Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 ON LINE
SPM: www.spm.uem.br/bspm

(3s.) **v. 2023 (41)** : 1–9.
ISSN-0037-8712 IN PRESS
doi:10.5269/bspm.62481

Some Identities in Quotient Rings

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ABSTRACT: Let R be an associative ring, P a prime ideal of R. In this paper, we study the structure of the ring R/P and describe the possible forms of the generalized derivations satisfying certain algebraic identities on R. As a consequence of our theorems, we first investigate strong commutativity preserving generalized derivations of prime rings, and then examine the generalized derivations acting as (anti)homomorphisms in prime rings. Some commutativity theorems are also given in prime rings.

Key Words: Generalized derivation, prime ideal, semi-prime ring.

Contents

1. Introduction

Lets R an associative ring with center $Z(R)$, Q its Martindale quotient ring and U its left Utumi quotient ring with center denoted by C which called the extended centroid of R (we refer the reader to [2] for more information about these objects) and the symbols $s \circ t$ and $[s, t]$ will denote the anti-commutator $st - ts$ and commutator $st + ts$, respectively. Recall that an ideal P of R is said to be prime if for any $x, y \in R, xRy \subseteq P$ implies that $x \in P$ or $y \in P$. Therefore, R is called a prime ring if and only if $\{0\}$ is prime ideal of R. A additive mapping $d : R \longrightarrow R$ is said to be a derivation of a ring R if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. Moreover an additive mapping F is called a generalized derivation of R associated with derivation d if $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$, (derivation d is a generalized derivation associated with it self). An additive mapping $H : R \to R$ is called a left (resp. right) multiplier if $H(xy) = H(x)y$ (resp. $H(xy) = xH(y)$), holds for all $x, y \in R$. A multiplier is an additive mapping which is both right as well as left multiplier.

Over the last few decades, several authors have investigated the relationship between the commutativity of the ring R and certain specific types of additives mappings, as automorphisms, generalized derivations acting on appropriate subsets of the rings, see [1], [5], [6], [10] and [12]. In [3], Bell and Daif investigated the commutativity in rings admitting a derivation which is SCP on a nonzero right ideal. Precisely, they proved that if a semiprime ring R admits a derivation d satisfying $[d(x), d(y)] = [x, y]$ for all x, y in a right ideal I of R, then $I \subseteq Z(R)$. In particular, R is commutative if $I = R$. In [11] Nadeem ur Rehman generalized this result in the prime ring such he proved the commutativity of R/P and some useful results if R admits two derivations d_1 and d_2 satisfies $[d_1(x), d_2(y)] - [x, y] \in P$ for all $x, y \in R$. (Many authors have also worked on commutativity-preserving maps on rings). In this paper, we investigate the commutativity of quotient ring R/P by using generalized derivations F_1 and F_2 satisfying algebraic identities acting on prime ideals P. More precisely, we will establish a relationship between the structure of rings R/P and the behavior of generalized derivations satisfying the following algebraic identities:

(i) $[F_1(x), F_2(y)] + H([x, y]) \in P$, for all $x, y \in R$.

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2010 *Mathematics Subject Classification:* 16N60, 16W10, 16W25.
Submitted February 11, 2022. Published April 09, 2022

(ii) $F_1(x) \circ F_2(y) + H(x \circ y) \in P$, for all $x, y \in R$.
(iii) $[F_1(x), F_2(y)] + H(x \circ y) \in P$, for all $x, y \in R$.
(iv) $F_1(x) \circ F_2(y) + H([x, y]) \in P$, for all $x, y \in R$.
(v) $F(xy) \mp F(x)F(y) + H([x, y]) \in P$, for all $x, y \in R$.
(vi) $F(xy) \mp F(x)F(y) + H(x \circ y) \in P$, for all $x, y \in R$.
Finally, examples are given to demonstrate that the restrictions imposed on the hypothesis of our results are not superfluous.

2. Some preliminaries

In this section, we recall some basic identities and lemmas, which are useful in demonstration of our results.

(i) $[x, yz] = y[x, z] + [x, y]z$.

(ii) $[xy, z] = [x, z]y + x[y, z]$.

(iii) $xy \circ z = (x \circ z)y + x[y, z] = x(y \circ z) - [x, z]y$.

(vi) $x \circ yz = y(x \circ z) + [x, y]z = (x \circ y)z + y[z, x]$.

Lemma 2.1. [11, proposition 3.1] *Let R be a ring, P be a prime ideal of R. If R admits a generalized derivation F associated with derivation d satisfying $[x, F(x)] \in P$ for all $x \in R$, then either R/P is an integral domain or $d(R) \subseteq P$.*

Lemma 2.2. [4, lemma] *Let R be a prime ring. If $F : R \longrightarrow R$ and $G : R \longrightarrow R$ are functions such that $F(x)yG(z) = G(x)yF(z)$ for all $x, y, z \in R$, then there exists λ in the extended centroid of R such that $F(x) = \lambda G(x)$ for all $x \in R$.*

3. Mains results

Lemma 3.1. *Let R be a ring, P be a prime ideal of R. If R admits a multiplicative generalized derivation F associated with derivation d satisfying $[y, F(x)] \in P$ for all $x, y \in R$, then either R/P is an integral domain or $d(R) \subseteq P$.*

Proof. Suppose that
$$[x, F(y)] \in P \quad \text{for all } x, y \in R. \tag{3.1}$$
Replacing y by yt in (3.1) and using it again we obtain
$$F(y)[x, t] + [x, y]d(t) + y[x, d(t)] \in P \text{ for all } x, y, t \in R. \tag{3.2}$$
For $x = t$, it follows that
$$[t, y]d(t) + y[t, d(t)] \in P \quad \text{for all } y, t \in R. \tag{3.3}$$
Taking ry instead of y in (3.3), we get $[t, ry]d(t) + ry[t, d(t)] \in P$ for all $r, y, t \in R$ which implies that
$$r[t, y]d(t) + [t, r]yd(t) + ry[t, d(t)] \in P \quad \text{for all } r, y, t \in R. \tag{3.4}$$
Using (3.3) and (3.4), we conclude that
$$[t, r]yd(t) \in P \quad \text{for all } r, y, t \in R. \tag{3.5}$$
Equivalently,
$$[t, r]Rd(t) \subset P \quad \text{for all } r, t \in R. \tag{3.6}$$
By primness of P, we arrive at
$$[t, r] \in P \quad \text{or } d(t) \in P \quad \text{for all } r, t \in P. \tag{3.7}$$
If we set $A = \{t \in R \mid [t, r] \in P, r \in R\}$ and $B = \{t \in R \mid d(t) \in P\}$. Then A and B are two additives subgroups of R such $A \cap B = \emptyset$, and $A \cup B = R$. Since a group cannot be a union of two of its proper additives subgroups, we are forced to conclude that $R = A$ or $R = B$. Therefore, $d(R) \subseteq P$ or R/P is commutative. □

Motivated by the papers [7] and [8], we will study the case in the identity is replaced by a more general algebraic identity by connecting the identities with a multiplier H.

Theorem 3.2. *Let R be a ring, P be a prime ideal of R. If R admits a multiplier H and F_1, F_2 generalized derivations associated respectively with derivations d_1, d_2. If $[F_1(x), F_2(y)]+H([x,y]) \in P$ for all $x, y \in R$, then one of the following holds true:*

(i) *$H(R) \subseteq P$,*

(ii) *$d_1(R) \subseteq P$ or $d_2(R) \subseteq P$,*

(iii) *R/P is an integral domain.*

Proof. Suppose that

$$[F_1(x), F_2(y)] + H([x,y]) \in P \text{ for all } x, y \in R. \tag{3.8}$$

Replacing x by xt in (3.8) and using it, we obtain for all $x, y, t \in R$

$$F_1(x)[t, F_2(y)] + [x, F_2(y)]d_1(t) + x[d_1(t), F_2(y)] + H(x)[t, y] \in P. \tag{3.9}$$

In particular taking $t = F_2(y)$ in (3.9), we get for all $x, y \in R$

$$[x, F_2(y)]d_1(F_2(y)) + x[d_1(F_2(y)), F_2(y)] + H(x)[F_2(y), y] \in P. \tag{3.10}$$

Replacing x by sx in (3.10), we obtain for all $x, y \in R$

$$s[x, F_2(y)]d_1(F_2(y)) + [s, F_2(y)]xd_1(F_2(y)) + sx[d_1(F_2(y)), F_2(y)] + sH(x)[F_2(y), y] \in P.$$

Using (3.10) in the above expression, it is easy to get for all $x, y, s \in R$

$$[s, F_2(y)]xd_1(F_2(y)) \in P. \tag{3.11}$$

By primness of P, we find that for each $y \in R$

$$[s, F_2(y)] \in P \text{ for all } s \in R \text{ or } \ d_1(F_2(y)) \in P. \tag{3.12}$$

Let us set $A = \{y \in R \mid [s, F_2(y)] \in P, s \in R\}$ and $B = \{y \in R \mid d_1(F_2(y)) \in P\}$. A and B are two additives subgroups of R such that $A \cap B = \emptyset$, and $A \cup B = R$. Since a group cannot be a union of two of its proper subgroups, thus we have either $R = A$ or $R = B$. If $R = A$, that mean $[s, F_2(y)] \in P$ for all $y, s \in R$, using Lemma 3.1, we get R/P is commutative or $d_2(R) \subseteq P$. In case of $R = B$, we have $d_1F_2(y) \in P$ for all $y \in R$. From (3.10) we find $H(x)[F_2(y), y] \in P$ for all $x, y \in R$, replacing x by xt in this relation, we find that $H(x)R[F_2(y), y] \in P$. By primness of P, we arrive $H(R) \subseteq P$ or $[F_2(y), y] \in P$, in the last case, using lemma 2.1, we find that $d_2(R) \subseteq P$ or R/P is commutative. If we replace y by yt in (3.8) and we continue by the same techniques, we will find the same results with $d_1(R) \subseteq P$. It completes the proof.

□

Denote the identity map of R by I_R. It is easy to verify that I_R and $-I_R$ are multipliers of R, thus by replacing H with $\mp I_R$, where I_R is the additive map, we find the following consequent results of Theorem 3.2:

Corollary 3.3. *Let R be a ring, P be a prime ideal of R. If R admits generalized derivations F_1 and F_2 associated respectively with derivations d_1 and d_2. If $[F_1(x), F_2(y)] \mp [x, y] \in P$ for all $x, y \in R$, then one of the following holds true:*

(i) *$d_1(R) \subseteq P$ or $d_2(R) \subseteq P$,*

(ii) *R/P is an integral domain.*

Corollary 3.4. *Let R be a semiprime ring. If R admits two generalized derivations F_1, F_2 associated respectively with nonzero derivations d_1, d_2, then $[F_1(x), F_2(y)] \mp [x, y] = 0$ for all $x, y \in R$ if only if R is commutative.*

Proof. Suppose that $[F_1(x), F_2(y)] + [x, y] = 0$ for all $x, y \in R$. Since R is semiprime ring, there exits a family $\mathfrak{P}$ of prime ideals of R such that $\cap_{P\in\mathfrak{P}} P = \{0\}$. Thus for each $x, y \in R$, we have $[F_1(x), F_2(y)] + [x, y] \in P$ for all $P \in \mathfrak{P}$. Invoking Theorem 3.2 and using $H = I_R$, we get that ($d_1(R) \subseteq P$ or $d_2(R) \subseteq P$ or $R \subseteq P$ for all $P \in \mathfrak{P}$) or R is commutative, which, because of $\cap_{P\in\mathfrak{P}} P = \{0\}$, that means ($d_1 = 0$ or $d_2 = 0$ or $R = \{0\}$) or R is commutative, since the first is impossible, we get R is commutative.
If R is commutative it is easy to verify that the identity equal a zero. □

Now from Theorem 3.2, one can drive the following commutativity theorems:

Theorem 3.5. *Let R be a ring, P be a prime ideal of R. If R admits a multiplier H and generalized derivations F_1, F_2 associated respectively with derivations d_1, d_2 such that R satisfies any of the following assertions:*

(a) $F_1(x) \circ F_2(y) + H([x, y]) \in P$ *for all* $x, y \in R$.

(b) $F_1(x) \circ F_2(y) + H(x \circ y) \in P$, *for all* $x, y \in R$.

(c) $[F_1(x), F_2(y)] + H(x \circ y) \in P$, *for all* $x, y \in R$,

then one of the following holds true:

(i) $H(R) \subseteq P$.

(ii) $d_1(R) \subseteq P$ *or* $d_2(R) \subseteq P$.

(iii) R/P *is an integral domain.*

Proof. (a) Suppose that
$$F_1(x) \circ F_2(y) + H([x, y]) \in P \text{ for all } x, y \in R. \tag{3.13}$$
Replacing x by xt in (3.13) and using it, we get for all $x, y, t \in R$
$$F_1(x)[t, F_2(y)] + (x \circ F_2(y))d_1(t) + x[d_1(t), F_2(y)] + H(x)[t, y] \in P. \tag{3.14}$$
In particular taking $t = F_2(y)$ in (3.14), we arrive at
$$x \circ F_2(y)d_1(F_2(y)) + x[d_1(F_2(y)), F_2(y)] + H(x)[F_2(y), y] \in P \text{ for all } x, y \in R. \tag{3.15}$$
Replacing x by sx in (3.15), we get for all $x, y \in R$
$$s(x \circ F_2(y))d_1(F_2(y)) + [s, F_2(y)]xd_1(F_2(y)) + sx[d_1(F_2(y)), F_2(y)] + sH(x)[F_2(y), y] \in P.$$
Using (3.15) in the above expression, it is easy to get
$$[s, F_2(y)]xd_1(F_2(y)) \in P \text{ for all } x, y, s \in R. \tag{3.16}$$
By primness of P, we find that for each $y \in R$
$$[s, F_2(y)] \in P \text{ for all } s \in R \text{ or } d_1(F_2(y)) \in P. \tag{3.17}$$
We recalled the argument from Theorem 3.2 to get the results. This completes the proof.
Similarly, one can easily prove the conclusion from the identities (b) and (c). □

It is easy to verify that the maps I_R and $-I_R$ are multipliers of R, thus by replacing H with $-I_R, I_R$, we find the following consequent results of Theorem 3.2:

Corollary 3.6. *Let R be a ring, P be a prime ideal of R. If R admits generalized derivations F_1 and F_2 associated respectively with derivations d_1 and d_2, which satisfies any of the following assertions:*

(a) $F_1(x) \circ F_2(y) \mp [x, y] \in P$ *for all* $x, y \in R$.

(b) $F_1(x) \circ F_2(y) \mp x \circ y \in P$, *for all* $x, y \in R$.

(c) $[\, F_1(x), F_2(y)] \mp x \circ y \in P$, *for all* $x, y \in R$,

then one of the following holds true:

(i) $d_1(R) \subseteq P$ *or* $d_2(R) \subseteq P$.

(ii) R/P *is an integral domain.*

If we replace F with d and H with $\mp I_R$ in Theorem 3.2 and Theorem 3.5, we get the following corollary:

Corollary 3.7. *Let R be a ring, P be a prime ideal of R. If R admits derivations d_1 and d_2, then*

(a) [8, Theorem 1 (1)] $[d_1(x), d_2(y)] \mp [x, y] \in P$ *for all* $x, y \in R$ *if only if* R/P *is an integral domain.*

(b) [8, Theorem 3] *If* $d_1(x) \circ d_2(y) \mp [x, y] \in P$ *for all* $x, y \in R$ *then* R/P *is is an integral domain.*

(c) [8, Theorem 2 (1)] *If* $d_1(x) \circ d_2(y) \mp x \circ y \in P$, *for all* $x, y \in R$ *then* R/P *is an integral domain.*

(d) [8, Theorem 2 (2)] *If* $[\, d_1(x), d_2(y)] \mp x \circ y \in P$, *for all* $x, y \in R$ *then* R/P *is an integral domain.*

Proof. $(a) \Rightarrow$ We see that derivation is a generalized derivation associated with itself, so using theorem 3.2 with $H = \mp I_R$, we get one of the following holds true:

(i) $d_1(R) \subseteq P$ or $d_2(R) \subseteq P$.

(ii) R/P is an integral domain.

Using (i) with our assumption, we find that $[x, y] \in P$, then R/P is an integral domain.

$\Leftarrow$ trivial.

Using the same technique, we can prove the others results. □

Corollary 3.8. *Let R be a 2-torsion free semiprime ring. If R admits generalized derivations F_1 and F_2 associated respectively with nonzero derivations d_1 and d_2 such that R satisfies any of the following assertions:*

(a) $F_1(x) \circ F_2(y) \mp [x, y] = 0$ *for all* $x, y \in R$.

(b) $F_1(x) \circ F_2(y) \mp (x \circ y) = 0$ *for all* $x, y \in R$.

(c) $[F_1(x), F_2(y)] \mp (x \circ y) = 0$ *for all* $x, y \in R$,

then R is commutative.

Proof. By the same argument used in corollary 3.4, we can find our result. □

Theorem 3.9. *Let R be a ring, P be a prime ideal of R. If R admits a multiplier H (not necessarily additive) and a generalized derivation F associated with a nonzero derivation d such that R satisfies any of the following assertions:*

(a) $F(xy) \mp F(x)F(y) + H([x, y]) \in P$ *for all* $x, y \in R$.

(b) $F(xy) \mp F(x)F(y) + H(x \circ y) \in P$ *for all* $x, y \in R$,

then one of the following holds true:

(i) $H(R) \subseteq P$ *and* $F(R) \subseteq P$.

(ii) $H(R) \subseteq P$ *and* $(F \mp I_R)(R) \subseteq P$.

(iii) R/P *is an integral domain.*

Proof. (a) Suppose that

$$F(xy) + F(x)F(y) + H([x, y]) \in P \text{ for all } x, y \in R. \tag{3.18}$$

Replacing y by yx in (3.18) and using it, we obtain

$$(x + F(x))yd(x) \in P \text{ for all } x, y \in R. \tag{3.19}$$

By primeness of P, we have

$$x + F(x) \in P \text{ or } d(x) \in P \text{ for all } x \in R. \tag{3.20}$$

Since both the sets $A = \{x \in R \mid x + F(x) \in P\}$ and $B = \{x \in R \mid d(x) \in P\}$ form additive subgroups of R, it follows that either $R = A$ or $R = B$. Let us consider first $R = A$, which mean that $x + F(x) \in P$ for all $x \in R$. Writing rx for x, we find $(r + F(r))x + rd(x) \in P$ for all $x, r \in R$, which implies that $rd(x) \in P$ for all $x, r \in R$. Left multiplying by $d(x)$ and using primness of P, we obviously get $d(R) \in P$. Therefore, in each case we conclude that $d(x) \subseteq P$ for all $x \in R$.

In light of this fact, from (3.18), we notice that

$$F(x)(y + F(y)) + H([x, y]) \in P \text{ for all } x, y \in R. \tag{3.21}$$

Replacing y by yt in (3.21) and using it, we get $H(y)[x, t] \in P$ for all $x, y, t \in R$. It forces that either $H(R) \subseteq P$ or $[R, R] \subseteq P$. We have nothing to prove in the latter case, so let us assume that H maps R into P. Thereby from (3.21) we obtain $F(x)(y + F(y)) \in P$. It gives that $F(x)R(y + F(y)) \subseteq P$ for all $x, y \in P$. By primeness of P, we have either $F(R) \subseteq P$ or $(y + F(y)) \in P$ for all $y \in R$.
(b) By following the similar techniques as above with necessary variations we can get the conclusions. □

Corollary 3.10. [11, Theorem 1.4 (i)] *Let R be a ring, P be a prime ideal of R. If R admits a generalized derivation F associated with a nonzero derivation d such that $F(xy) \mp F(x)F(y) \in P$ for all $x, y \in R$, then $F(R) \subseteq P$ or $(F \mp I_R)(R) \subseteq P$ or R/P is an integral domain.*

Corollary 3.11. *Let R be a ring, P be a prime ideal of R. If R admits a generalized derivation F associated with a nonzero derivation d which satisfies one of the following assertions:*

(a) $F(xy) \mp F(x)F(y) \mp [x, y] \in P$ *for all* $x, y \in R$.

(b) $F(xy) \mp F(x)F(y) \mp x \circ y \in P$ *for all* $x, y \in R$,

then R/P is an integral domain.

Corollary 3.12. *Let R be a semiprime ring. If R admits a generalized derivation associated with a derivation d which satisfies one of following assertions:*

(a) $F(xy) + F(x)F(y) \mp [x, y] = 0$ *for all* $x, y \in R$.

(b) $F(xy) + F(x)F(y) \mp x \circ y = 0$ *for all* $x, y \in R$,

then R is commutative.

Corollary 3.13. [9, Theorem 1.2 (i)] *Let R be a prime ring and F be a generalized derivation of R associated with nonzero derivation d. If F acts as an homomorphism on R, then R is commutative.*

Proof. Remember that F cannot be zero or identity because $d \neq 0$. □

Theorem 3.14. *Let R be a ring, P be a prime ideal of R. If R admits a multiplier H (not necessarily additive) and a generalized derivation F associated with a nonzero derivation d such that R satisfies any of the following assertions:*

(a) $F(xy) \mp F(y)F(x) + H([x, y]) \in P$ *for all* $x, y \in R$.

(b) $F(xy) \mp F(y)F(x) + H(x \circ y) \in P$ *for all* $x, y \in R$,

then one of the following holds true:

(i) $F(R) \subseteq P$ *and* $H(R) \subseteq P$,

(ii) *there exists* $\lambda \in C$ *such that* $(F - \lambda)(R) \subseteq P$ *with* $(\lambda + H)(R) \subseteq P$,

(iii) R/P *is an integral domain.*

Proof. (a) Suppose that

$$F(xy) + F(y)F(x) + H([x, y]) \in P \text{ for all } x, y \in R. \tag{3.22}$$

Taking xy in place of x in (3.22) and using it, we find that

$$xyd(y) + F(y)xd(y) \in P \text{ for all } x, y \in R. \tag{3.23}$$

Putting rx for x in (3.23), it gives

$$rxyd(y) + F(y)rxd(y) \in P \text{ for all } x, y, r \in R. \tag{3.24}$$

Left multiplying (3.23) by r and then subtract from (3.24), we get

$$[F(y), r]xd(y) \in P \text{ for all } x, y, r \in R. \tag{3.25}$$

By Primness of P, for each $y \in R$ either $[F(y), r] \in P$ or $d(y) \in P$ for all $y, r \in R$. Since both of the sets $A = \{y \in R \mid [F(y), r] \in P, r \in R\}$ and $B = \{y \in R \mid d(y) \in P\}$ are additives subgroups of R, it follows that either $R = A$ or $R = B$.

Let $R = A$, i.e., $[F(y), r] \in P$ for all $y, r \in R$. Using Lemma 3.1, we get R/P is commutative or $d(R) \subseteq P$. If R/P is commutative, then we are done, so let us assume that $d(R) \subseteq P$. From our initial hypothesis, we have

$$F(x)y + F(y)F(x) + H([x, y]) \in P \text{ for all } x, y \in R. \tag{3.26}$$

Replacing x by xt in (3.26), we get

$$F(x)ty + F(y)F(x)t + H([x, y])t + H(x)[t, y] \in P \text{ for all } x, y, t \in R. \tag{3.27}$$

Right-multiplying (3.26) by t and then subtract from (3.27), we get $F(x)[t, y] + H(x)[t, y] \in P$ for all $x, y, t \in R$. This gives that $\{F + H\}(R)R[R, R] \subseteq P$, which, in lite of primness of P, yields that either R/P is commutative or $\{F + H\}(R) \subseteq P$. In the latter case, again from our initial hypothesis (3.22), we obtain

$$F(y)F(x) - H(y)x \in P \text{ for all } x, y \in R. \tag{3.28}$$

Replacing y by yr in (3.28), we get

$$F(y)rF(x) - H(y)rx \in P \text{ for all } x, y, r \in R. \tag{3.29}$$

Now taking rx instead of x in (3.28), we have

$$F(y)F(r)x - H(y)rx \in P \text{ for all } x, y, r \in R. \tag{3.30}$$

Comparing (3.29) and (3.30) to arrive at $F(y)(F(r)x - rF(x)) \in P$ for all $x, y, r \in R$. It forces that either $F(R) \subseteq P$ or $F(r)x - rF(x) \in P$ for all $x, r \in R$. Since $(F + H)(R) \subseteq P$, therefore in the first case, we get $H(R) \subseteq P$. We assume the case $F(u)x - uF(x) \in P$ for all $u, x \in R$. Replacing u by uy, we get

$$\overline{F(u)yI_R(x)} = \overline{I_R(u)yF(x)} \text{ for all } x, y \in R.$$

Using lemma 2.2, there exists $\overline{\lambda} \in \overline{C}$ such that $\overline{F(x)} = \overline{\lambda x}$ for all $x \in R$. It implies that $(F - \lambda)x \in P$ for all $x \in R$ and using our hypothesis $(F + H)(R) \subseteq P$, we get $[(F - \lambda) + (\lambda + H)](R) \subseteq P$, its forces that $(\lambda + H)(R) \subseteq P$.
Using similar techniques as above with necessary variations, we can prove the result from (b). It completes the proof. □

Corollary 3.15. [11, Theorem 1.4] *Let R be a ring, P be a prime ideal of R. If R admits a generalized derivation F associated with a nonzero derivation d such that $F(xy) \mp F(y)F(x) \in P$ for all $x, y \in R$, then either $F(R) \subseteq P$ or R/P is an integral domain.*

Corollary 3.16. [9, Theorem 1.2 (ii)] *Let R be a semiprime ring and F be a generalized derivation of R associated with nonzero derivation d. If F acts as an anti-homomorphism on R, then R is commutative.*

Corollary 3.17. *Let R be a semiprime ring. If R admits a generalized derivation associated with a nonzero derivation d such that R satisfies any of the following assertions:*

(a) *$F(xy) \mp F(y)F(x) - ([x, y]) = 0$ for all $x, y \in R$.*

(b) *$F(xy) \mp F(y)F(x) - (x \circ y) = 0$ for all $x, y \in R$,*

then R is commutative.

4. Examples

The following examples show that the condition "primness of P" in Theorems 3.2, 3.5 and 3.9 cannot be omitted.

Example 4.1. *Let $\mathbb{Z}$ be the set of integers. We define R, and $F : R \to R$ as follows:*

$$R = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & t \end{pmatrix} \mid x, y, z, t, 0 \in \mathbb{Z} \right\} P = \left\{ \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & z \end{pmatrix} \mid x, y, z, 0 \in \mathbb{Z} \right\}$$

,

$$d = F \text{ such that } F \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & t \end{pmatrix} = \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } H = \mp I_R.$$

It is clear that R is a ring, P is not a prime ideal, F is a generalized derivation associated with a nonzero derivation d such that $d(R) \subsetneq P$, and for all $X, X' \in R$, we have

(a) *$[F(X), F(X')] \mp [X, X'] \in P$ for all $X, X' \in R$.*

(b) *$F(X) \circ F(X') \mp X \circ X' \in P$ for all $X, X' \in R$.*

(c) *$[F(X), F(X')] \mp X \circ X' \in P$ for all $X, X' \in R$.*

(d) *$F(X) \circ F(X') \mp [X, X'] \in P$ for all $X, X' \in R$,*

but R/P is not an integral domain.

Example 4.2. *Let $\mathbb{Z}$ be the set of integers. We define R, and $F, d, H : R \to R$ as follows:*

$$R = \left\{ \begin{pmatrix} x & y & 0 \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} \mid x, y, z, 0 \in \mathbb{Z} \right\}, P = \left\{ \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & y & 0 \end{pmatrix} \mid x, y \in \mathbb{Z} \right\},$$

$$F \begin{pmatrix} x & y & 0 \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} = \begin{pmatrix} x & y & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, d \begin{pmatrix} x & y & 0 \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix},$$

$$\text{and } H \begin{pmatrix} x & y & 0 \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} = \begin{pmatrix} x & y & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easy to verify that R is a ring, P is not a prime ideal, F is a generalized derivation associated with a nonzero derivation d, H is a multiplier such that $H(R) \nsubseteq P$. Then for all $X, X^{'} \in R$, we have

(i) $F(XX') - F(X)F(X') + H[X, X'] \in P$.

(ii) $F(XX') + F(X)F(X') - H(X \circ X') \in P$,

but R/P not an integral domain.

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