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Existence and Controllability Results for Fuzzy Neutral Stochastic Differential Equations with Impulses

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abstract: In this paper, the existence, uniqueness and controllability of solutions for fuzzy neutral stochastic differential equations (FNSDEs) with impulses are considered based on the Banach fixed point theorem.

Key Words: Fuzzy neutral stochastic differential equations, Banach fixed point Theorem, controllability results.

Contents

1. Introduction

In this paper, we will study the existence, uniqueness and controllability of solutions for FNSDEs with impulses given by

$$\begin{cases} d\Big[y(t) - f(t, y(t))\Big] = \Big[Ay(t) + g(t, y(t)) + v(t)\Big]dt + \Big\langle h(t)d\mathbf{B}_H(t)\Big\rangle, t \in \mathcal{J} = [0, T]. \\ \Delta y(t_k) = I_k(y(t_k^-)), \qquad k = 1, ..., m \qquad t \neq t_k. \\ y(t) = y_0. \end{cases} \tag{1.1}$$

Where $A : \mathcal{J} \longrightarrow \mathcal{F}_{\mathbb{R}^n}$ is a fuzzy coefficient, $f, g : \mathcal{J} \times \mathcal{F}_{\mathbb{R}^n} \longrightarrow \mathcal{F}_{\mathbb{R}^n}$ are nonlinear continuous functions, $h : \mathcal{J} \longrightarrow \mathbb{R}^n$, $v : \mathcal{J} \longrightarrow \mathcal{F}_{\mathbb{R}^n}$ is an admissible control function and $\mathbf{B}_H$ is a fractional Browian motion defined on a filtered probability space $\big(\Omega, \mathcal{A}, \{\mathcal{A}_t^H\}_{0\leq t\leq T}, \mathbb{P}\big)$, the initial data $y_0 \in \mathcal{F}_{\mathbb{R}^n}$ and $I_k \in C\big(\mathcal{F}_{\mathbb{R}^n}, \mathcal{F}_{\mathbb{R}^n}\big)$ are bounded functions, $\Delta y(t_k) = y(t_k^+) - y(t_k^-)$ represents the left and right limits of $y(t)$ at $t = t_k$ respectively, $k = 1, .., m$.

We wish to mention that the theory of fuzzy neutral stochastic differential equations with impulses have recently been the subject of important studies. As, for the controllability of fuzzy stochastic differential equations, even less has been done, with only a few works published in this topic as far as we know. In [18,19] Sakthivel et al studied the approximate controllability of nonlinear impulsive differential systems and stochastic systems with unbounded delay. Bouffoussi et al [11] studied the existence, uniqueness and asymptotic behavior of mild solutions for the neutral stochastic differential equations with finite delay. In [6] Arhrrabi et al studied the existence and stability of solutions of fuzzy fractional stochastic differential equations with fractional brownian motions. Park et al [20] proved the existence and uniqueness of fuzzy solutions and controllability for the impulsive semilinear fuzzy integrodifferential equations. In [12] Bouzahir et al discussed the controllability of neutral functional differential equations with infinite delay. Ahmed [2,3] studied the controllability of impulsive neutral stochastic differential equations with finite

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delay and fractional Browian motion in a Hilbert space. Achary et al in [1] studied the controllability of fuzzy solutions for first order nonlocal impulsive neutral functional differential equations by using the Banach fixed point theorem. Chalishajar et al [13] proved the existence, uniqueness and controllability for impulsive fuzzy neutral functional integrodifferential equations.
Our results are inspired by the one in [16] where the approximate controllability of impulsive neutral fuzzy stochastic differential equations with nonlocal condition in Banach space is studied. The rest of this paper is organized as follows, Section 2 summarizes the fundamental aspects. The existence, uniqueness and controllability results of solution for fuzzy neutral stochastic differential equations with impulses are proved in Section 3, an example in Section 4 is given to illustrate the results and we conclude the results in Section 5.

2. Preliminaries

Let $\mathcal{F}_{\mathbb{R}^n}$ denote the set of fuzzy subsets of the real axis, if $\Lambda : \mathbb{R}^n \longrightarrow [0,1]$, satisfying the following properties:

(i) Λ is normal, that is, there exists $z_0 \in \mathbb{R}^n$ such that $\Lambda(z_0) = 1$,

(ii) Λ is fuzzy convex, that is, for $0 \leq \lambda \leq 1$

$$\Lambda\big(\lambda z_1 + (1-\lambda) z_2\big) \geq min\left\{\Lambda(z_1), \Lambda(z_2)\right\}, \ for\ any\ z_1, z_2 \in \mathbb{R}^n,$$

(iii) Λ is upper semicontinous on $\mathbb{R}^n$,

(iv) $[\Lambda]^0 = cl\{z \in \mathbb{R}^n : \Lambda(z) > 0\}$ is compact, where cl denotes the closure in $(\mathbb{R}^n, |.|)$.
Then $\mathcal{F}_{\mathbb{R}^n}$ is called the space of fuzzy number. For $\alpha \in (0,1]$, we denote $[\Lambda]^\alpha = \{z \in \mathbb{R}^n | \Lambda(z) \geq \alpha\}$ and $[\Lambda]^0 = \{z \in \mathbb{R}^n | \Lambda(z) > 0\}$. From the conditions (i) to (iv), it follows that the $\alpha - level$ set of Λ, $[\Lambda]^\alpha$, is a nonempty compact interval, for all $\alpha \in [0,1]$ and any $\Lambda \in \mathcal{F}_{\mathbb{R}^n}$.
The notation $[\Lambda]^\alpha = [\underline{\Lambda}(\alpha), \overline{\Lambda}(\alpha)]$, denotes explicitly the $\alpha - level$ set of Λ, for $\alpha \in [0,1]$. We refer to $\underline{\Lambda}$ and $\overline{\Lambda}$ as the lower and upper branches of Λ, respectively. For $\Lambda \in \mathcal{F}_{\mathbb{R}^n}$, we define the lengh of the $\alpha - level$ set of Λ as $len([\Lambda]^\alpha) = \overline{\Lambda}(\alpha) - \underline{\Lambda}(\alpha)$. For addition and scalar multiplication in fuzzy set space $\mathcal{F}_{\mathbb{R}^n}$, we have $[\Lambda_1 + \Lambda_2]^\alpha = [\Lambda_1]^\alpha + [\Lambda_2]^\alpha$, $[\lambda\Lambda]^\alpha = \lambda[\Lambda]^\alpha$.
The Hausdorff distance between fuzzy numbers is given by

$$\begin{aligned} \mathcal{D}_\infty\big(\Lambda_1, \Lambda_2\big) &= \sup_{0 \leq \alpha \leq 1} \left\{|\underline{\Lambda}_1(\alpha) - \underline{\Lambda}_2(\alpha)|, |\overline{\Lambda}_1(\alpha) - \overline{\Lambda}_2(\alpha)|\right\}, \\ &= \sup_{0 \leq \alpha \leq 1} \mathcal{D}_H\big([\Lambda_1]^\alpha, [\Lambda_2]^\alpha\big). \end{aligned}$$

The metric space $(\mathcal{F}_{\mathbb{R}^n}, \mathcal{D}_\infty)$ is complet metric space and the following properties of the metric $\mathcal{D}_\infty$ are valid.

$$\mathcal{D}_\infty\big(\Lambda_1 + \Lambda_3, \Lambda_2 + \Lambda_3\big) = \mathcal{D}_\infty\big(\Lambda_1, \Lambda_2\big),$$

$$\mathcal{D}_\infty\big(\lambda\Lambda_1, \lambda\Lambda_2\big) = |\lambda| \mathcal{D}_\infty\big(\Lambda_1, \Lambda_2\big),$$

$$\mathcal{D}_\infty\big(\Lambda_1, \Lambda_2\big) \leq \mathcal{D}_\infty\big(\Lambda_1, \Lambda_3\big) + \mathcal{D}_\infty\big(\Lambda_3, \Lambda_2\big),$$

for all $\Lambda_1, \Lambda_2, \Lambda_3 \in \mathcal{F}_{\mathbb{R}^n}$ and $\lambda \in \mathbb{R}^n$.

Definition 2.1. *[1] The metric D on $C\big(\mathcal{J}, \mathcal{F}_{\mathbb{R}^n}\big)$ is given by*

$$D\big(\Lambda_1, \Lambda_2\big) = \sup_{0 \leq t \leq T} \mathcal{D}_\infty\big(\Lambda_1(t), \Lambda_2(t)\big).$$

Remark 2.2. *$\big(C(\mathcal{J}, \mathcal{F}_{\mathbb{R}^n}), D\big)$ is a complete metric space.*

Definition 2.3. *[1] • The derivative $v'(t)$ of a fuzzy processus u is defined by*

$$[v'(t)]^{\alpha} = [(\underline{v}^{\alpha})'(t), (\overline{v}^{\alpha})'(t)],$$

provided that the equation define a fuzzy set $u'(t) \in \mathfrak{F}_{\mathbb{R}^n}$.
• The fuzzy integral $\int_a^b v(t)dt$, $a, b \in [0, T]$ is defined by

$$\left[\int_a^b v(t)dt\right]^{\alpha} = \left[\int_a^b \underline{v}^{\alpha}(t)dt, \int_a^b \overline{v}^{\alpha}(t)dt\right],$$

provided that the Lebesgue integral on the right hand side exist.

Definition 2.4. *[1] A mapping $f : \mathfrak{J} \longrightarrow \mathfrak{F}_{\mathbb{R}^n}$ is strongly measurable if for all $\alpha \in [0, 1]$ the set-valued function $F_{\alpha} : \mathfrak{J} \longrightarrow \mathbf{K}(\mathbb{R}^n)$ defined by $F_{\alpha}(t) = [f(t)]^{\alpha}$ is Lebesgue measurable when $\mathbf{K}(\mathbb{R}^n)$ has the topology induced by the Hausdorff metric.*

Definition 2.5. *[1] • A mapping $f : \mathfrak{J} \longrightarrow \mathfrak{F}_{\mathbb{R}^n}$ is called level wise continuous at $t_0 \in \mathfrak{J}$ if the multivalued mapping $F_{\alpha}(t) = [f(t)]^{\alpha}$ is continuous at $t = t_0$ with respect to the Hausdorff metric for all $\alpha \in [0, 1]$.*
• A mapping $f : \mathfrak{J} \longrightarrow \mathfrak{F}_{\mathbb{R}^n}$ is said to be integrably bounded if there is an integrable function $g(t)$ such that $\|y(t)\| \leq g(t)$ for every $y(t) \in F_0(t)$.
• A strongly measurable and integrably bounded mapping $f : \mathfrak{J} \longrightarrow \mathfrak{F}_{\mathbb{R}^n}$ is said to be integrable over $\mathfrak{J}$ if $\int_0^T f(t)dt \in \mathfrak{F}_{\mathbb{R}^n}$.

Remark 2.6. *If $f : \mathfrak{J} \longrightarrow \mathfrak{F}_{\mathbb{R}^n}$ is strongly measurable and integrably bounded, then f is integrable.*

Let $\langle . \rangle : \mathbb{R}^n \longrightarrow \mathfrak{F}_{\mathbb{R}^n}$ denote the embedding of $\mathbb{R}^n$ into $\mathfrak{F}_{\mathbb{R}^n}$, i.e. for $r \in \mathbb{R}^n$ we have

$$\langle r \rangle(a) = \begin{cases} 1\ , & if \quad a = r, \\ 0\ , & if \quad a \neq r. \end{cases}$$

Remark 2.7. *It is easy to see that if $y : \Omega \longrightarrow \mathbb{R}^n$ is an $\mathbb{R}^n$-valued random variable on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, then $\langle y \rangle : \Omega \longrightarrow \mathfrak{F}_{\mathbb{R}^n}$ is a fuzzy random variable. For stochastic processes we have a similar property.*

3. Main results

3.1. Existence and uniqueness result

In this subsection, we show the existence and uniqueness of fuzzy solution for FNSDEs with impulses given by (1.1) ($v \equiv 0$).

Definition 3.1. *A fuzzy process $\{y(t),\ t \in \mathfrak{J}\}$ is said to be solution of System (1.1) if:*

(i) $y(\cdot) \in C\big(\mathfrak{J}, \mathfrak{F}_{\mathbb{R}^n}\big)$,

(ii) $y(0) = y_0$,

(iii) for $t \in \mathfrak{J}$, we have

$$\begin{aligned} y(t) =& G(t)\Big[y_0 - f(0, y_0)\Big] + f(t, y(t)) + \int_0^t AG(t-s)f(s, y(s))ds + \int_0^t G(t-s)g(s, y(s))ds \\ &+ \left\langle \int_0^t G(t-s)h(s)d\mathbf{B}_H(s) \right\rangle + \sum_{t_k=0}^{t} G(t-t_k)I_k(y(t_k^-)). \end{aligned} \tag{3.1}$$

Let us introduce the following hypotheses.

($\mathcal{H}1$) Let $G(t) \in \mathcal{F}_{\mathbb{R}^n}$, such that

$$\big[G(t)\big]^{\alpha} = \big[\underline{G}^{\alpha}(t), \overline{G}^{\alpha}(t)\big], \qquad G(0) = I,$$

and $\underline{G}^{\alpha}, \overline{G}^{\alpha}$ are continuous such that $\max\{|\underline{G}^{\alpha}(t)|, |\overline{G}^{\alpha}(t)|\} \leq M$ and $|AG(t)| \leq N$, $\forall t \in \mathcal{J}$.

($\mathcal{H}2$) The functions $f, g : \mathcal{J} \times \mathcal{F}_{\mathbb{R}^n} \longrightarrow \mathcal{F}_{\mathbb{R}^n}$ are continuous and there exists a finite constants $\lambda, \gamma > 0$ such that

$$\mathcal{D}_H\Big([f(t, y(t))]^{\alpha}, [f(t, z(t))]^{\alpha} \Big) \leq \lambda \mathcal{D}_H\Big([y(t)]^{\alpha}, [z(t)]^{\alpha}\Big),$$

and

$$\mathcal{D}_H\Big([g(t, y(t))]^{\alpha}, [g(t, z(t))]^{\alpha} \Big) \leq \gamma \mathcal{D}_H\Big([y(t)]^{\alpha}, [z(t)]^{\alpha}\Big),$$

for all $y, z \in \mathcal{F}_{\mathbb{R}^n}$ and $t \in \mathcal{J}$.

($\mathcal{H}3$) For each $y, z \in \mathcal{F}_{\mathbb{R}^n}$, there exist a constant $\mu > 0$ such that

$$\mathcal{D}_H\Big([I_k(y(t_k^-))]^{\alpha}, [I_k(z(t_k^-))]^{\alpha}\Big) \leq \mu \mathcal{D}_H\Big([y(t)]^{\alpha}, [z(t)]^{\alpha}\Big), \; k = 1, .., m.$$

($\mathcal{H}4$) $\lambda + \lambda NT + (\gamma + \mu)MT < 1$.

Theorem 3.2. *Suppose that the hypotheses* ($\mathcal{H}1$) $-$ ($\mathcal{H}4$) *holds, then, for all* $T > 0$*, the System* (1.1) ($v \equiv 0$) *has a unique fuzzy solution on* $\mathcal{J}$*.*

Proof: For each $y \in \mathcal{F}_{\mathbb{R}^n}$ and $t \in \mathcal{J}$, define $\phi : C(\mathcal{J}, \mathcal{F}_{\mathbb{R}^n}) \longrightarrow C(\mathcal{J}, \mathcal{F}_{\mathbb{R}^n})$ by

$$\begin{aligned} \phi y(t) = G(t)\Big[y_0 - f(0, y_0)\Big] + f(t, y(t)) + \int_0^t AG(t-s) f(s, y(s)) ds + \int_0^t G(t-s) g(s, y(s)) ds \\ + \Big\langle \int_0^t G(t-s) h(s) d\mathbf{B}_H(s) \Big\rangle + \sum_{t_k=0}^{t} G(t - t_k) I_k(y(t_k^-)). \end{aligned}$$

Since $\underline{G}^{\alpha}, \overline{G}^{\alpha}$ are continuous, $\phi y : \mathcal{J} \longrightarrow \mathcal{F}_{\mathbb{R}^n}$ is continuous, so ϕ is a mapping from $C(\mathcal{J}, \mathcal{F}_{\mathbb{R}^n})$ into itself.

Now, for $y, z \in C(\mathcal{J}, \mathcal{F}_{\mathbb{R}^n})$, we have:

$$\begin{aligned}
&\mathcal{D}_H\Big(\big[\phi y(t)\big]^\alpha, \big[\phi z(t)\big]^\alpha\Big) = \mathcal{D}_H\bigg(\Big[G(t)\{y_0 - f(0,y_0)\} + f(t,y(t)) + \int_0^t AG(t-s)f(s,y(s))ds \\
&\quad + \int_0^t G(t-s)g(s,y(s))ds + \Big\langle \int_0^t G(t-s)h(s)d\mathbf{B}_H(s)\Big\rangle \\
&\quad + \sum_{t_k=0}^{t} G(t-t_k)I_k(y(t_k^-))\Big]^\alpha, \Big[G(t)\{y_0 - f(0,y_0)\} + f(t,z(t)) + \int_0^t AG(t-s)f(s,z(s))ds \\
&\quad + \int_0^t G(t-s)g(s,z(s))ds + \Big\langle \int_0^t G(t-s)h(s)d\mathbf{B}_H(s)\Big\rangle + \sum_{t_k=0}^{t} G(t-t_k)I_k(z(t_k^-))\Big]^\alpha\bigg), \\
&\quad = \mathcal{D}_H\bigg(\Big[G(t)\{y_0 - f(0,y_0)\}\Big]^\alpha + \Big[f(t,y(t))\Big]^\alpha + \Big[\int_0^t AG(t-s)f(s,y(s))ds\Big]^\alpha \\
&\quad + \Big[\int_0^t G(t-s)g(s,y(s))ds\Big]^\alpha + \Big[\Big\langle \int_0^t G(t-s)h(s)d\mathbf{B}_H(s)\Big\rangle\Big]^\alpha \\
&\quad + \Big[\sum_{t_k=0}^{t} G\big(t-t_k\big)I_k\big(y(t_k^-)\big)\Big]^\alpha, \Big[G(t)\{y_0 - f(0,y_0)\}\Big]^\alpha + \Big[f(t,z(t))\Big]^\alpha \\
&\quad + \Big[\int_0^t AG(t-s)f(s,z(s))ds\Big]^\alpha + \Big[\int_0^t G(t-s)g(s,z(s))ds\Big]^\alpha \\
&\quad + \Big[\Big\langle \int_0^t G(t-s)h(s)d\mathbf{B}_H(s)\Big\rangle\Big]^\alpha + \Big[\sum_{t_k=0}^{t} G(t-t_k)I_k(z(t_k^-))\Big]^\alpha\bigg),
\end{aligned}$$

then:

$$\begin{aligned}
&\mathcal{D}_H\Big(\Big[\phi y(t)\Big]^\alpha, \Big[\phi z(t)\Big]^\alpha\Big) \le \int_0^t \mathcal{D}_H\Big(\Big[AG(t-s)f(s,y(s))\Big]^\alpha, \Big[AG(t-s)f(s,y(s))\Big]^\alpha\Big)ds \\
&\quad + \mathcal{D}_H\Big(\Big[f(t,y(t))\Big]^\alpha, \Big[f(t,z(t))\Big]^\alpha\Big) + \int_0^t \mathcal{D}_H\Big(\Big[G(t-s)g(s,y(s))\Big]^\alpha, \Big[G(t-s)g(s,z(s))\Big]^\alpha\Big)ds \\
&\quad + \mathcal{D}_H\Big(\Big[\sum_{t_k=0}^{t} G(t-t_k)I_k(y(t_k^-))\Big]^\alpha, \Big[\sum_{t_k=0}^{t} G(t-t_k)I_k(z(t_k^-))\Big]^\alpha\Big), \\
&\quad \le \int_0^t \Big|AG(t-s)\Big|\mathcal{D}_H\Big(\Big[f(s,y(s))\Big]^\alpha, \Big[f(s,z(s))\Big]^\alpha\Big)ds + \lambda\mathcal{D}_H\Big(\Big[y(t)\Big]^\alpha, \Big[z(t)\Big]^\alpha\Big) \\
&\quad + \int_0^t \Big|G(t-s)\Big|\mathcal{D}_H\Big(\Big[g(s,y(s))\Big]^\alpha, \Big[g(s,z(s))\Big]^\alpha\Big)ds \\
&\quad + \sum_{t_k=0}^{t} \Big|G(t-t_k)\Big|\mathcal{D}_H\Big(\Big[I_k(y(t_k^-))\Big]^\alpha, \Big[I_k(z(t_k^-))\Big]^\alpha\Big), \\
&\quad \le \lambda N \int_0^t \mathcal{D}_H\Big(\Big[y(s)\Big]^\alpha, \Big[z(s)\Big]^\alpha\Big)ds + \lambda\mathcal{D}_H\Big(\Big[y(t)\Big]^\alpha, \Big[z(t)\Big]^\alpha\Big) \\
&\quad + \gamma M \int_0^t \mathcal{D}_H\Big(\Big[y(s)\Big]^\alpha, \Big[z(s)\Big]^\alpha\Big)ds + \mu M\mathcal{D}_H\Big(\Big[y(t)\Big]^\alpha, \Big[z(t)\Big]^\alpha\Big),
\end{aligned}$$

therefore:

$$\begin{aligned}\mathcal{D}_\infty\Big(\phi y(t),\phi z(t)\Big) &= \sup_{0\le\alpha\le1}\mathcal{D}_H\Big(\Big[\phi y(t)\Big]^\alpha,\Big[\phi z(t)\Big]^\alpha\Big)\le\lambda\sup_{0\le\alpha\le1}\mathcal{D}_H\Big(\Big[y(t)\Big]^\alpha,\Big[z(t)\Big]^\alpha\Big)\\
&+\lambda N\int_0^t\sup_{0\le\alpha\le1}\mathcal{D}_H\Big(\Big[y(s)\Big]^\alpha,\Big[z(s)\Big]^\alpha\Big)ds\\
&+\gamma M\int_0^t\sup_{0\le\alpha\le1}\mathcal{D}_H\Big(\Big[y(s)\Big]^\alpha,\Big[z(s)\Big]^\alpha\Big)ds+\mu M\sup_{0\le\alpha\le1}\mathcal{D}_H\Big(\Big[y(t)\Big]^\alpha,\Big[z(t)\Big]^\alpha\Big),\\
&\le\lambda\mathcal{D}_\infty\Big(y(t),z(t)\Big)+\lambda N\int_0^t\mathcal{D}_\infty\Big(y(s),z(s)\Big)ds\\
&+\gamma M\int_0^t\mathcal{D}_\infty\Big(y(s),z(s)\Big)ds+\mu M\mathcal{D}_\infty\Big(y(t),z(t)\Big),\end{aligned}$$

or we have

$$D\big(\phi y,\phi z\big)=\sup_{0\le t\le T}\mathcal{D}_\infty\big(\phi y(t),\phi z(t)\big),$$

then:

$$\begin{aligned}D\big(\phi y,\phi z\big)&\le\lambda\sup_{0\le t\le T}\mathcal{D}_\infty\Big(y(t),z(t)\Big)+\lambda N\int_0^t\sup_{0\le s\le T}\mathcal{D}_\infty\Big(y(s),z(s)\Big)ds\\
&+\gamma M\int_0^t\sup_{0\le s\le T}\mathcal{D}_\infty\Big(y(s),z(s)\Big)ds+\mu M\sup_{0\le t\le T}\mathcal{D}_\infty\Big(y(t),z(t)\Big),\\
&\le\Big(\lambda+\lambda NT+(\gamma+\mu)MT\Big)D\big(y,z\big).\end{aligned}$$

So, by the hypotheses $(\mathcal{H}4)$, ϕ is a contraction mapping. Hence, by Banach fixed point theorem, the fuzzy neutral stochastic differential equations (1.1) has a unique fixed point $y\in C(\mathcal{J},\mathcal{F}_{\mathbb{R}^n})$. □

3.2. Controllability result

In this subsection, we state the controllability result for fuzzy neutral stochastic differential equations with impulses given by (1.1) by using Banach fixed point theorem.

Definition 3.3. *[12] The System (1.1) is said to be controllable on $\mathcal{J}$, if there exist a fuzzy control function $v(t)$ such that a fuzzy solution $y(t)$ of (1.1) satisfies $y(T)=y_1$, i.e. $[y(T)]^\alpha=[y_1]^\alpha$, where $y_1\in\mathcal{F}_{\mathbb{R}^n}$ is a target set.*

Defined the fuzzy mapping $\mathcal{W}:P(\mathbb{R})\longrightarrow\mathcal{F}_{\mathbb{R}^n}$ by

$$\mathcal{W}(v)=\begin{cases}\int_0^T G(T-s)v(s)ds\ ,\ v\subset\overline{\Gamma}_v,\\ \quad 0\ ,\ \text{otherwise},\end{cases}$$

where $\overline{\Gamma}_v$ is the closure of support v and $P(\mathbb{R})$ is a nonempty fuzzy subset of $\mathbb{R}$. Then, the α-level of $\mathcal{W}$ is given by

$$\underline{\mathcal{W}}^\alpha(\underline{v})=\int_0^T\underline{G}^\alpha(T-s)\underline{v}(s)ds,\qquad \underline{v}(s)\in[\underline{v}^\alpha(s),v^1(s)],$$

$$\overline{\mathcal{W}}^\alpha(\overline{v})=\int_0^T\overline{G}^\alpha(T-s)\overline{v}(s)ds,\qquad \overline{v}(s)\in[v^1(s),\overline{v}^\alpha(s)].$$

We assume that $\underline{\mathcal{W}}$,$\overline{\mathcal{W}}$ are bijective mappings. Hence, We can introduce α-level set of $v(s)$ given by:

$$
\begin{aligned}
[v(s)]^\alpha &= [\underline{v}^\alpha(s), \overline{v}^\alpha(s)], \\
&= \Big[(\underline{\mathcal{W}}^\alpha)^{-1} \Big(\underline{y}_1^\alpha - \underline{G}^\alpha(T) \left\{ \underline{y_0}^\alpha - \underline{f}^\alpha(0, y_0) \right\} - \int_0^T \underline{A}^\alpha \underline{G}^\alpha(T-s) \underline{f}^\alpha(s, \underline{y}^\alpha(s)) ds \\
&- \underline{f}^\alpha(T, \underline{y}^\alpha(T)) - \int_0^T \underline{G}^\alpha(T-s) \underline{g}^\alpha(s, \underline{y}^\alpha(s)) ds - \int_0^T \underline{G}^\alpha(T-s) \underline{h}^\alpha(s) d\mathbf{B}_H(s) \\
&- \sum_{t_k=0}^{T} \underline{G}^\alpha(T-t_k) \underline{I}_k^\alpha(\underline{y}^\alpha(t_k^-)) \Big), (\overline{\mathcal{W}}^\alpha)^{-1} \Big(\overline{y}_1^\alpha - \overline{G}^\alpha(T) \Big\{ \overline{y_0}^\alpha - \overline{f}^\alpha(0, y_0) \Big\} \\
&- \int_0^T \overline{A}^\alpha \overline{G}^\alpha(T-s) \overline{f}^\alpha(s, \overline{y}^\alpha(s)) ds - \overline{f}^\alpha(T, \overline{y}^\alpha(T)) - \int_0^T \overline{G}^\alpha(T-s) \overline{g}^\alpha(s, \overline{y}^\alpha(s)) ds \\
&- \int_0^T \overline{G}^\alpha(T-s) \overline{h}^\alpha(s) d\mathbf{B}_H(s) - \sum_{t_k=0}^{T} \overline{G}^\alpha(T-t_k) \overline{I}_k^\alpha(\overline{y}^\alpha(t_k^-)) \Big) \Big].
\end{aligned}
$$

Then, substitute this expression into the Eq. (3.1) yields α-level of $y(T)$, we get

$$
\begin{aligned}
\big[y(T)\big]^{\alpha} = &\Bigg[G(T)\big[y_0 - f(0,y_0)\big] + f(T,y(t)) + \int_0^T AG(T-s)f(s,y(s))ds \\
&+ \int_0^T G(T-s)g(s,y(s))ds + \Big\langle \int_0^T G(T-s)h(s)d\mathbf{B}_H(s)\Big\rangle \\
&+ \sum_{t_k=0}^{T} G(T-t_k)I_k(y(t_k^-)) + \int_0^T G(T-s)\mathcal{W}^{-1}\Big(y_1 - G(T)[y_0 - f(0,y_0)] \\
&- f(T,y(t)) - \int_0^T AG(T-s)f(s,y(s))ds - \int_0^T G(T-s)g(s,y(s))ds \\
&- \Big\langle \int_0^T G(T-s)h(s)d\mathbf{B}_H(s)\Big\rangle - \sum_{t_k=0}^{T} G(T-t_k)I_k(y(t_k^-))\Big)ds\Bigg]^{\alpha}, \\
= &\Bigg[\underline{G}^{\alpha}(T)\left\{\underline{y_0}^{\alpha} - \underline{f}^{\alpha}(0,y_0)\right\} + \underline{f}^{\alpha}(T,\underline{y}^{\alpha}(T)) + \int_0^T \underline{A}^{\alpha}\underline{G}^{\alpha}(T-s)\underline{f}^{\alpha}(s,\underline{y}^{\alpha}(s))ds \\
&+ \int_0^T \underline{G}^{\alpha}(T-s)\underline{g}^{\alpha}(s,\underline{y}^{\alpha}(s))ds + \int_0^T \underline{G}^{\alpha}(T-s)\underline{h}^{\alpha}(s)d\mathbf{B}_H(s) \\
&+ \sum_{t_k=0}^{T} \underline{G}^{\alpha}(T-t_k)\underline{I}_k^{\alpha}(\underline{y}^{\alpha}(t_k^-)) + \int_0^T \underline{G}(T-s)(\underline{\mathcal{W}}^{\alpha})^{-1}\Big(\underline{y}_1^{\alpha} - \underline{G}^{\alpha}(T)\left\{\underline{y_0}^{\alpha} - \underline{f}^{\alpha}(0,y_0)\right\} \\
&- \underline{f}^{\alpha}(T,\underline{y}^{\alpha}(T)) - \int_0^T \underline{A}^{\alpha}\underline{G}^{\alpha}(T-s)\underline{f}^{\alpha}(s,\underline{y}^{\alpha}(s))ds - \int_0^T \underline{G}^{\alpha}(T-s)\underline{g}^{\alpha}(s,\underline{y}^{\alpha}(s))ds \\
&- \int_0^T \underline{G}^{\alpha}(T-s)\underline{h}^{\alpha}(s)d\mathbf{B}_H(s) - \sum_{t_k=0}^{T} \underline{G}^{\alpha}(T-t_k)\underline{I}_k^{\alpha}(\underline{y}^{\alpha}(t_k^-))\Big)ds, \overline{G}^{\alpha}(T)\left\{\overline{y_0}^{\alpha} - \overline{f}^{\alpha}(0,y_0)\right\} \\
&+ \int_0^T \overline{A}^{\alpha}\overline{G}^{\alpha}(T-s)\overline{f}^{\alpha}(s,\overline{y}^{\alpha}(s))ds + \overline{f}^{\alpha}(T,\overline{y}^{\alpha}(T)) + \int_0^T \overline{G}^{\alpha}(T-s)\overline{g}^{\alpha}(s,\overline{y}^{\alpha}(s))ds \\
&+ \int_0^T \overline{G}^{\alpha}(T-s)\overline{h}^{\alpha}(s)d\mathbf{B}_H(s) + \sum_{t_k=0}^{T} \overline{G}^{\alpha}(T-t_k)\overline{I}_k^{\alpha}(\overline{y}^{\alpha}(t_k^-)) \\
&+ \int_0^T \overline{G}((T-s))(\overline{\mathcal{W}}^{\alpha})^{-1}\Big(\overline{y}_1^{\alpha} - \overline{G}^{\alpha}(T)\left\{\overline{y_0}^{\alpha} - \overline{f}^{\alpha}(0,y_0)\right\} \\
&- \int_0^T \overline{A}^{\alpha}\overline{G}^{\alpha}(T-s)\overline{f}^{\alpha}(s,\overline{y}^{\alpha}(s))ds - \overline{f}^{\alpha}(T,\overline{y}^{\alpha}(T)) - \int_0^T \overline{G}^{\alpha}(T-s)\overline{g}^{\alpha}(s,\overline{y}^{\alpha}(s))ds \\
&- \int_0^T \overline{G}^{\alpha}(T-s)\overline{h}^{\alpha}(s)d\mathbf{B}_H(s) - \sum_{t_k=0}^{T} \overline{G}^{\alpha}(T-t_k)\overline{I}_k^{\alpha}(\overline{y}^{\alpha}(t_k^-))\Big)ds\Bigg],
\end{aligned}
$$

then, we have

$$
\begin{aligned}
\big[y(T)\big]^{\alpha} = \Bigg[& \underline{G}^{\alpha}(T)\left\{\underline{y_0}^{\alpha} - \underline{f}^{\alpha}(0,y_0)\right\} + \underline{f}^{\alpha}(T,\underline{y}^{\alpha}(T)) + \int_0^T \underline{A}^{\alpha}\underline{G}^{\alpha}(T-s)\underline{f}^{\alpha}(s,\underline{y}^{\alpha}(s))ds \\
& + \int_0^T \underline{G}^{\alpha}(T-s)\underline{g}^{\alpha}(s,\underline{y}^{\alpha}(s))ds + \int_0^T \underline{G}^{\alpha}(T-s)\underline{h}^{\alpha}(s)d\mathbf{B}_H(s) \\
& + \sum_{t_k=0}^{T} \underline{G}^{\alpha}(T-t_k)\underline{I}_k^{\alpha}(\underline{y}^{\alpha}(t_k^-)) + (\underline{\mathcal{W}}^{\alpha})(\underline{\mathcal{W}}^{\alpha})^{-1}\Bigg(\underline{y}_1^{\alpha} - \underline{G}^{\alpha}(T)\left\{\underline{y_0}^{\alpha} - \underline{f}^{\alpha}(0,y_0)\right\} \\
& - \underline{f}^{\alpha}(T,\underline{y}^{\alpha}(T)) - \int_0^T \underline{A}^{\alpha}\underline{G}^{\alpha}(T-s)\underline{f}^{\alpha}(s,\underline{y}^{\alpha}(s))ds - \int_0^T \underline{G}^{\alpha}(T-s)\underline{g}^{\alpha}(s,\underline{y}^{\alpha}(s))ds \\
& - \int_0^T \underline{G}^{\alpha}(T-s)\underline{h}^{\alpha}(s)d\mathbf{B}_H(s) - \sum_{t_k=0}^{T} \underline{G}^{\alpha}(T-t_k)\underline{I}_k^{\alpha}(\underline{y}^{\alpha}(t_k^-))\Bigg), \overline{G}^{\alpha}(T)\Big\{\overline{y_0}^{\alpha} - \overline{f}^{\alpha}(0,y_0)\Big\} \\
& + \int_0^T \overline{A}^{\alpha}\overline{G}^{\alpha}(T-s)\overline{f}^{\alpha}(s,\overline{y}^{\alpha}(s))ds + \overline{f}^{\alpha}(T,\overline{y}^{\alpha}(T)) + \int_0^T \overline{G}^{\alpha}(T-s)\overline{g}^{\alpha}(s,\overline{y}^{\alpha}(s))ds \\
& + \int_0^T \overline{G}^{\alpha}(T-s)\overline{h}^{\alpha}(s)d\mathbf{B}_H(s) + \sum_{t_k=0}^{T} \overline{G}^{\alpha}(T-t_k)\overline{I}_k^{\alpha}(\overline{y}^{\alpha}(t_k^-)) \\
& + (\overline{\mathcal{W}}^{\alpha})(\overline{\mathcal{W}}^{\alpha})^{-1}\Bigg(\overline{y}_1^{\alpha} - \overline{G}^{\alpha}(T)\Big\{\overline{y_0}^{\alpha} - \overline{f}^{\alpha}(0,y_0)\Big\} \\
& - \int_0^T \overline{A}^{\alpha}\overline{G}^{\alpha}(T-s)\overline{f}^{\alpha}(s,\overline{y}^{\alpha}(s))ds - \overline{f}^{\alpha}(T,\overline{y}^{\alpha}(T)) - \int_0^T \overline{G}^{\alpha}(T-s)\overline{g}^{\alpha}(s,\overline{y}^{\alpha}(s))ds \\
& - \int_0^T \overline{G}^{\alpha}(T-s)\overline{h}^{\alpha}(s)d\mathbf{B}_H(s) - \sum_{t_k=0}^{T} \overline{G}^{\alpha}(T-t_k)\overline{I}_k^{\alpha}(\overline{y}^{\alpha}(t_k^-))\Bigg)\Bigg] = \Big[\underline{y}_1^{\alpha}, \overline{y}_1^{\alpha}\Big] = \Big[y_1\Big]^{\alpha}.
\end{aligned}
$$

Now, we set

$$
\begin{aligned}
(\mathit{\Pi} y)(t) = & G(t)\big[y_0 - f(0,y_0)\big] + f(t,y(t)) + \int_0^t AG(t-s)f(s,y(s))ds \\
& + \int_0^t G(t-s)g(s,y(s))ds + \Big\langle \int_0^t G(t-s)h(s)d\mathbf{B}_H(s)\Big\rangle \\
& + \sum_{t_k=0}^{t} G(t-t_k)I_k(y(t_k^-)) + \int_0^t G(t-s)\mathcal{W}^{-1}\Big(y_1 - G(T)[y_0 - f(0,y_0)] \\
& - f(T,y(t)) - \int_0^T AG(T-s)f(s,y(s))ds - \int_0^T G(T-s)g(s,y(s))ds \\
& - \Big\langle \int_0^T G(T-s)h(s)d\mathbf{B}_H(s)\Big\rangle - \sum_{t_k=0}^{T} G(T-t_k)I_k(y(t_k^-))\Big)ds, \qquad t \in \mathcal{J}. \qquad (3.2)
\end{aligned}
$$

In the following theorem, the controllability of fuzzy solutions for (1.1) is established.

Theorem 3.4. *If the hypotheses* $(\mathcal{H}1)-(\mathcal{H}3)$ *are satisfied and* $\lambda(1+NT) + MT\big(\gamma+\mu+\lambda-\lambda NT - \gamma MT - \mu M\big) < 1$*, then for all* $T>0$*, the System* (1.1) *is controllable on* $\mathcal{J}$*.*

Proof: We can easily check that $\mathit{\Pi}$ is continuous mapping from $C(\mathcal{J},\mathcal{F}_{\mathbb{R}^n})$ to itself. For $y,z \in C(\mathcal{J},\mathcal{F}_{\mathbb{R}^n})$, we have

$$
\begin{aligned}
&\mathcal{D}_H\big([\Pi y(t)]^\alpha, [\Pi z(t)]^\alpha\big) = \mathcal{D}_H\Bigg(\bigg[G(t)\{y_0 - f(0,y_0)\}\bigg]^\alpha + \bigg[\int_0^t AG(t-s)f(s,y(s))ds\bigg]^\alpha \\
&+ \bigg[f(t,y(t))\bigg]^\alpha + \bigg[\int_0^t G(t-s)g(s,y(s))ds\bigg]^\alpha + \bigg[\bigg\langle \int_0^t G(t-s)h(s)d\mathbf{B}_H(s)\bigg\rangle\bigg]^\alpha \\
&+ \bigg[\sum_{t_k=0}^{t} G(t-t_k)I_k(y(t_k^-))\bigg]^\alpha + \bigg[\int_0^t G(t-s)\mathcal{W}^{-1}\bigg(y_1 - G(T)\{y_0 - f(0,y_0)\} \\
&- f(T,y(t)) - \int_0^T AG(T-s)f(s,y(s))ds - \int_0^T G(T-s)g(s,y(s))ds \\
&- \bigg\langle \int_0^T G(T-s)h(s)d\mathbf{B}_H(s)\bigg\rangle - \sum_{t_k=0}^{t} G(T-t_k)I_k(y(t_k^-))\bigg)ds\bigg]^\alpha, \bigg[G(t)\{y_0 \\
&- f(0,y_0)\}\bigg]^\alpha + \bigg[\int_0^t AG(t-s)f(s,z(s))ds\bigg]^\alpha + \bigg[\int_0^t G(t-s)g(s,z(s))ds\bigg]^\alpha \\
&+ \bigg[f(t,z(t))\bigg]^\alpha + \bigg[\bigg\langle \int_0^t G(t-s)h(s)d\mathbf{B}_H(s)\bigg\rangle\bigg]^\alpha + \bigg[\sum_{t_k=0}^{t} G(t-t_k)I_k(z(t_k^-))\bigg]^\alpha \\
&+ \bigg[\int_0^t G(t-s)\mathcal{W}^{-1}\bigg(y_1 - G(T)\{y_0 - f(0,y_0)\} - f(T,z(t)) - \int_0^T AG(T-s)f(s,z(s))ds \\
&- \int_0^T G(T-s)g(s,z(s))ds - \bigg\langle \int_0^T G(T-s)h(s)d\mathbf{B}_H(s)\bigg\rangle - \sum_{t_k=0}^{t} G(T-t_k)I_k(y(t_k^-))\bigg)ds\bigg]^\alpha\Bigg), \\
&\leq \lambda\mathcal{D}_H\bigg([y(t)]^\alpha, [z(t)]^\alpha\bigg) + \lambda N\int_0^t \mathcal{D}_H\bigg([y(s)]^\alpha, [z(s)]^\alpha\bigg)ds \\
&+ \gamma M\int_0^t \mathcal{D}_H\bigg([y(s)]^\alpha, [z(s)]^\alpha\bigg)ds + \mu M\mathcal{D}_H\bigg([y(t)]^\alpha, [z(t)]^\alpha\bigg) \\
&+ M\int_0^t \bigg(\lambda\mathcal{D}_H\bigg([y(t)]^\alpha, [z(t)]^\alpha\bigg) - \lambda N\int_0^T \mathcal{D}_H\bigg([y(s)]^\alpha, [z(s)]^\alpha\bigg)ds \\
&- \gamma M\int_0^T \mathcal{D}_H\bigg([y(s)]^\alpha, [z(s)]^\alpha\bigg)ds - \mu M\mathcal{D}_H\bigg([y(s)]^\alpha, [z(s)]^\alpha\bigg)\bigg)ds.
\end{aligned}
$$

Therefore,

$$\begin{aligned}
\mathcal{D}_\infty\Big(\Pi y(t), \Pi z(t)\Big) &= \sup_{0\le\alpha\le 1} \mathcal{D}_H\Big([\Pi y(t)]^\alpha, [\Pi z(t)]^\alpha\Big),\\
&\le \lambda \sup_{0\le\alpha\le 1} \mathcal{D}_H\Big([y(t)]^\alpha, [z(t)]^\alpha\Big) + \lambda N \int_0^t \sup_{0\le\alpha\le 1} \mathcal{D}_H\Big([y(s)]^\alpha, [z(s)]^\alpha\Big) ds\\
&+ \gamma M \int_0^t \sup_{0\le\alpha\le 1} \mathcal{D}_H\Big([y(s)]^\alpha, [z(s)]^\alpha\Big) ds + \mu M \sup_{0\le\alpha\le 1} \mathcal{D}_H\Big([y(t)]^\alpha, [z(t)]^\alpha\Big)\\
&+ M \int_0^t \Big(\lambda \sup_{0\le\alpha\le 1} \mathcal{D}_H\Big([y(t)]^\alpha, [z(t)]^\alpha\Big) - \mu M \sup_{0\le\alpha\le 1} \mathcal{D}_H\Big([y(s)]^\alpha, [z(s)]^\alpha\Big)\\
&- \lambda N \int_0^T \sup_{0\le\alpha\le 1} \mathcal{D}_H\Big([y(s)]^\alpha, [z(s)]^\alpha\Big) ds\\
&- \gamma M \int_0^T \sup_{0\le\alpha\le 1} \mathcal{D}_H\Big([y(s)]^\alpha, [z(s)]^\alpha\Big) ds\Big) ds,\\
&\le \lambda \mathcal{D}_\infty\Big(y(t), z(t)\Big) + \lambda N \int_0^t \mathcal{D}_\infty\Big(y(s), z(s)\Big) ds\\
&+ \gamma M \int_0^t \mathcal{D}_\infty\Big(y(s), z(s)\Big) ds + \mu M \mathcal{D}_\infty\Big(y(t), z(t)\Big)\\
&+ M \int_0^t \Big(\lambda \mathcal{D}_\infty\Big(y(t), z(t)\Big) - \lambda N \int_0^T \mathcal{D}_\infty\Big(y(s), z(s)\Big) ds\\
&- \gamma M \int_0^T \mathcal{D}_\infty\Big(y(s), z(s)\Big) ds - \mu M \mathcal{D}_\infty\Big(y(s), z(s)\Big)\Big) ds.
\end{aligned}$$

Hence,

$$\begin{aligned}
D\Big(\Pi y, \Pi z\Big) &= \sup_{0\le t\le T} \mathcal{D}_\infty\Big(\Pi y(t), \Pi z(t)\Big)\\
&\le \lambda \sup_{0\le t\le T} \mathcal{D}_\infty\Big(y(t), z(t)\Big) + \lambda N \int_0^t \sup_{0\le s\le T} \mathcal{D}_\infty\Big(y(s), z(s)\Big) ds\\
&+ \gamma M \int_0^t \sup_{0\le s\le T} \mathcal{D}_\infty\Big(y(s), z(s)\Big) ds + \mu M \sup_{0\le t\le T} \mathcal{D}_\infty\Big(y(t), z(t)\Big)\\
&+ M \int_0^t \Big(\lambda \sup_{0\le s\le T} \mathcal{D}_\infty\Big(y(t), z(t)\Big) - \lambda N \int_0^T \sup_{0\le s\le T} \mathcal{D}_\infty\Big(y(s), z(s)\Big) ds\\
&- \gamma M \int_0^T \sup_{0\le s\le T} \mathcal{D}_\infty(y(s), z(s)) ds - \mu M \sup_{0\le s\le T} \mathcal{D}_\infty\Big(y(s), z(s)\Big)\Big) ds,\\
&\le \lambda D(y, z) + \lambda NTD(y, z) + \gamma MTD(y, z) + \mu MD(y, z) + MT\Big(\lambda D(y, z)\\
&- \lambda NTD(y, z) - \gamma MTD(y, z) - \mu MD(y, z)\Big),\\
&\le \Big(\lambda(1 + NT) + MT\big(\gamma + \mu + \lambda - \lambda NT - \gamma MT - \mu M\big)\Big) D(y, z).
\end{aligned}$$

Then, Π is a contraction mapping. So, by Banach fixed point theorem, Eq (3.2) has a unique fixed point $x \in C(\mathcal{J}, \mathcal{F}_{\mathbb{R}^n})$. Thus, the System (1.1) is controllable on $\mathcal{J}$. □

4. Example

In this section, we give an example to illustrate our results. let

$$\begin{cases} \frac{d}{dt}\big[y(t) - 3ty^3(t)\big] = 2y(t) + 2ty^3(t) + v(t) + \big\langle h(t)d\mathbf{B}_H(t)\big\rangle, t \in \mathcal{J}, \\ I_k(y(t_k^-)) = \frac{2}{2+y(t_k)}, \\ y(0) = 0 \in \mathcal{F}_{\mathbb{R}^n}. \end{cases} \tag{4.1}$$

Then, we set $f(t, y(t)) = 3ty^3(t)$ and $g(t, y(t)) = 2ty^3(t)$. Hence, α-level of f is

$$\begin{aligned} \Big[f(t, y(t))\Big]^\alpha &= \Big[3ty^3(t)\Big]^\alpha, \\ &= \Big[t(\alpha+2)(\underline{y}^\alpha(t))^3, t(4-\alpha)(\overline{y}^\alpha(t))^3\Big], \qquad \forall\alpha \in [0, 1]. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \mathcal{D}_H\Big(\big[f(t,y(t))\big]^\alpha, \big[f(t, z(t))\big]^\alpha\Big) &= \mathcal{D}_H\Big(\big[t(\alpha+2)(\underline{y}^\alpha(t))^3, t(4-\alpha)\big(\overline{y}^\alpha(t)\big)^3\big]^\alpha, \big[t(\alpha+2) \\ &\big(\underline{z}^\alpha(t)\big)^3, t(4-\alpha)\big(\overline{z}^\alpha(t)\big)^3\big]^\alpha\Big), \\ &\le t\max\left\{(\alpha+2)|(\underline{y}^\alpha(t))^3 - (\underline{z}^\alpha(t))^3|, (4-\alpha)|(\overline{y}^\alpha(t))^3 - (\overline{z}^\alpha(t))^3|\right\}, \\ &\le 4t\Big((\overline{y}^\alpha(t))^2 + \overline{y}^\alpha(t)\overline{z}^\alpha(t) + (\overline{z}^\alpha(t))^2\Big)\max\left\{|\underline{y}^\alpha(t) - \underline{z}^\alpha(t)|, |\overline{y}^\alpha(t) - \overline{z}^\alpha(t)|\right\}, \\ &\le K_1\mathcal{D}_H\Big([y(t)]^\alpha, [z(t)]^\alpha\Big), \end{aligned}$$

where $K_1 = 4t\Big((\overline{y}^\alpha(t))^2 + \overline{y}^\alpha(t)\overline{z}^\alpha(t) + (\overline{z}^\alpha(t))^2\Big) > 0$.
On the other hand, the α-level of g is given by:

$$\begin{aligned} \Big[g(t, y(t))\Big]^\alpha &= \Big[2ty^3(t)\Big]^\alpha \\ &= \Big[t(\alpha+1)(\underline{y}^\alpha(t))^3, t(3-\alpha)(\overline{y}^\alpha(t))^3\Big], \qquad \forall\alpha \in [0, 1]. \end{aligned}$$

Therefore, we get:

$$\begin{aligned} \mathcal{D}_H\Big(\big[g(t,y(t))\big]^\alpha, \big[g(t, z(t))\big]^\alpha\Big) &= \mathcal{D}_H\Big(\big[t(\alpha+1)(\underline{y}^\alpha(t))^3, t(3-\alpha)(\overline{y}^\alpha(t))^3\big], \big[t(\alpha+1) \\ &(\underline{z}^\alpha(t))^3, t(3-\alpha)(\overline{z}^\alpha(t))^3\big]\Big), \\ &\le t\max\left\{(\alpha+1)|(\underline{y}^\alpha(t))^3 - (\underline{z}^\alpha(t))^3|, (3-\alpha)|(\overline{y}^\alpha(t))^3 - (\overline{z}^\alpha(t))^3|\right\}, \\ &\le 3t\Big((\overline{y}^\alpha(t))^2 + \overline{y}^\alpha(t)\overline{z}^\alpha(t) + (\overline{z}^\alpha(t))^2\Big)\max\left\{|\underline{y}^\alpha(t) - \underline{z}^\alpha(t)|, |\overline{y}^\alpha(t) - \overline{z}^\alpha(t)|\right\}, \\ &\le K_2\mathcal{D}_H\Big([y(t)]^\alpha, [z(t)]^\alpha\Big), \end{aligned}$$

where $K_2 = 3t\Big((\overline{y}^\alpha(t))^2 + \overline{y}^\alpha(t)\overline{z}^\alpha(t) + (\overline{z}^\alpha(t))^2\Big) > 0$. And, the α-level of I is given by:

$$\Big[I_k(y(t_k^-))\Big]^\alpha = \left[\frac{2}{2+\underline{y}^\alpha(t_k)}, \frac{2}{2+\overline{y}^\alpha(t_k)}\right], \qquad \forall\alpha \in [0, 1].$$

Then, we have:

$$\begin{aligned} \mathcal{D}_H\Big(\big[I_k(y(t_k^-))\big]^\alpha, \big[I_k(z(t_k^-))\big]^\alpha\Big) &= \mathcal{D}_H\Big(\Big[\frac{2}{2+\underline{y}^\alpha(t_k)}, \frac{2}{2+\overline{y}^\alpha(t_k)}\Big], \Big[\frac{2}{2+\underline{z}^\alpha(t_k)}, \frac{2}{2+\overline{z}^\alpha(t_k)}\Big]\Big), \\ &\le \max\left\{\left|\frac{2}{2+\underline{y}^\alpha(t_k^-)} - \frac{2}{2+\underline{z}^\alpha(t_k^-)}\right|, \left|\frac{2}{2+\overline{y}^\alpha(t_k^-)} - \frac{2}{2+\overline{z}^\alpha(t_k^-)}\right|\right\}, \\ &\le K_3\mathcal{D}_H\Big(\big[y(t)\big]^\alpha, \big[z(t)\big]^\alpha\Big), \end{aligned}$$

where $K_3 = \frac{2}{(2+\overline{y}^\alpha(t_k^-))(2+\overline{z}^\alpha(t_k^-))} > 0$. Hence, the constants k_1, K_2, K_3 are satisfied the hypotheses ($\mathcal{H}1$)-($\mathcal{H}4$). Thus, the conditions of Theorem 3.2 are satisfied. Therefore, Systeme (4.1) has a unique fuzzy solution.
Now, we examine the controllability result, then, let's consider the target state $y_1 = 3 \in \mathcal{F}_{\mathbb{R}^n}$.

$$\begin{aligned}
\Big[v(s)\Big]^\alpha &= \Big[\underline{v}^\alpha(s), \overline{v}^\alpha(s)\Big],\\
&= \Bigg[(\underline{\mathcal{W}}^\alpha)^{-1}\Big((\alpha+2) - T(\alpha+2)\underline{y}^\alpha(T) - \int_0^T G(\alpha+1)\underline{G}^\alpha(T-s)(\alpha+2)\underline{y}^\alpha(s)ds\\
&- \int_0^T s\underline{G}^\alpha(T-s)(\alpha+1)\underline{y}^\alpha(s)ds - \int_0^T \underline{G}^\alpha(T-s)\underline{h}^\alpha(s)d\mathbf{B}_H(s)\\
&- 2\sum_{t_k=0}^{t} \frac{\underline{G}^\alpha(T-t_k)}{2+\underline{y}^\alpha(t_k^-)}\Big), (\overline{\mathcal{W}}^\alpha)^{-1}\Big((4-\alpha) - T(4-\alpha)\overline{y}^\alpha(T)\\
&- \int_0^T G(3-\alpha)\overline{G}^\alpha(T-s)(4-\alpha)\overline{y}^\alpha(s)ds - \int_0^T s\overline{G}^\alpha(T-s)(3-\alpha)\overline{y}^\alpha(s)ds\\
&- \int_0^T \overline{G}^\alpha(T-s)\overline{h}^\alpha(s)d\mathbf{B}_H(s) - 2\sum_{t_k=0}^{t} \frac{\overline{G}^\alpha(T-t_k)}{2+\overline{y}^\alpha(t_k^-)}\Big)\Bigg].
\end{aligned}$$

Further, by remplacing the above derived values into the integral equation with respect to System (4.1), we obtain $[y(T)]^\alpha = [3]^\alpha$. Hence, all conditions of Theorem 3.4 are fulfilled. Therefore, the System (4.1) is controllable.

5. Conclusion

In this work, we have proved the existence, uniqueness and controllability results for FNSDEs with impulses via Banach fixed point analysis approach and using the fuzzy numbers whose values are normal, upper semicontinuous, convex and compact.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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