Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 2023 (41)** : 1–11.
ISSN-0037-8712 in press
doi:10.5269/bspm.52666

$\tilde{\gamma}$-open Sets and ($\tilde{\gamma}$, $\tilde{\beta}$)-continuous Mappings *

D. Saravanakumar and M. Mohamed Riyazdeen

ABSTRACT: In this paper, we introduce a new class of open sets namely $\tilde{\gamma}$-open sets in a topological space. In addition, we define $\tilde{\gamma}$-T_i $(i = 0, \frac{1}{2}, 1, 2)$ spaces, ($\tilde{\gamma}$, $\tilde{\beta}$)-continuous mapping and study their basic properties.

Key Words: $\tilde{\gamma}$-open set, $\tilde{\gamma}$-closed set, $\tilde{\gamma}$-T_i $(i = 0, \frac{1}{2}, 1, 2)$ spaces, ($\tilde{\gamma}$, $\tilde{\beta}$)-continuous mapping.

Contents

1. Introduction

Kasahara [3] initiated the concept of α-operation in topological spaces, Jankovic [2] obtained the α-closed graphs and Ogata [7,8] defined the γ-operation (resp. γ-closed set) and investigated the relationships between $cl_\gamma(A)$ and $\tau_\gamma cl(A)$ for a subset A of a topological space X. Further, he obtained the concept of γ-T_i $(i = 0, \frac{1}{2}, 1, 2)$ spaces through the γ-closed and γ-open sets. Levine [4] defined a generalized closed set and Maki et al. [6] extended this generalized closed set to a γ-g.closed set. Dunham [1] introduced $T_{\frac{1}{2}}$ spaces and Levine [5] defined a semi-open set. Sai sundara Krishnan et al. [9,10] modified the concept of semi-open to γ-semi-open (resp. γ^*-pre-open) and he was studied γ-semi-separation axioms. Saravanakumar et al. [11,12,13] introduced a $\tilde{\mu}$-open set, $\tilde{\mu}$-separation axioms and some continuous mappings with respect to γ and β operations on the topological spaces (X, τ) and (Y, σ) respectively and investigated their basic properties.

In this paper, we define a $\tilde{\gamma}$-open set in a topological space with respect to γ-operation and its family denoted by $\tilde{\gamma}O(X)$. Further, we obtain a $\tilde{\gamma}$-closed set, $\tilde{\gamma}$-interior, $\tilde{\gamma}$-closure and $\tilde{\gamma}$-boundary in a topological space. Moreover, we discuss the basic properties of $\tilde{\gamma}$-T_i $(i = 0, \frac{1}{2}, 1, 2)$ spaces and ($\tilde{\gamma}$, $\tilde{\beta}$)-continuous mappings.

2. Preliminaries

Let the topological spaces (X, τ) and (Y, σ) be respectively denoted by X and Y. An operation γ [7] on the topology τ is a mapping from τ into the power set $P(X)$ of X such that $V \subseteq V^\gamma$ for each $V \in \tau$, where V^γ denotes the value of γ at V. Similarly, an operation β on the topology σ is a mapping from σ into the power set $P(Y)$ of Y such that $W \subseteq W^\beta$ for each $W \in \sigma$, where W^β denotes the value of β at W. A subset A of X is γ-open [7], if for each $x \in A$, there exist an open neighborhood U such that $x \in U$ and $U^\gamma \subseteq A$. Its complement is called γ-closed and τ_γ [7] denotes set of all γ-open sets in X. For a subset A of X, γ-interior [7] of A is $int_\gamma(A) = \{x \in A : x \in N \in \tau \text{ and } N^\gamma \subseteq A \text{ for some } N\}$; γ-closure [7] of A is $cl_\gamma(A) = \{x \in X : x \in U \in \tau \text{ and } U^\gamma \cap A \neq \emptyset \text{ for all } U\}$; τ_γ-$int(A)$ [7] $= \cup\{G : G \subseteq A$ and

* We wish to thank the referees for their suggestions and support in completing this paper.
2010 *Mathematics Subject Classification:* 54C05, 54C10 and 54D10.
Submitted March 17, 2020. Published July 15, 2020

$G \in \tau_\gamma\}$; τ_γ-$cl(A)$ [7] $= \cap\{F : A \subseteq F$ and $X \setminus F \in \tau_\gamma\}$. A subset A of X is γ-g.closed [7] if $cl_\gamma(A) \subseteq U$ whenever $A \subseteq U$ and U is γ-open in X. For a subset A of X, $cl^*_\gamma(A)$ [6] denotes the intersection of all γ-g.closed sets containing A, that is the smallest γ-g.closed set containing A; $int^*_\gamma(A)$ [6] denotes the union of all γ-g.open sets contained in A, that is the largest γ-g.open set contained in A. If A is a subset of X and $x \in X$, then (i) $x \in cl^*_\gamma(A)$ [6] if and only if $M \cap A \neq \emptyset$ for each γ-g.open set M containing x; (ii) $cl^*_\gamma(X \setminus A)$ [6] $= X \setminus int^*_\gamma(A)$ and (iii) $cl^*_\gamma(cl^*_\gamma(A))$ [6] $= cl^*_\gamma(A)$. A subset A of X is γ-semi-open [9] if $A \subseteq \tau_\gamma cl(\tau_\gamma int(A))$ and $\gamma SO(X)$ [9] denotes set of all γ-semi-open sets in X.

3. $\tilde{\gamma}$-open sets

Definition 3.1. *Let X be a topological space and $\gamma : \tau \to P(X)$ be an operation on τ. A subset A of X is said to be a $\tilde{\gamma}$-open set, if there exists a set $U \in \tau_\gamma$ such that $U \subseteq A \subseteq cl^*_\gamma(U)$. The collection of all $\tilde{\gamma}$-open sets is denoted by $\tilde{\gamma}O(X)$.*

Example 3.2. *Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}\}$ and define an operation $\gamma : \tau \to P(X)$ by*

$$\gamma(A) = \begin{cases} A \cup \{c\} \text{ if } A = \{a\} \\ cl(A) \text{ if } A \neq \{a\} \end{cases} \text{ for every } A \in \tau.$$

Then $\tilde{\gamma}O(X) = \{\emptyset, X, \{c\}, \{a, b\}\}$.

Theorem 3.3. *Let X be a topological space and $\gamma : \tau \to P(X)$ be an operation on τ. If A is a subset of X, then $A \in \tilde{\gamma}O(X)$ if and only if $A \subseteq cl^*_\gamma(\tau_\gamma int(A))$.*

Proof. If $A \in \tilde{\gamma}O(X)$, then there exists a set $U \in \tau_\gamma$ such that $U \subseteq A \subseteq cl^*_\gamma(U)$. Since $U \in \tau_\gamma$, we have that $U = \tau_\gamma int(U) \subseteq \tau_\gamma int(A)$. Therefore $A \subseteq cl^*_\gamma(U) \subseteq cl^*_\gamma(\tau_\gamma int(A))$ and hence $A \subseteq cl^*_\gamma(\tau_\gamma int(A))$. Conversely, assume that $A \subseteq cl^*_\gamma(\tau_\gamma int(A))$. To prove that $A \in \tilde{\gamma}O(X)$. Take $U = \tau_\gamma int(A)$. Then $\tau_\gamma int(A) \subseteq A \subseteq cl^*_\gamma(\tau_\gamma int(A))$. Hence $A \in \tilde{\gamma}O(X)$. □

Theorem 3.4. *Let X be a topological space and $\gamma : \tau \to P(X)$ be an operation on τ. If A is a subset of X and $A \in \tau_\gamma$, then $A \in \tilde{\gamma}O(X)$.*

Proof. If $A \in \tau_\gamma$, then $A = \tau_\gamma int(A)$. Since $A \subseteq cl^*_\gamma(A)$, we have that $A \subseteq cl^*_\gamma(\tau_\gamma int(A))$. Then by Theorem 3.3, $A \in \tilde{\gamma}O(X)$. □

Remark 3.5. *The converse of the Theorem 3.4 need not be true.*

Consider $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{d\}, \{a, d\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}\}$ and define an operation $\gamma : \tau \to P(X)$ by

$$\gamma(A) = \begin{cases} A \text{ if } A = \{b, c\} \\ A \cup \{b, d\} \text{ if } A \neq \{b, c\} \end{cases} \text{ for every } A \in \tau.$$

Then $\tau_\gamma = \{\emptyset, X, \{b, c\}, \{b, c, d\}\}$ and $\tilde{\gamma}O(X) = \{\emptyset, X, \{b, c\}, \{a, b, c\}, \{b, c, d\}\}$. Hence $A = \{a, b, c\} \in \tilde{\gamma}O(X)$, but $A \notin \tau_\gamma$.

Theorem 3.6. *Let X be a topological space and $\gamma : \tau \to P(X)$ be an operation on τ. If A is a subset of X and $A \in \tilde{\gamma}O(X)$, then $A \in \gamma SO(X)$.*

Proof. If $A \in \tilde{\gamma}O(X)$, then by Theorem 3.3, we have that $A \subseteq cl^*_\gamma(\tau_\gamma int(A))$. Since every γ-closed set is γ-g.closed and $cl^*_\gamma(\tau_\gamma int(A))$ is a least γ-g.closed set containing $\tau_\gamma int(A)$, this implies that $cl^*_\gamma(\tau_\gamma int(A)) \subseteq \tau_\gamma cl(\tau_\gamma int(A))$. Thus $A \subseteq \tau_\gamma cl(\tau_\gamma int(A))$ and hence $A \in \gamma SO(X)$. □

Remark 3.7. *The converse of the Theorem 3.6 need not be true.*

Consider $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{d\}, \{a, d\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}\}$ and define an operation $\gamma : \tau \to P(X)$ by

$$\gamma(A) = \begin{cases} A \text{ if } d \in A \\ A \cup \{d\} \text{ if } d \notin A \end{cases} \quad \text{for every } A \in \tau.$$

Then $\gamma SO(X) = \{\emptyset, X, \{d\}, \{a,d\}, \{b,d\}, \{c,d\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}\}$ *and* $\tilde{\gamma}O(X) = \{\emptyset, X, \{d\}, \{a,d\}, \{b,c,d\}\}$*. Hence* $A = \{b,d\} \in \gamma SO(X)$*, but* $A \notin \tilde{\gamma}O(X)$*.*

Theorem 3.8. *Let* X *be a topological space and* $\gamma : \tau \to P(X)$ *be an operation on* τ*. If* $A_\alpha \in \tilde{\gamma}O(X)$ *for each* $\alpha \in J$*, then* $\bigcup_{\alpha\in J} A_\alpha \in \tilde{\gamma}O(X)$*.*

Proof. Since $A_\alpha \in \tilde{\gamma}O(X)$, then there exists a set $U_\alpha \in \tau_\gamma$ such that $U_\alpha \subseteq A \subseteq cl^*_\gamma(U_\alpha)$. This implies that $\bigcup_{\alpha\in J} U_\alpha \subseteq \bigcup_{\alpha\in J} A_\alpha \subseteq \bigcup_{\alpha\in J} cl^*_\gamma(U_\alpha) \subseteq cl^*_\gamma(\bigcup_{\alpha\in J} U_\alpha)$ since union of all γ-open sets is γ-open. Therefore $\bigcup_{\alpha\in J} A_\alpha \in \tilde{\gamma}O(X)$. □

Note that intersection of any two sets in $\tilde{\gamma}O(X)$ need not be $\tilde{\gamma}$-open. Let $X = \{a,b,c,d\}$, $\tau = \{\emptyset, X, \{a\}, \{d\}, \{a,d\}, \{b,c\}, \{a,b,c\}, \{b,c,d\}\}$ and define an operation $\gamma : \tau \to P(X)$ by

$$\gamma(A) = \begin{cases} A \cup \{b\} \text{ if } a \in A \\ A \cup \{d\} \text{ if } a \notin A \end{cases} \quad \text{for every } A \in \tau.$$

Take $A = \{a,b,c\}$ and $B = \{b,c,d\}$. Then $A, B \in \tilde{\gamma}O(X)$, but $A \cap B = \{b,c\} \notin \tilde{\gamma}O(X)$.

Theorem 3.9. *Let* X *be a topological space and* $\gamma : \tau \to P(X)$ *be an operation on* τ*. If a set* $A \in \tilde{\gamma}O(X)$ *and* B *is a subset of* X *such that* $A \subseteq B \subseteq cl^*_\gamma(\tau_\gamma int(A))$*, then* $B \in \tilde{\gamma}O(X)$*.*

Proof. If $A \in \tilde{\gamma}O(X)$, then by Theorem 3.3, $A \subseteq cl^*_\gamma(\tau_\gamma int(A))$. Since $A \subseteq B$, this implies that $cl^*_\gamma(\tau_\gamma int(A)) \subseteq cl^*_\gamma(\tau_\gamma int(B))$. By hypothesis $B \subseteq cl^*_\gamma(\tau_\gamma int(A)) \subseteq cl^*_\gamma(\tau_\gamma int(B))$ and hence $B \subseteq cl^*_\gamma(\tau_\gamma int(B))$. This shows that $B \in \tilde{\gamma}O(X)$. □

Definition 3.10. *Let* X *be a topological space and* $\gamma : \tau \to P(X)$ *be an operation on* τ*. A subset* A *of* X *is called a* $\tilde{\gamma}$*-closed set if its complement* $X \setminus A \in \tilde{\gamma}O(X)$*. The collection of all* $\tilde{\gamma}$*-closed sets is denoted by* $\tilde{\gamma}C(X)$*.*

Theorem 3.11. *Let* X *be a topological space and* $\gamma : \tau \to P(X)$ *be an operation on* τ*. If* A *is a subset of* X*, then* $A \in \tilde{\gamma}C(X)$ *if and only if* $int^*_\gamma(\tau_\gamma cl(A)) \subseteq A$*.*

Proof. If $A \in \tilde{\gamma}C(X)$, then $X \setminus A \in \tilde{\gamma}O(X)$ is $\tilde{\gamma}$-open. Therefore $X \setminus A \subseteq cl^*_\gamma(\tau_\gamma int(X \setminus A))$ (by Theorem 3.3) $= cl^*_\gamma(X \setminus \tau_\gamma cl(A)) = X \setminus int^*_\gamma(\tau_\gamma cl(A))$. This implies that $int^*_\gamma(\tau_\gamma cl(A)) \subseteq A$. Conversely, suppose that $int^*_\gamma(\tau_\gamma cl(A)) \subseteq A$. Then $X \setminus A \subseteq X \setminus int^*_\gamma(\tau_\gamma cl(A)) = cl^*_\gamma(X \setminus \tau_\gamma cl(A)) = cl^*_\gamma(\tau_\gamma int(X \setminus A))$. Therefore $X \setminus A \in \tilde{\gamma}O(X)$ and this shows that $A \in \tilde{\gamma}C(X)$. □

Theorem 3.12. *Let* X *be a topological space and* $\gamma : \tau \to P(X)$ *be an operation on* τ*. If a set* $F \in \tau^c_\gamma$ *and* A *is a subset of* X *such that* $int^*_\gamma(F) \subseteq A \subseteq F$*, then* $A \in \tilde{\gamma}C(X)$*.*

Proof. Let $int^*_\gamma(F) \subseteq A \subseteq F$ where $F \in \tau^c_\gamma$. Then $X \setminus F \subseteq X \setminus A \subseteq X \setminus int^*_\gamma(F) = cl^*_\gamma(X \setminus F)$. Let $U = X \setminus F$. Then $U \in \tau_\gamma$ and $U \subseteq X \setminus A \subseteq cl^*_\gamma(U)$. This implies that $X \setminus A \in \tilde{\gamma}O(X)$ and hence $A \in \tilde{\gamma}C(X)$. □

Remark 3.13. *The converse of the Theorem 3.12 need not be true.*

Consider $X = \{a,b,c,d\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{b,d\}, \{a,b,c\}, \{a,b,d\}, \{b,c,d\}\}$ *and define an operation* $\gamma : \tau \to P(X)$ *by*

$$\gamma(A) = \begin{cases} A \text{ if } A = \{a\} \\ cl(A) \text{ if } A = \{a,d\} \\ A \cup \{c\} \text{ if } A \neq \{a\} \text{ and } \{a,b\} \end{cases} \quad \text{for every } A \in \tau.$$

Thus for the set $\{b\} \in \tilde{\gamma}C(X)$*, does not exist any* γ*-closed set in* X*.*

Theorem 3.14. *Let* X *be a topological space and* $\gamma : \tau \to P(X)$ *be an operation on* τ*. If* A *is a subset of* X*, then*
(i) $int^*_\gamma(\tau_\gamma cl(A))$ *is* $\tilde{\gamma}$*-closed;*
(ii) $cl^*_\gamma(\tau_\gamma int(A))$ *is* $\tilde{\gamma}$*-open.*

Proof. (i) Clearly, we have that $int^*_\gamma(\tau_\gamma cl(int^*_\gamma(\tau_\gamma cl(A)))) \subseteq int^*_\gamma(\tau_\gamma cl(\tau_\gamma cl(A))) = int^*_\gamma(\tau_\gamma cl(A))$. Hence $int^*_\gamma(\tau_\gamma cl(A))$ is $\tilde{\gamma}$-closed.

(ii) Follows from (i) and Theorem 3.3. □

Theorem 3.15. *Let* X *be a topological space and* $\gamma : \tau \to P(X)$ *be an operation on* τ*. If* $A_\alpha \in \tilde{\gamma}C(X)$ *for each* $\alpha \in J$*, then* $\bigcap_{\alpha \in J} A_\alpha \in \tilde{\gamma}C(X)$*.*

Proof. Let $A_\alpha \in \tilde{\gamma}C(X)$. Then $X \setminus A_\alpha \in \tilde{\gamma}O(X)$. By Theorem 3.8, $\bigcup_{\alpha \in J}(X \setminus A_\alpha) \in \tilde{\gamma}O(X)$. This implies that $\bigcup_{\alpha \in J}(X \setminus A_\alpha) = X \setminus \bigcap_{\alpha \in J} A_\alpha \in \tilde{\gamma}O(X)$ and hence $\bigcap_{\alpha \in J} A_\alpha \in \tilde{\gamma}C(X)$. □

Definition 3.16. *Let* X *be a topological space and* $\gamma : \tau \to P(X)$ *be an operation on* τ*. If* A *is a subset of* X*, then* $\tilde{\gamma}$*-interior of* A *is defined as union of all* $\tilde{\gamma}$*-open sets contained in* A*. Thus* $int_{\tilde{\gamma}}(A) = \cup\{U : U \in \tilde{\gamma}O(X) \text{ and } U \subseteq A\}$*.*

Definition 3.17. *Let* X *be a topological space and* $\gamma : \tau \to P(X)$ *be an operation on* τ*. If* A *is a subset of* X*, then* $\tilde{\gamma}$*-closure of* A *is defined as intersection of all* $\tilde{\gamma}$*-closed sets containing* A*. Thus* $cl_{\tilde{\gamma}}(A) = \cap\{F : X \setminus F \in \tilde{\gamma}O(X) \text{ and } A \subseteq F\}$*.*

Theorem 3.18. *Let* X *be a topological space and* $\gamma : \tau \to P(X)$ *be an operation on* τ*. If* A *is a subset of* X*, then*
(i) $int_{\tilde{\gamma}}(A)$ *is a* $\tilde{\gamma}$*-open set contained in* A*;*
(ii) $cl_{\tilde{\gamma}}(A)$ *is a* $\tilde{\gamma}$*-closed set containing* A*;*
(iii) A *is* $\tilde{\gamma}$*-closed if and only if* $cl_{\tilde{\gamma}}(A) = A$*;*
(iv) A *is* $\tilde{\gamma}$*-open if and only if* $int_{\tilde{\gamma}}(A) = A$*;*
(v) $int_{\tilde{\gamma}}(int_{\tilde{\gamma}}(A)) = int_{\tilde{\gamma}}(A)$*;*
(vi) $cl_{\tilde{\gamma}}(cl_{\tilde{\gamma}}(A)) = cl_{\tilde{\gamma}}(A)$*;*
(vii) $int_{\tilde{\gamma}}(A) = X \setminus cl_{\tilde{\gamma}}(X \setminus A)$*;*
(viii) $cl_{\tilde{\gamma}}(A) = X \setminus int_{\tilde{\gamma}}(X \setminus A)$*.*

Proof. (i) Follows from the Definition 3.16 and Theorem 3.18.
(ii) Follows from the Definition 3.17 and Theorem 3.15.
(iii) and (iv) Follows from the condition (ii), Definition 3.17 and the condition (i), Definition 3.16 respectively.
(v) and (vi) Follows from the conditions (i), (iv) and the conditions (ii), (iii) respectively.
(vii) and (viii) Follows from the Definitions 3.10, 3.16 and 3.17. □

Theorem 3.19. *Let* X *be a topological space and* $\gamma : \tau \to P(X)$ *be an operation on* τ*. If* A *and* B *are two subsets of* X*, then the following are hold:*
(i) If $A \subseteq B$*, then* $int_{\tilde{\gamma}}(A) \subseteq int_{\tilde{\gamma}}(B)$*;*
(ii) If $A \subseteq B$*, then* $cl_{\tilde{\gamma}}(A) \subseteq cl_{\tilde{\gamma}}(B)$*;*
(iii) $int_{\tilde{\gamma}}(A \cup B) \supseteq int_{\tilde{\gamma}}(A) \cup int_{\tilde{\gamma}}(B)$*;*
(iv) $cl_{\tilde{\gamma}}(A \cap B) \subseteq cl_{\tilde{\gamma}}(A) \cap cl_{\tilde{\gamma}}(B)$*;*
(v) $int_{\tilde{\gamma}}(A \cap B) \subseteq int_{\tilde{\gamma}}(A) \cap int_{\tilde{\gamma}}(B)$*.*
(vi) $cl_{\tilde{\gamma}}(A \cup B) \supseteq cl_{\tilde{\gamma}}(A) \cup cl_{\tilde{\gamma}}(B)$*.*

Proof. (i) and (ii) Follows from the Definition 3.16 and Definition 3.17 respectively.
(iii) and (iv) Follows from the condition (i), Theorem 3.8 and the condition (ii), Theorem 3.15 respectively.
(v) and (vi) Follows from the condition (i) and condition (ii) respectively. □

Theorem 3.20. *Let X be a topological space, $\gamma : \tau \to P(X)$ be an operation on τ and A be a subset of X. (i) If $A \subseteq int^*_{\gamma}(\tau_{\gamma}cl(A))$, then $cl_{\tilde{\gamma}}(A) \subseteq int^*_{\gamma}(\tau_{\gamma}cl(A))$;*
(ii) If $cl^*_{\gamma}(\tau_{\gamma}int(A)) \subseteq A$, then $int_{\tilde{\gamma}}(A) \supseteq cl^*_{\gamma}(\tau_{\gamma}int(A))$.

Proof. (i) Since $cl_{\tilde{\gamma}}(A)$ is the least $\tilde{\gamma}$-closed set containing A and Theorem 3.14(i) shows that $int^*_{\gamma}(\tau_{\gamma}cl(A)) \in \tilde{\gamma}C(X)$. Therefore $cl_{\tilde{\gamma}}(A) \subseteq int^*_{\gamma}(\tau_{\gamma}cl(A))$.

(ii) Since $int_{\tilde{\gamma}}(A)$ is the greatest $\tilde{\gamma}$-open set containing A and Theorem 3.14(ii) shows that $cl^*_{\gamma}(\tau_{\gamma}int(A)) \in \tilde{\gamma}O(X)$. Therefore $int_{\tilde{\gamma}}(A) \supseteq cl^*_{\gamma}(\tau_{\gamma}int(A))$. □

Definition 3.21. *Let X be a topological space and $\gamma : \tau \to P(X)$ be an operation on τ. A subset A of X is called $\tilde{\gamma}$-regular if it is both $\tilde{\gamma}$-open and $\tilde{\gamma}$-closed. The collection of all $\tilde{\gamma}$-regular set of X is denoted by $\tilde{\gamma}R(X)$.*

Note that a set $A \in \tilde{\gamma}R(X)$, then its complement $X \setminus A \in \tilde{\gamma}R(X)$.

Definition 3.22. *Let X be a topological space and $\gamma : \tau \to P(X)$ be an operation on τ and A be a subset of X. Then $\tilde{\gamma}$-boundary of A is denoted by $bd_{\tilde{\gamma}}(A)$ and is defined as $bd_{\tilde{\gamma}}(A) = cl_{\tilde{\gamma}}(A) \cap cl_{\tilde{\gamma}}(X \setminus A)$.*

Theorem 3.23. *Let X be a topological space and $\gamma : \tau \to P(X)$ be an operation on τ. For a subset A of X, $bd_{\tilde{\gamma}}(A) = \emptyset$ if and only if $A \in \tilde{\gamma}R(X)$.*

Proof. Let $bd_{\tilde{\gamma}}(A) = \emptyset$. Then $cl_{\tilde{\gamma}}(A) \cap cl_{\tilde{\gamma}}(X \setminus A) = \emptyset$. This implies that $cl_{\tilde{\gamma}}(A) \subseteq X \setminus cl_{\tilde{\gamma}}(X \setminus A) = int_{\tilde{\gamma}}(A)$ (by Theorem 3.18(vii)). Therefore $cl_{\tilde{\gamma}}(A) = A = int_{\tilde{\gamma}}(A)$ and hence $A \in \tilde{\gamma}O(X)$ and $A \in \tilde{\gamma}C(X)$. Conversely, assume that $A \in \tilde{\gamma}R(X)$. Then $A \in \tilde{\gamma}O(X)$ and $A \in \tilde{\gamma}C(X)$. This implies that $cl_{\tilde{\gamma}}(A) = A = int_{\tilde{\gamma}}(A) = X \setminus cl_{\tilde{\gamma}}(X \setminus A)$ (by Theorem 3.18(vii)). Since $X \setminus cl_{\tilde{\gamma}}(X \setminus A) \cap cl_{\tilde{\gamma}}(X \setminus A) = \emptyset$, we have that $cl_{\tilde{\gamma}}(A) \cap cl_{\tilde{\gamma}}(X \setminus A) = \emptyset$. This shows that $bd_{\tilde{\gamma}}(A) = \emptyset$. □

Theorem 3.24. *Let X be a topological space, $\gamma : \tau \to P(X)$ be an operation on τ and A be a subset of X. Then the following are equivalent:*
(i) $X \setminus bd_{\tilde{\gamma}}(A) = int_{\tilde{\gamma}}(A) \cup int_{\tilde{\gamma}}(X \setminus A)$;
(ii) $cl_{\tilde{\gamma}}(A) = int_{\tilde{\gamma}}(A) \cup bd_{\tilde{\gamma}}(A)$;
(iii) $bd_{\tilde{\gamma}}(A) = cl_{\tilde{\gamma}}(A) \cap cl_{\tilde{\gamma}}(X \setminus A) = cl_{\tilde{\gamma}}(A) \setminus int_{\tilde{\gamma}}(A)$.

Proof. (i) ⇒ (ii). From (i) $X \setminus bd_{\tilde{\gamma}}(A) = int_{\tilde{\gamma}}(A) \cup int_{\tilde{\gamma}}(X \setminus A)$ implies that $bd_{\tilde{\gamma}}(A) = [X \setminus int_{\tilde{\gamma}}(A)] \cap [X \setminus int_{\tilde{\gamma}}(X \setminus A)]$. Therefore $int_{\tilde{\gamma}}(A) \cup bd_{\tilde{\gamma}}(A) = [int_{\tilde{\gamma}}(A) \cup (X \setminus int_{\tilde{\gamma}}(A))] \cap [int_{\tilde{\gamma}}(A) \cup cl_{\tilde{\gamma}}(A)] = X \cap cl_{\tilde{\gamma}}(A) = cl_{\tilde{\gamma}}(A)$. Hence $cl_{\tilde{\gamma}}(A) = int_{\tilde{\gamma}}(A) \cup bd_{\tilde{\gamma}}(A)$.

(ii) ⇒ (iii). From (ii) $cl_{\tilde{\gamma}}(A) \setminus int_{\tilde{\gamma}}(A) = [int_{\tilde{\gamma}}(A) \cup bd_{\tilde{\gamma}}(A)] \setminus int_{\tilde{\gamma}}(A) = bd_{\tilde{\gamma}}(A)$ ....(*1). Also from (ii) $X \cap cl_{\tilde{\gamma}}(A) = int_{\tilde{\gamma}}(A) \cup bd_{\tilde{\gamma}}(A)$ implies that $[int_{\tilde{\gamma}}(A) \cup (X \setminus int_{\tilde{\gamma}}(A))] \cap [int_{\tilde{\gamma}}(A) \cup cl_{\tilde{\gamma}}(A)] = int_{\tilde{\gamma}}(A) \cup bd_{\tilde{\gamma}}(A)$ implies that $int_{\tilde{\gamma}}(A) \cup [cl_{\tilde{\gamma}}(X \setminus A) \cap cl_{\tilde{\gamma}}(A)] = int_{\tilde{\gamma}}(A) \cup bd_{\tilde{\gamma}}(A)$. Therefore $bd_{\tilde{\gamma}}(A) = cl_{\tilde{\gamma}}(A) \cap cl_{\tilde{\gamma}}(X \setminus A)$ ....(*2). From (*1) and (*2), we have that $bd_{\tilde{\gamma}}(A) = cl_{\tilde{\gamma}}(A) \cap cl_{\tilde{\gamma}}(X \setminus A) = cl_{\tilde{\gamma}}(A) \setminus int_{\tilde{\gamma}}(A)$.

(iii) ⇒ (i). From (iii), we have that $X \setminus bd_{\tilde{\gamma}}(A) = X \setminus [cl_{\tilde{\gamma}}(X \setminus A) \cap cl_{\tilde{\gamma}}(A)] = [X \setminus cl_{\tilde{\gamma}}(X \setminus A)] \cup [X \setminus cl_{\tilde{\gamma}}(A)] = int_{\tilde{\gamma}}(A) \cup int_{\tilde{\gamma}}(X \setminus A)$. Therefore $X \setminus bd_{\tilde{\gamma}}(A) = int_{\tilde{\gamma}}(A) \cup int_{\tilde{\gamma}}(X \setminus A)$. □

Theorem 3.25. *Let X be a topological space and $\gamma : \tau \to P(X)$ be an operation on τ. For a subset A of X, we have the following conditions hold:*
(i) $bd_{\tilde{\gamma}}(A) = bd_{\tilde{\gamma}}(X \setminus A)$;
(ii) $bd_{\tilde{\gamma}}(A) = cl_{\tilde{\gamma}}(A) \setminus int_{\tilde{\gamma}}(A)$;
(iii) $bd_{\tilde{\gamma}}(A) \cap int_{\tilde{\gamma}}(A) = \emptyset$;
(iv) $cl_{\tilde{\gamma}}(A) = int_{\tilde{\gamma}}(A) \cup bd_{\tilde{\gamma}}(A)$;
(v) $bd_{\tilde{\gamma}}(int_{\tilde{\gamma}}(A)) \subseteq bd_{\tilde{\gamma}}(A)$;
(vi) $bd_{\tilde{\gamma}}(cl_{\tilde{\gamma}}(A)) \subseteq bd_{\tilde{\gamma}}(A)$;
(vii) $X \setminus bd_{\tilde{\gamma}}(A) = int_{\tilde{\gamma}}(A) \cup int_{\tilde{\gamma}}(X \setminus A)$;
(viii) $X = int_{\tilde{\gamma}}(A) \cup int_{\tilde{\gamma}}(X \setminus A) \cup bd_{\tilde{\gamma}}(A)$.

Proof. (i) By Definition 3.22, we have that $bd_{\tilde{\gamma}}(A) = cl_{\tilde{\gamma}}(X \setminus A) \cap cl_{\tilde{\gamma}}(A) = cl_{\tilde{\gamma}}(X \setminus A) \cap cl_{\tilde{\gamma}}(X \setminus (X \setminus A)) = bd_{\tilde{\gamma}}(X \setminus A)$. Therefore $bd_{\tilde{\gamma}}(A) = bd_{\tilde{\gamma}}(X \setminus A)$.

(ii) By Definition 3.22, we have that $bd_{\tilde{\gamma}}(A) = cl_{\tilde{\gamma}}(A) \cap cl_{\tilde{\gamma}}(X \setminus A) = cl_{\tilde{\gamma}}(A) \setminus (X \setminus cl_{\tilde{\gamma}}(X \setminus A)) = cl_{\tilde{\gamma}}(A) \setminus int_{\tilde{\gamma}}(A)$ (by Theorem 3.18 (vii)). Therefore $bd_{\tilde{\gamma}}(A) = cl_{\tilde{\gamma}}(A) \setminus int_{\tilde{\gamma}}(A)$.

(iii) By Definition 3.22, we have that $bd_{\tilde{\gamma}}(A) \cap int_{\tilde{\gamma}}(A) = (cl_{\tilde{\gamma}}(A) \setminus int_{\tilde{\gamma}}(A)) \cap int_{\tilde{\gamma}}(A)$ (by (ii)) $= \emptyset$. Hence $bd_{\tilde{\gamma}}(A) \cap int_{\tilde{\gamma}}(A) = \emptyset$.

(iv) Follows from (ii) and Theorem 3.24.

(v) By Definition 3.22, we have that $bd_{\tilde{\gamma}}(int_{\tilde{\gamma}}(A)) = cl_{\tilde{\gamma}}(X \setminus int_{\tilde{\gamma}}(A)) \cap cl_{\tilde{\gamma}}(int_{\tilde{\gamma}}(A)) = cl_{\tilde{\gamma}}(cl_{\tilde{\gamma}}(X \setminus A)) \cap cl_{\tilde{\gamma}}(int_{\tilde{\gamma}}(A)) = cl_{\tilde{\gamma}}(X \setminus A) \cap cl_{\tilde{\gamma}}(int_{\tilde{\gamma}}(A))$ (by Theorem 3.18 (vi)) $\subseteq cl_{\tilde{\gamma}}(X \setminus A) \cap cl_{\tilde{\gamma}}(A) = bd_{\tilde{\gamma}}(A)$. This shows that $bd_{\tilde{\gamma}}(int_{\tilde{\gamma}}(A)) \subseteq bd_{\tilde{\gamma}}(A)$.

(vi) By Definition 3.22, we have that $bd_{\tilde{\gamma}}(cl_{\tilde{\gamma}}(A)) = cl_{\tilde{\gamma}}(X \setminus cl_{\tilde{\gamma}}(A)) \cap cl_{\tilde{\gamma}}(cl_{\tilde{\gamma}}(A)) = cl_{\tilde{\gamma}}(int_{\tilde{\gamma}}(X \setminus A)) \cap cl_{\tilde{\gamma}}(A)$ (by Theorem 3.18 (vi)) $\subseteq cl_{\tilde{\gamma}}(X \setminus A) \cap cl_{\tilde{\gamma}}(A) = bd_{\tilde{\gamma}}(A)$. Therefore $bd_{\tilde{\gamma}}(cl_{\tilde{\gamma}}(A)) \subseteq bd_{\tilde{\gamma}}(A)$.

(vii) Follows from (iv) and Theorem 3.24.

(viii) Using (vii) $(X \setminus bd_{\tilde{\gamma}}(A)) \cup bd_{\tilde{\gamma}}(A) = [int_{\tilde{\gamma}}(A) \cup int_{\tilde{\gamma}}(X \setminus A)] \cup bd_{\tilde{\gamma}}(A)$. This implies that $X = int_{\tilde{\gamma}}(A) \cup int_{\tilde{\gamma}}(X \setminus A) \cup bd_{\tilde{\gamma}}(A)$. □

Theorem 3.26. *Let X be a topological space, $\gamma : \tau \to P(X)$ be an operation on τ and A be a subset of X. Then*
(i) $A \in \tilde{\gamma}O(X)$ if and only if $A \cap bd_{\tilde{\gamma}}(A) = \emptyset$;
(ii) $A \in \tilde{\gamma}C(X)$ if and only if $bd_{\tilde{\gamma}}(A) \subseteq A$.

Proof. Let $A \in \tilde{\gamma}O(X)$. Then $X \setminus A \in \tilde{\gamma}C(X)$ and $cl_{\tilde{\gamma}}(X \setminus A) = X \setminus A$. Also $A \neq cl_{\tilde{\gamma}}(A)$. By Definition 3.22, $A \cap bd_{\tilde{\gamma}}(A) = A \cap (cl_{\tilde{\gamma}}(A) \cap cl_{\tilde{\gamma}}(X \setminus A)) = A \cap cl_{\tilde{\gamma}}(A \cap (X \setminus A)) = A \cap \emptyset = \emptyset$. Thus $A \cap bd_{\tilde{\gamma}}(A) = \emptyset$. Conversely, assume that $A \cap bd_{\tilde{\gamma}}(A) = \emptyset$. Then $A \cap (cl_{\tilde{\gamma}}(A) \cap cl_{\tilde{\gamma}}(X \setminus A)) = \emptyset$. This implies that $A \cap cl_{\tilde{\gamma}}(X \setminus A) = \emptyset$ and hence $cl_{\tilde{\gamma}}(X \setminus A) \subseteq X \setminus A$. Therefore $cl_{\tilde{\gamma}}(X \setminus A) = X \setminus A$. This shows that $X \setminus A \in \tilde{\gamma}C(X)$ and hence $A \in \tilde{\gamma}O(X)$

(ii) Let $A \in \tilde{\gamma}C(X)$. Then $A = cl_{\tilde{\gamma}}(A)$. Since $bd_{\tilde{\gamma}}(A) = (cl_{\tilde{\gamma}}(A) \cap cl_{\tilde{\gamma}}(X \setminus A)) \subseteq cl_{\tilde{\gamma}}(A) = A$. Therefore $bd_{\tilde{\gamma}}(A) \subseteq A$. Conversely, let $bd_{\tilde{\gamma}}(A) \subseteq A$. Then $bd_{\tilde{\gamma}}(A) \cap (X \setminus A) = \emptyset$. By Theorem 3.25(i), $bd_{\tilde{\gamma}}(X \setminus A) \cap (X \setminus A) = \emptyset$. By (i), $X \setminus A \in \tilde{\gamma}O(X)$. Hence $A \in \tilde{\gamma}C(X)$. □

4. $\tilde{\gamma}$-separation axioms

Definition 4.1. *A topological space X is called a $\tilde{\gamma}$-T_0 space if for each pair of distinct points $x, y \in X$, there exists a set $U \in \tilde{\gamma}O(X)$ such that either $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$.*

Definition 4.2. *A topological space X is called a $\tilde{\gamma}$-T_1 space if for each pair of distinct points $x, y \in X$, there exists sets $U, V \in \tilde{\gamma}O(X)$ containing x and y respectively such that $y \notin U$ and $x \notin V$.*

Definition 4.3. *A topological space X is called a $\tilde{\gamma}$-T_2 space if for each pair of distinct points $x, y \in X$, there exists sets $U, V \in \tilde{\gamma}O(X)$ such that $x \in U$ and $y \in V$ and $U \cap V = \emptyset$.*

Definition 4.4. *Let X be a topological space, $\gamma : \tau \to P(X)$ be an operation on τ and A be a subset of X. Then A is called a $\tilde{\gamma}$-generalized closed (briefly $\tilde{\gamma}$-g.closed) set if $cl_{\tilde{\gamma}}(A) \subseteq U$ whenever $A \subseteq U$ and $U \in \tilde{\gamma}O(X)$. The collection of all $\tilde{\gamma}$-g.closed sets is denoted by $\tilde{\gamma}GC(X)$.*

Note that every $\tilde{\gamma}$-closed set is $\tilde{\gamma}$-g.closed, but the converse need not be true.

Definition 4.5. *A topological space X is called a $\tilde{\gamma}$-$T_{\frac{1}{2}}$ space if each $\tilde{\gamma}$-g.closed set of X is $\tilde{\gamma}$-closed.*

Theorem 4.6. *Let X be a topological space and $\gamma : \tau \to P(X)$ be an operation on τ . Then for a point $x \in X$, $x \in cl_{\tilde{\gamma}}(A)$ if and only if $V \cap A \neq \emptyset$ for any $V \in \tilde{\gamma}O(X)$ such that $x \in V$.*

Proof. Let F_0 be the set of all $y \in X$ such that $V \cap A \neq \emptyset$ for any $V \in \tilde{\gamma}O(X)$ and $y \in V$. Now, we prove that $cl_{\tilde{\gamma}}(A) = F_0$. Let us assume $x \in cl_{\tilde{\gamma}}(A)$ and $x \notin F_0$. Then there exists a set $U \in \tilde{\gamma}O(X)$ containing x such that $U \cap A = \emptyset$. This implies that $A \subseteq X \setminus U$. Therefore $cl_{\tilde{\gamma}}(A) \subseteq X \setminus U$. Hence $x \notin cl_{\tilde{\gamma}}(A)$. This is a contradiction. Hence $cl_{\tilde{\gamma}}(A) \subseteq F_0$. Conversely, let F be a set such that $A \subseteq F$ and $X \setminus F \in \tilde{\gamma}O(X)$. Let $x \notin F$. Then we have that $x \in X \setminus F$ and $(X \setminus F) \cap A = \emptyset$. This implies that $x \notin F_0$. Therefore $F_0 \subseteq F$. Hence $F_0 \subseteq cl_{\tilde{\gamma}}(A)$. □

Theorem 4.7. *Let X be a topological space and $\gamma : \tau \to P(X)$ be an operation on τ and A be a subset of X. Then $A \in \tilde{\gamma}GC(X)$ if and only if $cl_{\tilde{\gamma}}(\{x\}) \cap A \neq \emptyset$ holds for every $x \in cl_{\tilde{\gamma}}(A)$.*

Proof. Let $U \in \tilde{\gamma}O(X)$ such that $A \subseteq U$. Let $x \in cl_{\tilde{\gamma}}(A)$. By assumption there exists a point $z \in cl_{\tilde{\gamma}}(\{x\})$ and $z \in A \subseteq U$. Then by Theorem 4.6, we have that $U \cap \{x\} \neq \emptyset$. This implies that $x \in U$ and hence $A \in \tilde{\gamma}GC(X)$. Conversely, suppose there exists a point $x \in cl_{\tilde{\gamma}}(A)$ such that $cl_{\tilde{\gamma}}(\{x\}) \cap A = \emptyset$. Since $cl_{\tilde{\gamma}}(\{x\}) \in \tilde{\gamma}C(X)$ implies that $X \setminus cl_{\tilde{\gamma}}(\{x\}) \in \tilde{\gamma}O(X)$. Since $A \subseteq X \setminus cl_{\tilde{\gamma}}(\{x\})$ and $A \in \tilde{\gamma}GC(X)$, implies that $cl_{\tilde{\gamma}}(A) \subseteq X \setminus cl_{\tilde{\gamma}}(\{x\})$. Hence $x \notin cl_{\tilde{\gamma}}(A)$. This is a contradiction. □

Theorem 4.8. *Let X be a topological space and $\gamma : \tau \to P(X)$ be an operation on τ and A be a subset of X. Then $cl_{\tilde{\gamma}}(\{x\}) \cap A \neq \emptyset$ for every $x \in cl_{\tilde{\gamma}}(A)$ if and only if $cl_{\tilde{\gamma}}(A) \subseteq ker_{\tilde{\gamma}}(A)$ holds, where $ker_{\tilde{\gamma}}(E) = \cap\{V : V \in \tilde{\gamma}O(X)$ and $E \subseteq V\}$ for any subset E of X.*

Proof. Let $x \in cl_{\tilde{\gamma}}(A)$. By hypothesis, there exists a point z such that $z \in cl_{\tilde{\gamma}}(\{x\})$ and $z \in A$. Let $U \in \tilde{\gamma}O(X)$ be a subset of X such that $A \subseteq U$. Since $z \in U$ and $z \in cl_{\tilde{\gamma}}(\{x\})$. By Theorem 4.7, we have that $U \cap \{x\} \neq \emptyset$, this implies that $x \in ker_{\tilde{\gamma}}(A)$. Hence $cl_{\tilde{\gamma}}(A) \subseteq ker_{\tilde{\gamma}}(A)$. Conversely, let $U \in \tilde{\gamma}O(X)$ such that $A \subseteq U$. Let x be a point such that $x \in cl_{\tilde{\gamma}}(A)$. By hypothesis, $x \in ker_{\tilde{\gamma}}(A)$ holds. Namely, we have that $x \in U$, because $A \subseteq U$ and $U \in \tilde{\gamma}O(X)$. Therefore $cl_{\tilde{\gamma}}(A) \subseteq U$. By Definition 4.4, $A \in \tilde{\gamma}GC(X)$. Then by Theorem 4.7, $cl_{\tilde{\gamma}}(\{x\}) \cap A \neq \emptyset$ holds for every $x \in cl_{\tilde{\gamma}}(A)$. □

Theorem 4.9. *Let X be a topological space and $\gamma : \tau \to P(X)$ be an operation on τ and A be a subset of X. If $A \in \tilde{\gamma}GC(X)$, then $cl_{\tilde{\gamma}}(A) \setminus A$ does not contain a non empty $\tilde{\gamma}$-closed set.*

Proof. Suppose there exists a non empty set $F \in \tilde{\gamma}C(X)$ such that $F \subseteq cl_{\tilde{\gamma}}(A) \setminus A$. Let $x \in F$. Then $x \in cl_{\tilde{\gamma}}(A)$, implies that $F \cap A = cl_{\tilde{\gamma}}(A) \cap A \supseteq cl_{\tilde{\gamma}}(\{x\}) \cap A \neq \emptyset$ and hence $F \cap A \neq \emptyset$. This is a contradiction. □

Theorem 4.10. *For each $x \in X$, $\{x\} \in \tilde{\gamma}C(X)$ or $X \setminus \{x\} \in \tilde{\gamma}GC(X)$.*

Proof. Suppose that $\{x\} \notin \tilde{\gamma}C(X)$. Then $X \setminus \{x\} \notin \tilde{\gamma}O(X)$. This implies that $X \in \tilde{\gamma}O(X)$ and the set X only containing $X \setminus \{x\}$. Hence $X \setminus \{x\} \in \tilde{\gamma}GC(X)$. □

Theorem 4.11. *A topological space X is a $\tilde{\gamma}$-$T_{\frac{1}{2}}$ space if and only if for each $x \in X$, $\{x\} \in \tilde{\gamma}O(X)$ or $\{x\} \in \tilde{\gamma}C(X)$.*

Proof. Suppose that $\{x\} \notin \tilde{\gamma}C(X)$. Then it follows from the assumption and Theorem 4.10, $\{x\} \in \tilde{\gamma}O(X)$. Conversely, let $F \in \tilde{\gamma}GC(X)$. Let $x \in cl_{\tilde{\gamma}}(F)$. Then by the assumption $\{x\} \in \tilde{\gamma}O(X)$ or $\{x\} \in \tilde{\gamma}C(X)$.

Case(i): Suppose that $\{x\} \in \tilde{\gamma}O(X)$. Then by Theorem 4.6, $\{x\} \cap F \neq \emptyset$. This implies that $cl_{\tilde{\gamma}}(F) = F$. Therefore X is a $\tilde{\gamma}$-$T_{\frac{1}{2}}$ space.

Case(ii): Suppose that $\{x\} \in \tilde{\gamma}C(X)$. Let us assume $x \notin F$. Then $x \in cl_{\tilde{\gamma}}(F) \setminus F$. This is a contradiction. Hence $x \in F$. Therefore X is a $\tilde{\gamma}$-$T_{\frac{1}{2}}$ space. □

Theorem 4.12. *A space X is $\tilde{\gamma}$-T_1 if and only if for any $x \in X$, $\{x\} \in \tilde{\gamma}C(X)$.*

Proof. Follows from Definitions 3.10 and 4.2. □

Remark 4.13. *(i) From the Theorems 4.10, 4.11 and 4.12, we have that every $\tilde{\gamma}$-$T_{\frac{1}{2}}$ space is $\tilde{\gamma}$-T_0, every $\tilde{\gamma}$-T_1 space is $\tilde{\gamma}$-$T_{\frac{1}{2}}$ and every $\tilde{\gamma}$-T_2 space is $\tilde{\gamma}$-T_1.*

(ii) Let X be the set of real numbers, τ be the co-finite topology on X and define an operation $\gamma : \tau \to P(X)$ for a particular point $p \in X$ by

$$\gamma(A) = \begin{cases} A \text{ if } p \in A \\ cl(A) \text{ if } p \notin A \end{cases} \text{ for every } A \in \tau.$$

Then X is a $\tilde{\gamma}$-T_0 space but not $\tilde{\gamma}$-$T_{\frac{1}{2}}$.

(iii) Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}\}$ and define an operation $\gamma : \tau \to P(X)$ by $\gamma(A) = A \cup \{a\}$ for every $A \in \tau$. Then X is a $\tilde{\gamma}$-$T_{\frac{1}{2}}$ space but not $\tilde{\gamma}$-T_1.

(iv) Let $X = \{a, b, c\}$, $\tau = P(X)$ and define an operation $\gamma : \tau \to P(X)$ by

$$\gamma(A) = \begin{cases} A \cup \{c\} \text{ if } A = \{a\} \text{ or } \{b\} \\ A \cup \{a\} \text{ if } A = \{c\} \\ A \text{ if } A \neq \{a\}, \{b\} \text{ and } \{c\} \end{cases} \text{ for every } A \in \tau.$$

Then X is a $\tilde{\gamma}$-T_1 space but not $\tilde{\gamma}$-T_2.

5. $(\tilde{\gamma}, \tilde{\beta})$-continuous mappings

Definition 5.1. *A mapping $f : X \to Y$ is said to be $(\tilde{\gamma}, \tilde{\beta})$-continuous if $f^{-1}(V) \in \tilde{\gamma}O(X)$ whenever $V \in \tilde{\beta}O(Y)$.*

Example 5.2. *Let $X = \{a, b, c\}$, $Y = \{1, 2, 3\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$ and $\sigma = \{\emptyset, Y, \{1\}, \{3\}, \{1, 2\}, \{1, 3\}\}$ and define operations $\gamma : \tau \to P(X)$ and $\beta : \sigma \to P(Y)$ by*

$$\gamma(A) = \begin{cases} A \text{ if } c \notin A \\ A \cup \{a\} \text{ if } c \in A \end{cases} \text{ for every } A \in \tau \text{ and}$$

$$\beta(A) = \begin{cases} A \cup \{3\} \text{ if } A = \{1\} \\ cl(A) \text{ if } A \neq \{1\} \end{cases} \text{ for every } A \in \sigma \text{ respectively.}$$

Define $f : X \to Y$ by $f(a) = 3$, $f(b) = 1$ and $f(c) = 2$. Then the inverse image of every $\tilde{\beta}$-open set is $\tilde{\gamma}$-open under f. Hence f is $(\tilde{\gamma}, \tilde{\beta})$-continuous.

Definition 5.3. *(i) Let X be a topological space and $\gamma : \tau \to P(X)$ be an operation on τ. A subset A of a topological space X is said to be a $\tilde{\gamma}$-neighborhood of a point $x \in X$ if there exist a set $U \in \tilde{\gamma}O(X)$ such that $x \in U \subseteq A$.*

Note that $\tilde{\gamma}$-neighborhood of x may be replaced by $\tilde{\gamma}$-open neighborhood of x.

(ii) Let X be a topological space. $A \subseteq X$ and $p \in X$. Then p is called a $\tilde{\gamma}$-limit point of A if $U \cap (A - \{p\}) \neq \emptyset$, for any set $U \in \tilde{\gamma}O(X)$ containing p. The set of all $\tilde{\gamma}$-limit points of A is called a $\tilde{\gamma}$-derived set of A and is denoted by $d_{\tilde{\gamma}}(A)$. Clearly if $A \subseteq B$ then $d_{\tilde{\gamma}}(A) \subseteq d_{\tilde{\gamma}}(B)$.

Remark 5.4. *From the Definition 5.3(ii), it follows that p is a $\tilde{\gamma}$-limit point of A if and only if $p \in cl_{\tilde{\gamma}}(A - \{p\})$.*

Theorem 5.5. *For any $A, B \subseteq X$, the $\tilde{\gamma}$-derived sets have the following properties:*
(i) $cl_{\tilde{\gamma}}(A) \supseteq A \cup d_{\tilde{\gamma}}(A)$;
(ii) $\cup_i d_{\tilde{\gamma}}(A_i) = d_{\tilde{\gamma}}(\cup_i A_i)$;
(iii) $d_{\tilde{\gamma}}(d_{\tilde{\gamma}}(A)) \subseteq d_{\tilde{\gamma}}(A)$;
(iv) $cl_{\tilde{\gamma}}(d_{\tilde{\gamma}}(A)) = d_{\tilde{\gamma}}(A)$.

Proof. Follows from the Definition 5.4(ii) and Remark 5.4. □

Theorem 5.6. *Let $f : X \to Y$ be a mapping. Then the following statements are equivalent:*
(i) f is $(\tilde{\gamma}, \tilde{\beta})$-continuous;
(ii) for each point x in X, the inverse of every $\tilde{\beta}$-neighborhood of $f(x)$ is a $\tilde{\gamma}$-neighborhood of x;
(iii) for each point x in X and each $\tilde{\beta}$-neighborhood B of $f(x)$, there is a $\tilde{\gamma}$-neighborhood A of x such that $f(A) \subseteq B$;
(iv) for each $x \in X$ and each set $B \in \tilde{\beta}O(Y)$ contains $f(x)$, there is a set $A \in \tilde{\gamma}O(X)$ containing x such that $f(A) \subseteq B$;
(v) $f(cl_{\tilde{\gamma}}(A)) \subseteq cl_{\tilde{\beta}}(f(A))$ holds for every subset A of X;
(vi) for any set $H \in \tilde{\beta}C(Y)$, $f^{-1}(H) \in \tilde{\gamma}C(X)$.

Proof. (i) ⇒ (ii). Let $x \in X$ and B be a $\tilde{\beta}$-neighborhood of $f(x)$. By Definition 5.3(i), there exist $V \in \tilde{\beta}O(Y)$ such that $f(x) \in V \subseteq B$. This implies that $x \in f^{-1}(V) \subseteq f^{-1}(B)$. Since f is $(\tilde{\gamma}, \tilde{\beta})$-continuous, so $f^{-1}(V) \in \tilde{\gamma}O(X)$. Hence $f^{-1}(B)$ is a $\tilde{\gamma}$-neighborhood of x.

(ii) ⇒ (i). Let $B \in \tilde{\beta}O(Y)$. Put $A = f^{-1}(B)$. Let $x \in A$. Then $f(x) \in B$. Clearly, B (being $\tilde{\beta}$-open) is a $\tilde{\beta}$-neighborhood of $f(x)$. By (ii), $A = f^{-1}(B)$ is a $\tilde{\gamma}$-neighborhood of x. Hence by Definition 5.3(i), there exist $A_x \in \tilde{\gamma}O(X)$ such that $x \in A_x \subseteq A$. This implies that $A = \cup_{x\in A} A_x$. By Theorem 3.8, we have that $A \in \tilde{\gamma}O(X)$. Therefore f is $(\tilde{\gamma}, \tilde{\beta})$-continuous.

(i) ⇒ (iii). Let $x \in X$ and B be a $\tilde{\beta}$-neighborhood of $f(x)$. Then, there exist $O_{f(x)} \in \tilde{\beta}O(Y)$ such that $f(x) \in O_{f(x)} \subseteq B$. It follows that $x \in f^{-1}(O_{f(x)}) \subseteq f^{-1}(B)$. By (i), $f^{-1}(O_{f(x)}) \in \tilde{\gamma}O(X)$. Let $A = f^{-1}(B)$. Then it follows that A is $\tilde{\gamma}$-neighborhood of x and $f(A) = f(f^{-1}(B)) \subseteq B$.

(iii) ⇒ (i). Let $U \in \tilde{\beta}O(Y)$. Take $W = f^{-1}(U)$. Let $x \in W$. Then $f(x) \in U$. Thus U is a $\tilde{\beta}$-neighborhood of $f(x)$. By (iii), there exist a $\tilde{\gamma}$-neighborhood V_x of x such that $f(V_x) \subseteq U$. Thus it follows that $x \in V_x \subseteq f^{-1}(f(V_x)) \subseteq f^{-1}(U) = W$. Since V_x is a $\tilde{\gamma}$-neighborhood of x, which implies that there exist a $W_x \in \tilde{\gamma}O(X)$ such that $x \in W_x \subseteq W$. This implies that $W = \cup_{x\in W} W_x$. By Theorem 3.8, $W \in \tilde{\gamma}O(X)$. Thus f is $(\tilde{\gamma}, \tilde{\beta})$-continuous.

(iii) ⇒ (iv). We may replace the $\tilde{\gamma}$-neighborhood of x as $\tilde{\gamma}$-open neighborhood of x in condition (iii). Straightforward.

(iv) ⇒ (v). Let $y \in f(cl_{\tilde{\gamma}}(A))$ and any set $V \in \tilde{\beta}O(Y)$ containing y. Then, there exist a point $x \in X$ and a set $U \in \tilde{\gamma}O(X)$ such that $x \in U$ with $f(x) = y$ and $f(U) \subseteq V$. Since $x \in cl_{\tilde{\gamma}}(A)$, we have that $U \cap A \neq \emptyset$ and hence $\emptyset \neq f(U \cap A) \subseteq f(U) \cap f(A) \subseteq V \cap f(A)$. This implies that $y \in cl_{\tilde{\beta}}(f(A))$. Therefore, we have that $f(cl_{\tilde{\gamma}}(A)) \subseteq cl_{\tilde{\beta}}(f(A))$.

(v) ⇒ (vi). Let $H \in \tilde{\beta}C(Y)$. Then $cl_{\tilde{\beta}}(H) = H$. By (v), $f(cl_{\tilde{\gamma}}(f^{-1}(H))) \subseteq cl_{\tilde{\beta}}(f(f^{-1}(H))) \subseteq cl_{\tilde{\beta}}(H) = H$ holds. Therefore $cl_{\tilde{\gamma}}(f^{-1}(H)) \subseteq f^{-1}(H)$ and thus $f^{-1}(H) = cl_{\tilde{\gamma}}(f^{-1}(H))$. Hence $f^{-1}(H) \in \tilde{\gamma}C(X)$.

(vi) ⇒ (i). Let $B \in \tilde{\beta}O(X)$. We take $H = Y - B$. Then $H \in \tilde{\beta}C(Y)$. By (iv), $f^{-1}(H) \in \tilde{\gamma}C(X)$. Hence $f^{-1}(B) = X - f^{-1}(Y - B) = X - f^{-1}(H) \in \tilde{\gamma}O(X)$. □

Theorem 5.7. *A mapping $f : X \to Y$ is $(\tilde{\gamma}, \tilde{\beta})$-continuous if and only if $f(d_{\tilde{\gamma}}(A)) \subseteq cl_{\tilde{\beta}}(f(A))$, for all $A \subseteq X$.*

Proof. Let $f : X \to Y$ be $(\tilde{\gamma}, \tilde{\beta})$-continuous. Let $A \subseteq X$ and $x \in d_{\tilde{\gamma}}(A)$. Assume that $f(x) \notin f(A)$ and let V denote a $\tilde{\beta}$-neighborhood of $f(x)$. Since f is $(\tilde{\gamma}, \tilde{\beta})$-continuous, so by Theorem 5.6(iii), there exist a $\tilde{\gamma}$-neighborhood U of x such that $f(U) \subseteq V$. From $x \in d_{\tilde{\gamma}}(A)$, it follows that $U \cap A \neq \emptyset$; there exist, therefore, at least one element $a \in U \cap A$ such that $f(a) \in f(A)$ and $f(a) \in V$. Since $f(x) \notin f(A)$, we have that $f(a) \neq f(x)$. Thus every $\tilde{\beta}$-neighborhood of $f(x)$ contains an element $f(a)$ of $f(A)$ different from $f(x)$. Consequently, $f(x) \in d_{\tilde{\beta}}(f(A))$. Conversely, suppose that f is not $(\tilde{\gamma}, \tilde{\beta})$-continuous. Then by Theorem 5.6(iii), there exist $x \in X$ and a $\tilde{\beta}$-neighborhood V of $f(x)$ such that every $\tilde{\gamma}$-neighborhood U of x contains at least one element $a \in U$ for which $f(a) \notin V$. Put $A = \{a \in X : f(a) \notin V\}$. Since $f(x) \in V$, therefore $x \notin A$ and hence $f(x) \notin f(A)$. Since $f(A) \cap (V - \{f(x)\}) = \emptyset$, therefore $f(x) \notin d_{\tilde{\beta}}(f(A))$. It follows that $f(x) \in f(d_{\tilde{\gamma}}(A)) - (f(A) \cup d_{\tilde{\beta}}(f(A))) \neq \emptyset$, which is a contradiction to the given condition. □

Theorem 5.8. *Let $f : X \to Y$ be an injective mapping. Then f is $(\tilde{\gamma}, \tilde{\beta})$-continuous if and only if $f(d_{\tilde{\gamma}}(A)) \subseteq d_{\tilde{\beta}}(f(A))$, for all $A \subseteq X$.*

Proof. Let $A \subseteq X$, $x \in d_{\tilde{\gamma}}(A)$ and V be a $\tilde{\beta}$-neighborhood of $f(x)$. Since f is $(\tilde{\gamma}, \tilde{\beta})$-continuous, so by Theorem 5.6(iii), there exist a $\tilde{\gamma}$- neighborhood U of x such that $f(U) \subseteq V$. But $x \in d_{\tilde{\gamma}}(A)$ gives there exist an element $a \in U \cap A$ such that $a \neq x$. Clearly $f(a) \in f(A)$ and since f is injective, $f(a) \neq f(x)$. Thus every $\tilde{\beta}$-neighborhood V of $f(x)$ contains an element $f(a)$ of $f(A)$ different from $f(x)$. Consequently, $f(x) \in d_{\tilde{\beta}}(f(A))$. Therefore, we have that $f(d_{\tilde{\gamma}}(A)) \subseteq d_{\tilde{\beta}}(f(A))$. Converse follows from the Theorem 5.7. □

Theorem 5.9. *Let $f : X \to Y$ be a $(\tilde{\gamma}, \tilde{\beta})$-continuous and injective mapping. If Y is $\tilde{\beta}$-T_2 (resp. $\tilde{\beta}$-T_1), then X is $\tilde{\gamma}$-T_2 (resp. $\tilde{\gamma}$-T_1).*

Proof. Suppose Y is $\tilde{\beta}$-T_2. Let x and y be two distinct points of X. Then, there exist two sets $U, V \in \tilde{\beta}O(X)$-open such that $f(x) \in U$, $f(y) \in V$ and $U \cap V = \emptyset$. Since f is $(\tilde{\gamma}, \tilde{\beta})$-continuous, for U and V, there exist two sets $W, S \in \tilde{\gamma}O(X)$ such that $x \in W$ and $y \in S$, $f(W) \subseteq U$ and $f(S) \subseteq V$, implies that $W \cap S = \emptyset$. Hence X is $\tilde{\gamma}$-T_2. In similar way we prove X is $\tilde{\gamma}$-T_1 whenever Y is $\tilde{\beta}$-T_1. □

References

1. W. Dunham, *$T_{\frac{1}{2}}$ spaces*, Kyungpook Math. J., 17, 161-169, (1977).
2. D. S. Jankovic, *On functions with α-closed graphs*, Glasnik Mat., 18, 141-148, (1983).
3. S. Kasahara, *Operation-compact spaces*, Math. Japonica, 24, 97-105, (1979).
4. N. Levine, *Generalized closed sets in topology*, Rend. Circ. Mat. Palermo, 19, 89-96, (1970).
5. N. Levine, *Semi-open sets and semi continuity in topological spaces*, Amer. Math. Monthly, 70, 36-41, (1963).
6. H. Maki, H. Ogata, K. Balachandran, P. Sundaram and R. Devi, *The digital line and operation approches of $T_{\frac{1}{2}}$ spaces*, Scientiae Math., 3, 345-352, (2000).
7. H. Ogata, *Operation on topological spaces and associated topology*, Math. Japonica, 36, 175-184, (1991).
8. H. Ogata, *Remarks on Some Operation-Separation Axioms*, Bull. Fukuoka Univ. Ed. Part III, 40, 41-43, (1991).
9. G. Sai Sundara Krishnan and K. Balachadran, *On γ-semi open sets in topological spaces*, Bull. Cal. Math. Soc., 98, 517-530, (2006).
10. G. Sai Sundara Krishnan, D. Saravanakumar, M. Ganster and K. Balachandran, *On a class of γ^*-pre-open sets in topological spaces*, Kyungpook Math. J., 54, 173-188, (2014).
11. D. Saravanakumar, M. Ganster, N. Kalaivani and G. Sai Sundara Krishnan, *On (γ^*, β^*)-almost-pre-continuous mappings in topological spaces*, J. Egypt. Math. Soc., 23, 180-189, (2015).
12. D. Saravanakumar, N. Kalaivani and G. Sai Sundara Krishnan, *$\tilde{\mu}$-open sets in generalized topological spaces*, Malaya J. Mat., 3, 268-276, (2015).
13. D. Saravanakumar, N. Kalaivani and G. Sai Sundara Krishnan, *Operations pre-continuous mappings in topological spaces*, Acta Sci. et Int., 2, 30-42, (2018).

D. Saravanakumar,
PG and Research Department of Mathematics,
Thiagarajar College,
India.
E-mail address: drdstop@gmail.com

and

D. Saravanakumar,
Department of Mathematics,
M.S.S. Wakf Board College,
India.
E-mail address: drdstop@gmail.com

and

M. Mohamed Riyazdeen,
Department of Mathematics,
M.S.S. Wakf Board College,
India.
E-mail address: mhmdriyazdeen@gmail.com