)^N\subset\left(L^{p'_+}(\Omega)\right)^N$, then from the hypothesis (H_3) the sequence $\{\Theta(T_L(u_n)\}$ is also bounded in $\left(L^{p(u_n)}(\Omega)\right)^N\subset(L^{p_-}(\Omega))^N$, which implies that $\left\{|\nabla T_L(u_n)-\Theta(T_L(u_n))|^{p(u_n)-2}(\nabla T_L(u_n)-\Theta(T_L(u_n)))\right\}$ is bounded in $\left(L^{p'(u_n)}(\Omega)\right)^N\subset\left(L^{p'_+}(\Omega)\right)^N$. On account of the fact that $u_n\to u$ a.e. in Ω and $\nabla u_n\to\nabla u$ a.e. in Ω,

$$\Theta(T_L(u_n))\longrightarrow\Theta(T_L(u))\ \text{ a.e. in }\Omega\tag{3.16}$$

and

$$\nabla T_L(u_n)\longrightarrow\nabla T_L(u)\text{ a.e. in }\Omega.\tag{3.17}$$

Hence follows that

$$\begin{aligned}&|\nabla T_L(u_n)-\Theta(T_L(u_n))|^{p(u_n)-2}(\nabla T_L(u_n)-\Theta(T_L(u_n)))\\&\quad\rightharpoonup|\nabla T_L(u)-\Theta(T_L(u))|^{p(u)-2}(\nabla T_L(u)-\Theta(T_L(u)))\quad\text{ in }\left(L^{p'_+}(\Omega)\right)^N.\end{aligned}$$

Therefore

$$\begin{aligned}&\int_\Omega|\nabla T_L(u_n)-\Theta(T_L(u_n))|^{p(u_n)-2}(\nabla T_L(u_n)-\Theta(T_L(u_n)))\cdot\nabla\varphi\chi_{\{|T_L(u_n)-\phi|\leqslant k\}}dx\\&\longrightarrow\int_\Omega|\nabla T_L(u)-\Theta(T_L(u))|^{p(u)-2}(\nabla T_L(u)-\Theta(T_L(u)))\cdot\nabla\varphi\chi_{\{|T_L(u)-\phi|\leqslant k\}}dx,\quad\text{ as }n\to\infty.\end{aligned}\tag{3.18}$$

From (3.16) and the Lebesgue dominated convergence theorem, we obtain

$$\int_\Omega\frac{1}{p_-}|\Theta(T_L(u_n)|^\gamma\chi_{\{|T_L(u_n)-\phi|\leqslant k\}}dx\to\int_\Omega\frac{1}{p_-}|\Theta(T_L(u)|^\gamma\chi_{\{|T_L(u)-\phi|\leqslant k\}}dx.$$

On the other hand, by using Lemma 2.6 we have

$$\begin{aligned}\Big[&|\nabla T_L(u_n)-\Theta(T_L(u_n))|^{p(u_n)-2}(\nabla T_L(u_n)-\Theta(T_L(u_n)))\cdot\nabla T_L(u_n)\\&+\frac{1}{p_-}|\Theta(T_L(u_n)|^\gamma\Big]\chi_{\{|T_L(u_n)-\phi|\leqslant k\}}\geqslant 0\text{ a.e. in }\Omega.\end{aligned}$$

By using Fatou's Lemma, we get

$$\begin{aligned}\int_\Omega \Bigg[& |\nabla T_L(u) - \Theta(T_L(u))|^{p(u)-2} (\nabla T_L(u) - \Theta(T_L(u))) \cdot \nabla T_L(u) \\ & + \frac{1}{p_-} |\Theta(T_L(u)|^\gamma \Bigg] \chi_{\{|T_L(u)-\phi| \leqslant k\}} dx \\ \leq \liminf_{n\to\infty} \int_\Omega \Bigg[& |\nabla T_L(u_n) - \Theta(T_L(u_n))|^{p(u_n)-2} (\nabla T_L(u_n) - \Theta(T_L(u_n))) \cdot \nabla T_L(u_n) \\ & + \frac{1}{p_-} |\Theta(T_L(u_n)|^\gamma \Bigg] \chi_{\{|T_L(u_n)-\phi| \leqslant k\}} dx. \end{aligned} \tag{3.19}$$

For the fifth term of the left hand side in (3.15), we prove that

$$\lim_{n\to+\infty} \varepsilon \int_\Omega \left[|\nabla u_n|^{p_+-2} \nabla u_n \nabla T_k(u_n - \phi) + |u_n|^{p_+-2} u_n T_k(u_n - \phi) \right] dx \geq 0 \text{ as } \varepsilon \to 0. \tag{3.20}$$

Setting $l = k + \|\phi\|_{L^\infty(\Omega)}$ we have,

$$\begin{aligned} & \varepsilon \int_\Omega |\nabla u_n|^{p_+-2} \nabla u_n \nabla T_k(u_n - \phi)\, dx \\ & = \varepsilon \int_{\{|u_n-\phi|<k\}} |\nabla T_l(u_n)|^{p_+-2} \nabla T_l(u_n) \nabla (T_l(u_n) - \phi)\, dx \\ & = \varepsilon \int_{\{|u_n-\phi|<k\}} |\nabla T_l(u_n)|^{p_+} dx - \varepsilon \int_{\{|u_n-\phi|<k\}} |\nabla T_l(u_n)|^{p_+-2} \nabla T_l(u_n) \nabla \phi dx \\ & \geq -\varepsilon \int_{\{|u_n-\varphi|<k\}} |\nabla T_l(u_n)|^{p_+-2} \nabla T_l(u_n) \nabla \phi dx. \end{aligned} \tag{3.21}$$

By taking $v = T_l(u_n)$ in (3.5) we get

$$\varepsilon \int_\Omega \left[|\nabla u_n|^{p_+-2} \nabla u_n \nabla T_l(u_n) + |u_n|^{p_+-2} u_n T_l(u_n) \right] dx \leq l \left(\|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)} \right),$$

hence

$$\varepsilon \int_\Omega |\nabla T_l(u_n)|^{p_+} dx \leq l \left(\|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)} \right),$$

which implies that the sequence $\varepsilon \nabla T_l(u_n)$ is uniformly bounded in $L^{p_+}(\Omega)$. From Lemma 3.6 $\nabla T_l(u_n)$ converges a.e. in Ω (up to a subsequence) to $\nabla T_l(u)$. So, by Vitali's Theorem, $\varepsilon \nabla T_l(u_n)$ converges to $\varepsilon \nabla T_l(u)$ in $L^{p_+}(\Omega)$. Thus,
$\varepsilon |\nabla T_l(u_n)|^{p_+-2} \nabla T_l(u_n) \chi_{\{|u_n-\phi|<k\}}$ converges to $\varepsilon |\nabla T_l(u)|^{p_+-2} \nabla T_l(u) \chi_{\{|u-\phi|<k\}}$ in $L^{p'_+}(\Omega)$. Using (3.21), we obtain

$$\lim_{n\to+\infty} \varepsilon \int_\Omega |\nabla u_n|^{p_+-2} \nabla u_n \nabla T_k(u_n - \phi)\, dx \geq -\varepsilon \int_{\{|u-\phi|<k\}} |\nabla T_l(u)|^{p_+-2} \nabla T_l(u) \nabla \varphi dx.$$

Therefore,

$$\lim_{n\to+\infty} \varepsilon \int_\Omega |\nabla u_n|^{p_+-2} \nabla u_n \nabla T_k(u_n - \phi)\, dx \geq 0, \text{ as } \varepsilon \to 0. \tag{3.22}$$

Now, we prove that

$$\lim_{n\to+\infty} \varepsilon \int_\Omega |u_n|^{p_+-2} u_n T_k(u_n - \phi)\, dx \geq 0, \text{ as } \varepsilon \to 0.$$

We have

$$\begin{aligned}\int_\Omega |u_n|^{p_+-2}u_n T_k(u_n-\phi)\,dx &= \int_\Omega \left(|u_n|^{p_+-2}u_n - |\phi|^{p_+-2}\phi\right) T_k(u_n-\phi)\,dx\\ &\quad+\int_\Omega |\phi|^{p_+-2}\phi T_k(u_n-\phi)\,dx\\ &\geq \int_\Omega |\phi|^{p_+-2}\phi T_k(u_n-\phi)\,dx,\end{aligned}\tag{3.23}$$

since $\left(|u_n|^{p_+-2}u_n - |\phi|^{p_+-2}\phi\right) T_k(u_n-\phi)$ is nonnegative. Furthermore, $T_k(u_n-\phi)$ converges weakly* to $T_k(u-\phi)$ in $L^\infty(\Omega)$ and $|\phi|^{p_+-2}\phi \in L^{p'_+}(\Omega)$, so

$$\lim_{n\to+\infty}\int_\Omega |\phi|^{p_+-2}\phi T_k(u_n-\phi)\,dx = \int_\Omega |\varphi|^{p_+-2}\varphi T_k(u-\varphi)dx.\tag{3.24}$$

Combining (3.23) and (3.24), we obtain

$$\lim_{n\to+\infty}\varepsilon\int_\Omega |u_n|^{p_+-2}u_n T_k(u_n-\phi)\,dx \geq 0,\ \text{ as } \varepsilon\to 0.\tag{3.25}$$

From (3.22) and (3.25), we get (3.20).
Now, we consider the first term in the right hand side of (3.15), since $T_n(f)\to f$ in $L^1(\Omega)$ then

$$\lim_{n\to\infty}\int_\Omega T_n(f)T_k(u_n-\phi)\,dx = \int_\Omega fT_k(u-\phi)\,dx.\tag{3.26}$$

Finally, by using the above results we can pass to the limit as $n\to\infty$ in the equality (3.15) to conclude that

$$\begin{aligned}\int_\Omega |\nabla u-\Theta(u)|^{p(u)-2}(\nabla u-\Theta(u))\nabla T_k(u-\phi)dx + \int_\Omega |u|^{p(u)-2}uT_k(u-\phi)dx + \int_\Omega \alpha(u)T_k(u-\phi)dx\\ +\lambda\int_{\partial\Omega} uT_k(u-\phi)d\sigma \leqslant \int_\Omega fT_k(u-\phi)dx + \int_{\partial\Omega} gT_k(u-\phi)d\sigma,\end{aligned}\tag{3.27}$$

for $\phi\in\mathcal{C}^\infty(\overline{\Omega})$.
Since $p(u(\cdot))$ verifies the log-Hölder condition, $C^\infty(\overline{\Omega})$ is dense in the space $W^{1,p(u(\cdot))}(\Omega)$. Moreover, $W^{1,p(u(\cdot))}(\Omega)\hookrightarrow W^{1,p_-}(\Omega)\hookrightarrow L^\infty(\Omega)$, since $p(u(\cdot))\geq p_- > N$ and Ω is a bounded open domain with Lipschitz boundary $\partial\Omega$. Hence, the inequality (3.27) holds true for $\phi\in W^{1,p(u(\cdot))}(\Omega)\cap L^\infty(\Omega)$. Therefore, u is an entropy solution of the problem (1.1).

References

1. A. Abbassi, C. Allalou, A. Kassidi, *Existence of Entropy Solutions for Anisotropic Elliptic Nonlinear Problem in Weighted Sobolev Space*, In The International Congress of the Moroccan Society of Applied Mathematics. Springer, Cham, pp. 102-122 (2019).

2. A. Abbassi, C. Allalou, A. Kassidi, *Anisotropic Elliptic Nonlinear Obstacle Problem with Weighted Variable Exponent*, J. Math. Study, 54(4), 337-356 (2021).

3. A. Abassi, A. El Hachimi, A. Jamea, *Entropy solutions to nonlinear Neumann problems with L^1-data*, Int. J. Math. Statist 2 (2008): 4-17.

4. Y. Akdim, C. Allalou, N. El Gorch, M. Mekkour, *Obstacle problem for nonlinear $p(x)$-parabolic inequalities*, In AIP Conference Proceeding, Vol. 2074, No. 1, p.020018, AIP Publishing LLC, (2019).

5. B. Andreianov, M. Bendahmane, S. Ouaro, *Structural stability for variable exponent elliptic problems. II. The $p(u)$-Laplacian and coupled problems*, Nonlinear Anal. 72(12), 4649-4660 (2010).

6. E. Azroul, M. B. Benboubker, S. Ouaro, *Entropy solutions for nonlinear nonhomogeneous Neumann problems involving the generalized $p(x)$-Laplace operator*, J. Appl. Anal. Comput 3.2 (2013): 105-121.

7. E. Azroul, F. Balaadich, *Generalized $p(x)$-elliptic system with nonlinear physical data*, Journal of Applied Analysis and Computation 10.5 (2020): 1995-2007.

8. P. Benilan, L. Boccardo, T. Gallouet, R. Gariepy, M. Pierre, J. L. Vazquez, *An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. 22 (1995), 241-273.
9. P. Blomgren, T. Chan, P. Mulet, C. Wong, *Total variation image restoration: Numerical methods and extensions. In: Proceedings of the IEEE International Conference on Image Processing*, vol. 3, 384-387. IEEE Computer Society Press, Piscataway (1997).
10. E. Bollt, R. Chartrand, S. Esedoglu, P. Schultz, K. Vixie, *Graduated, adaptive image denoising: local compromise between total-variation and isotropic diffusion*, Adv. Comput. Math. 31, 61-85 (2007).
11. C. Allalou, K. Hilal, S. Ait Temghart, *Existence of weak solutions for some local and nonlocal p-Laplacian problem. Journal of Elliptic and Parabolic Equations*, 1-19 (2022).
12. M. Chipot, H. B. de Oliveira, *Some Results On The $p(u)$-Laplacian Problem*, Mathematische Annalen (2019).
13. A. Jamea, A. Sabri, H. T. Alaoui, *Entropy solution for nonlinear degenerate elliptic problem with Dirichlet-type boundary condition in weighted Sobolev spaces*, Le Matematiche 76.1 (2021): 109-131.
14. O. Kovácık, and J. Rákosnık, *On spaces $L^{p(x)}$ and $W^{k,p(x)}$*, Czechoslovak Mathematical Journal 41.116 (1991): 592-618.
15. N. G. Meyers, and J Serrin, *H=W*, Proc. Nat. Acad. Sci USA 51 (1964): 1055-1056.
16. D. Edmunds, J. Rakosnik, *Sobolev embeddings with variable exponent*, 267-293, DOI:10.4064sm-143-3-267-293.
17. N. Ouaro, N. Sawadogo, *Nonlinear elliptic $p(u)$-Laplacian problem with Fourier boundary condition*, CUBO A Mathematical, Vol.22, No 01, 85-124 (2020).
18. N. Ouaro, N. Sawadogo, *Structural stability for nonlinear $p(u)$-Laplacian problem with Fourier boundary condition*, Gulf Journal of Mathematics 11.1 (2021): 1-37.
19. V. Calogero, *The Existence of Solutions for Local Dirichlet $(r(u), s(u))$-Problems*, Mathematics 10.2 (2022): 237.
20. J. Türola, *Image denoising using directional adaptive variable exponents model*, J. Math. Imaging. Vis. 57, 56-74 (2017).
21. C. Zhang, X. Zhang, *Some further results on the nonlocal $p-$Laplacian type problems*, Proceedings of the Royal Society of Edinburgh Section A: Mathematics, 151(3), 953-970 (2021).

S. Ait Temghart,
Laboratory LMACS,
FST of Beni-Mellal, Sultan Moulay slimane University, Morocco.
E-mail address: saidotmghart@gmail.com

and

C. Allalou,
Laboratory LMACS,
FST of Beni-Mellal, Sultan Moulay slimane University, Morocco.
E-mail address: chakir.allalou@yahoo.fr

and

K. Hilal, Laboratory LMACS,
FST of Beni-Mellal, Sultan Moulay slimane University, Morocco.
E-mail address: khalidhilal2003@gmail.com