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On Zero-power Valued Homoderivations in 3-prime Near-rings

Adel En-guady, Samir Mouhssine and Abdelkarim Boua*

ABSTRACT: We will start this article by proving a crucial concept, which will allow us to overcome a set of obstacles we encountered in previous articles concerning the commutativity of near-rings involving homoderivations and Jordan ideals. Furthermore, we present examples to demonstrate that the limitations imposed in the hypothesis of our results are necessary.

Key Words: 3-prime Near-ring, homoderivation, zero-power valued mappings, Jordan ideal.

Contents

1. Introduction

In the current paper, $\mathcal{N}$ will denote a left near-ring with center $Z(\mathcal{N})$. A left near-ring $\mathcal{N}$ is a triple $(\mathcal{N}, +, .)$ with two binary operations $+$ and $.$ such that (i) $(\mathcal{N}, +)$ is a group (not necessarily abelian), (ii) $(\mathcal{N}, .)$ is a semigroup, (iii) $a.(b + c) = a.b + a.c$, for all $a, b, c \in \mathcal{N}$. Note that $\mathcal{N}$ is a zero symmetric if $0.a = 0$ for all $a \in \mathcal{N}$, (recall that left distributive yields $a.0 = 0$). A near-ring $\mathcal{N}$ is said to be 3-prime if $a\mathcal{N}b = \{0\}$ for $a, b \in \mathcal{N}$ implies $a = 0$ or $b = 0$. For any pair of elements $a, b \in \mathcal{N}$, $[a, b] = ab - ba$ and $a \circ b = ab + ba$ stand for Lie product and Jordan product respectively. Recall that $\mathcal{N}$ is called 2-torsion free if $2a = 0$ implies $a = 0$ for all $a \in \mathcal{N}$. An additive subgroup J of $\mathcal{N}$ is said to be Jordan left (resp. right) ideal of $\mathcal{N}$ if $n \circ i \in J$ (resp. $i \circ n \in J$) for all $i \in J, n \in \mathcal{N}$ and J is said to be a Jordan ideal of $\mathcal{N}$ if $n \circ i \in J$ and $i \circ n \in J$ for all $i \in J, n \in \mathcal{N}$. For $S \subseteq \mathcal{N}$, a mapping $f : \mathcal{N} \longrightarrow \mathcal{N}$ is called zero-power valued on S if for each $x \in S$, there exists a positive integer $k(x) > 1$ such that $f^{k(x)}(x) = 0$. A mapping $f : \mathcal{N} \longrightarrow \mathcal{N}$ preserves S if $f(S) \subseteq S$. In [11] El Sofy (2000) defined a homoderivation on prime ring R to be an additive mapping h from R into itself such that $h(xy) = h(x)h(y) + h(x)y + xh(y)$ for all $x, y \in R$. An example of such mapping is to let $h(x) = f(x) - x$ for all $x \in R$ where f is an endomorphism on R. It is clear that a homoderivation h is also a derivation if and only if $h(x)h(y) = 0$ for all $x, y \in R$. In this case $h(x)Rh(y) = \{0\}$ for all $x, y \in R$. So, if R is a prime ring, then the only additive mapping which is both a derivation and a homoderivation is the zero mapping. In [13] A. Melaibari et al. studied the commutativity of rings admitting a homoderivation h such that $h([x, y]) = 0$ for all $x, y \in U$, where U a nonzero ideal of R. Also in [1] E. F. Al harfie et al. proved the commutativity of prime rings admitting a homoderivation h such that $h(xy) + xy \in Z(R)$ for all $x, y \in U$ or $h(xy) - xy \in Z(R)$ for all $x, y \in U$, where U a nonzero ideal of R. In [4], A. Boua have extended this study on 3-prime near-rings. Furthermore, he proved significant results on Jordan ideals.

This article is motivated by the previous results and here we continue this line of investigation to generalize and study this type of identity by invoking the concept of zero-power valued homoderivations and Jordan ideals in 3-prime near-rings.

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* Corresponding author
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2. Some preliminaries

We will begin this paragraph by introducing the following lemmas, which are essential for developing the proof of our results.

Lemma 2.1. [4, Lemma 2.4] *Let $\mathcal{N}$ be a 2-torsion 3-prime near-ring admits a nonzero homoderivation h of $\mathcal{N}$. If $h^2(\mathcal{N}) = \{0\}$, then $h = 0$.*

Lemma 2.2. [4, lemma 2.4] *Let $\mathcal{N}$ be a 3-prime near-ring. If $\mathcal{N}$ admits a nonzero homoderivation h, then for all $x, y, a \in \mathcal{N}$ we have*

$$h(xy)(h(a) + a) = h(x)h(y)(h(a) + a) + h(x)y(h(a) + a) + xh(y)(h(a) + a).$$

Lemma 2.3. [5, Lemma 2, Lemma 3] Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and J a nonzero Jordan ideal of $\mathcal{N}$.

(i) If $j^2 = 0$ for all $j \in J$, then $J = \{0\}$.

(ii) If $J \subseteq Z(\mathcal{N})$, then $\mathcal{N}$ is a commutative ring.

Lemma 2.4. [3, Lemma 1.2 (i), Lemma 1.2 (iii), Lemma 1.3 (iii)] *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring.*

(i) *If $z \in Z(\mathcal{N}) \setminus \{0\}$, then z is not a zero divisor.*

(ii) *If $Z(\mathcal{N})$ contains a nonzero element z for which $z + z \in Z(\mathcal{N})$, then $\mathcal{N}$ is abelian.*

(iii) *If $z \in Z(\mathcal{N}) \setminus \{0\}$ and $x \in \mathcal{N}$ such that $xz \in Z(\mathcal{N})$ or $zx \in Z(\mathcal{N})$, then $x \in Z(\mathcal{N})$.*

3. Homoderivation and Jordan ideal in 3-prime near-ring

In [2], A. Melaibari, N. Muthana, and A. Al-Kenani proved that if $\mathcal{R}$ is a prime ring, U is a nonzero ideal of $\mathcal{R}$ and h is a homoderivation of $\mathcal{R}$ satisfying any one of the following conditions (i) $h([x, y]) = 0$ for all $x, y \in U$, (ii) $[x, y] = [h(x), h(y)]$ for all $x, y \in \mathcal{R}$, then $\mathcal{R}$ is a commutative ring. In [4], A. Boua extended this study on 3-prime near-rings and he derived important results about its structure. In this section and according to the light of the above, we will continue this direction of the investigation by invoking the concept of the Jordan ideal.

Theorem 3.1. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and J be a nonzero Jordan ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a nonzero homoderivation h, then the following assertions are equivalent:*

(i) *$h([j, n]) = 0$ for all $j \in J, n \in \mathcal{N}$,*

(ii) *$\mathcal{N}$ is a commutative ring.*

Proof. It is clear that $(ii) \Rightarrow (i)$.
$(i) \Rightarrow (ii)$ Suppose that

$$h([j, n]) = 0 \quad \text{for all} \quad j \in J, n \in \mathcal{N}. \tag{3.1}$$

Replacing n by jn in (3.1), and using the fact that $[j, jn] = j[j, n]$, we get

$$h(j)jn = h(j)nj \quad \text{for all} \quad j \in J, n \in \mathcal{N}. \tag{3.2}$$

Taking nm in place of n in (3.2) and using it, we arrive at

$$h(j)n[j, m] = 0 \quad \text{for all} \quad j \in J, n, m \in \mathcal{N}.$$

Which leads to

$$h(j)\mathcal{N}[j, m] = \{0\} \quad \text{for all} \quad j \in J, m \in \mathcal{N}.$$

By the 3-primeness of $\mathcal{N}$, we obtain

$$h(j) = 0 \quad \text{or} \quad j \in Z(\mathcal{N}) \quad \text{for all} \quad j \in J. \tag{3.3}$$

Suppose there exists $j_0 \in J$ such that $h(j_0) = 0$. Using the definition of h and applying (3.1), we get

$$j_0 h(n) = h(n) j_0 \quad \text{for all} \quad n \in \mathcal{N}. \tag{3.4}$$

This allows us to write the following

$$(h(j_0) + j_0)h(n) = h(n)(h(j_0) + j_0) \quad \text{for all} \quad n \in \mathcal{N}.$$

For $n = h(x)y$, by Lemma 2.2, (3.4) implies that for all $x, y \in \mathcal{N}$ we have

$$j_0 h^2(x)h(y) + j_0 h^2(x)y + j_0 h(x)h(y) = h^2(x)h(y)j_0 + h^2(x)yj_0 + h(x)h(y)j_0. \tag{3.5}$$

Using (3.4), then (3.5) becomes

$$j_0 h^2(x)y = h^2(x)yj_0 \quad \text{for all} \quad x, y \in \mathcal{N}. \tag{3.6}$$

Replacing y by zm in (3.6) and using it again, we can easily arrive at $h^2(x)\mathcal{N}[j_0, m] = \{0\}$ for all $x, m \in \mathcal{N}$. By the 3-primeness of $\mathcal{N}$ and Lemma 2.1, we deduce that $j_0 \in Z(\mathcal{N})$. In this case, (3.3) becomes $J \subseteq Z(\mathcal{N})$, which forces that $\mathcal{N}$ is a commutative ring by Lemma 2.3 (ii). □

Theorem 3.2. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and J a nonzero Jordan ideal of $\mathcal{N}$. There is no nonzero homoderivations h satisfying any one of the following assertions:*

(i) $h(j \circ n) = 0$ *for all* $j \in J, n \in \mathcal{N}$.

(ii) $h(j \circ n) = [j, n]$ *for all* $j \in J, n \in \mathcal{N}$.

Proof. (i) Assume that

$$h(j \circ n) = 0 \quad \text{for all} \quad j \in J, n \in \mathcal{N}. \tag{3.7}$$

Replacing n by jn in (3.7), and using the definition of h together with the fact $j \circ jn = j(j \circ n)$, we get

$$h(j)(j \circ n) = 0 \quad \text{for all} \quad j \in J, n \in \mathcal{N}.$$

Which implies

$$h(j)jn = -h(j)nj \quad \text{for all} \quad j \in J, n \in \mathcal{N}. \tag{3.8}$$

Taking nm in place of n in (3.8), we obtain

$$h(j)jnm = -h(j)nmj \quad \text{for all} \quad j \in J, n, m \in \mathcal{N}.$$

Using (3.8), we can see

$$(-h(j)nj)m = -h(j)nmj \quad \text{for all} \quad j \in J, n, m \in \mathcal{N}.$$

Which implies that

$$h(j)n(-j)m = h(j)nm(-j) \quad \text{for all} \quad j \in J, n, m \in \mathcal{N}.$$

Putting $(-j)$ in place of j in the latter equation, we obtain

$$h(-j)njm = h(-j)nmj \quad \text{for all} \quad j \in J, n, m \in \mathcal{N}.$$

Which forces that

$$h(-j)\mathcal{N}[j, m] = \{0\} \quad \text{for all} \quad j \in J, m \in \mathcal{N}. \tag{3.9}$$

By the 3-primeness of $\mathcal{N}$, we arrive at

$$h(j)=0 \quad \text{or} \quad j\in Z(\mathcal{N}) \quad \text{for all} \quad j\in J. \tag{3.10}$$

Suppose there exists $j_0\in J$ such that $j_0\in Z(\mathcal{N})$, then $h(2j_0n)=0$ for all $n\in\mathcal{N}$ and by the 2-torsion freeness of $\mathcal{N}$, we have

$$h(j_0n)=0 \quad \text{for all} \quad n\in\mathcal{N}. \tag{3.11}$$

Using the definition of h, then (3.11) becomes

$$h(j_0)h(n)+h(j_0)n+j_0h(n)=0 \quad \text{for all} \quad n\in\mathcal{N}.$$

Substituting nj_0 instead of n in last equation and using (3.11), we obtain

$$h(j_0)nj_0=0 \quad \text{for all} \quad n\in\mathcal{N},$$

which leads to $h(j_0)\mathcal{N}j_0=\{0\}$. By the 3-primeness of $\mathcal{N}$, we deduce that $h(j_0)=0$, then (3.10) implies $h(J)=\{0\}$. And therefore, after using the definition of h, (3.7) becomes

$$jh(n)=h(n)(-j) \quad \text{for all} \quad j\in J, n\in\mathcal{N}. \tag{3.12}$$

This allows us to write the following:

$$(h(j)+j)h(n)=h(n)(h(-j)+(-j)) \quad \text{for all} \quad j\in J, n\in\mathcal{N}.$$

Taking $h(n)t$ instead of n in last equation, then by Lemma 2.2, we have

$$jh^2(n)h(t)+jh^2(n)t+jh(n)h(t)=h^2(n)h(t)(-j)+h^2(n)t(-j)+h(n)h(t)(-j). \tag{3.13}$$

Using (3.12), then (3.13) becomes

$$jh^2(n)t=h^2(n)t(-j) \quad \text{for all} \quad j\in J, n,t\in\mathcal{N}. \tag{3.14}$$

Replacing t by tm in (3.14) and using, we arrive at $h^2(n)\mathcal{N}[j,m]=\{0\}$ for all $j\in J, n,m\in\mathcal{N}$. By the 3-primeness of $\mathcal{N}$ and Lemma 2.1, we deduce that $J\subseteq Z(\mathcal{N})$, then $\mathcal{N}$ is a commutative ring by Lemma 2.3 (ii). In this case, (3.7) becomes $h(jn)=0$ for all $j\in J, n\in\mathcal{N}$. Thus $jh(n)=0$ for all $j\in J, n\in\mathcal{N}$. Replacing n by nm in last equation and using definition of h, we arrive at $j\mathcal{N}h(m)=\{0\}$ for all $j\in J, m\in\mathcal{N}$, which gives a contradiction with our hypotheses by using the 3-primeness of $\mathcal{N}$.

(ii) Suppose that

$$h(j\circ n)=[n,j] \text{ for all } j\in J, n\in\mathcal{N}. \tag{3.15}$$

Replacing n by jn in (3.15) and using the fact that $[jn,j]=j[n,j]$ and $j\circ jn=j(j\circ n)$, we have

$$h(j\circ jn)=[jn,j] \text{ for all } j\in J, n\in\mathcal{N}.$$

Which implies that

$$h(j)h(j\circ n)+h(j)(j\circ n)+jh(j\circ n)=j[n,j] \text{ for all } j\in J, n\in\mathcal{N}.$$

Then

$$h(j)h(j\circ n)+h(j)(j\circ n)=0 \text{ for all } j\in J, n\in\mathcal{N}.$$

Using (3.15), we can see that

$$h(j)([n,j]+(j\circ n))=0 \text{ for all } j\in J, n\in\mathcal{N}.$$

Which is easily reduced to

$$2h(j)nj=0 \text{ for all } j\in J, n\in\mathcal{N}.$$

Using the 2-torsion freeness of $\mathcal{N}$, we can write the following equation

$$h(j)\mathcal{N}j=\{0\} \text{ for all } j\in J.$$

By the 3-primeness of $\mathcal{N}$, we can observe $h(J)=\{0\}$. In this case, (3.15) becomes $[j,n]=0$ for all $j\in J, n\in\mathcal{N}$, then $J\subseteq Z(\mathcal{N})$, and by Lemma 2.1, we deduce that $\mathcal{N}$ is a commutative ring. Reasoning identical to those employed in (i), we arrive to a contradiction. $\square$

4. Zero-power valued homoderivations and Jordan ideals

In this paragraph, we will introduce a new concept called "Zero-power valued homoderivations" and we will start by proving an important Theorem, which is the foundation of our results and the key to prove the theorems below and allow us to use the right distributivity without any problem by invoking Lemma 2.2.

Theorem 4.1. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and J be a nonzero Jordan ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a nonzero additive map f on $\mathcal{N}$ which is zero-power valued on $\mathcal{N}$ and preserves J. Then the following assertions are equivalent:*

(i) $f(j) + j \in Z(\mathcal{N})$ *for all* $j \in J$.

(ii) $j + f(j) \in Z(\mathcal{N})$ *for all* $j \in J$.

(iii) $\mathcal{N}$ *is a commutative ring.*

Proof. It is clear that the implications $(iii) \Rightarrow (i)$ and $(iii) \Rightarrow (ii)$ are trivial.
$(i) \Rightarrow (iii)$ Suppose that

$$f(j) + j \in Z(\mathcal{N}) \text{ for all } j \in J. \tag{4.1}$$

If $f(j) \neq 0$ for all $j \in J \setminus \{0\}$. By recurrence we have $f^n(j) \neq 0$ for all $j \in J \setminus \{0\}$ and $n \in \mathbb{N}^*$. Since f is zero-power valued, for each $j \in J$, there exists a positive integer $k(j) > 1$ such that $f^{k(j)}(j) = 0$, it follows that for $z = f^{k(j)-1}(j) \neq 0$, $f(z) = f^{k(j)}(j) = 0$ which is a contradiction. Hence there exists $i \in J \setminus \{0\}$ such that $f(i) = 0$, so we get $i = f(i) + i \in Z(\mathcal{N}) \setminus \{0\}$ and $i + i = f(i + i) + i + i \in Z(\mathcal{N})$ which forces that $\mathcal{N}$ is abelian by Lemma 2.4 (ii). Now by replacing j by $j - f(j) + f^2(j) + ... + (-1)^{k(j)-1} f^{k(j)-1}(j)$ in (4.1), we get $j \in Z(\mathcal{N})$ for all $j \in J$, thus $J \subseteq Z(\mathcal{N})$. Hence $\mathcal{N}$ is commutative ring by Lemma 2.3 (ii).

$(ii) \Rightarrow (iii)$ Using the same techniques as used in the proof of $(i) \Rightarrow (iii)$, we can easily get the required result. □

Theorem 4.2. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and J be a nonzero Jordan ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a nonzero homoderivation h which is zero-power valued on $\mathcal{N}$ and preserves J, then the following assertions are equivalent:*

(i) $h([j, n]) + [j, n] = [h(j) + j, n]$ *for all* $j \in J, n \in \mathcal{N}$.

(ii) $\mathcal{N}$ *is a commutative ring.*

Proof. It's obvious that $(ii) \Rightarrow (i)$. $(i) \Rightarrow (ii)$. Assume that

$$h([j, n]) + [j, n] = [h(j) + j, n] \quad \text{for all} \quad j \in J, n \in \mathcal{N}. \tag{4.2}$$

For $n = j$, we get $[h(j) + j, j] = 0$ for all $j \in J$, and replacing n by jn in (4.2), we arrive at

$$h(j)h([j, n]) + h(j)[j, n] + j(h([j, n]) + [j, n]) = j[h(j) + j, n]. \tag{4.3}$$

Using (4.2), then (4.3) becomes

$$h(j)n(h(j) + j) = h(j)(h(j) + j)n \quad \text{for all} \quad j \in J, n \in \mathcal{N}. \tag{4.4}$$

Replacing n by nt in (4.4), we arrive at

$$h(j)\mathcal{N}[t, h(j) + j] = \{0\} \quad \text{for all} \quad j \in J, t \in \mathcal{N}. \tag{4.5}$$

By the 3-primeness of $\mathcal{N}$, we obtain

$$h(j) = 0 \quad \text{or} \quad h(j) + j \in Z(\mathcal{N}) \quad \text{for all} \quad j \in J. \tag{4.6}$$

If there exists $j_0 \in J$ such that $h(j_0) = 0$, by using (4.4) we get $h([j_0, n]) = 0$ for all $j \in J, n \in \mathcal{N}$. Thus

$$j_0 h(n) = h(n) j_0 \quad \text{for all} \quad n \in \mathcal{N}. \tag{4.7}$$

Which means that $(h(j_0) + j_0)h(n) = h(n)(h(j_0) + j_0)$ for all $n \in \mathcal{N}$. Taking $h(n)t$ instead of n, then by Lemma 2.2, we have for all $n, t \in \mathcal{N}$

$$j_0 h^2(n)h(t) + j_0 h^2(n)t + j_0 h(n)h(t) = h^2(n)h(t)j_0 + h^2(n)tj_0 + h(n)h(t)j_0. \tag{4.8}$$

Using (4.7), (4.8) becomes

$$j_0 h^2(n)t = h^2(n)tj_0 \quad \text{for all} \quad n, t \in \mathcal{N}. \tag{4.9}$$

Replacing t by tm in (4.9) and using it, we obtain $h^2(n)\mathcal{N}[j_0, m] = \{0\}$ for all $n, m \in \mathcal{N}$. Since $\mathcal{N}$ is 3-prime, by Lemma 2.1, we obtain $j_0 \in Z(\mathcal{N})$. In this case, (4.6) becomes $h(j) + j \in Z(\mathcal{N})$ for all $j \in J$ which forces that $\mathcal{N}$ is a commutative ring by Theorem 4.1. □

Theorem 4.3. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and J be a nonzero Jordan ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a nonzero homoderivation h which is zero-power valued on $\mathcal{N}$ and preserves J, then the following assertions are equivalent:*

(i) $(h(j) + j) \circ n \in Z(\mathcal{N})$ *for all* $j \in J, n \in \mathcal{N}$.

(ii) $[h(j) + j, n] \in Z(\mathcal{N})$ *for all* $j \in J, n \in \mathcal{N}$.

(iii) $h(j \circ n) + j \circ n \in Z(\mathcal{N})$ *for all* $j \in J, n \in \mathcal{N}$.

(iv) $\mathcal{N}$ *is a commutative ring.*

Proof. It is easy to see that $(iv) \Rightarrow (i)$, $(iv) \Rightarrow (ii)$ and $(iv) \Rightarrow (iii)$.

$(i) \Rightarrow (iv)$. Assume that

$$(h(j) + j) \circ n \in Z(\mathcal{N}) \text{ for all } j \in J, n \in \mathcal{N}. \tag{4.10}$$

Replacing n by $(h(j) + j)n$ in (4.10), we get

$$(h(j) + j)((h(j) + j) \circ n) \in Z(\mathcal{N}) \text{ for all } j \in J, n \in \mathcal{N}.$$

By Lemma 2.4 (iii), it follows that

$$(h(j) + j) \circ n = 0 \text{ or } h(j) + j \in Z(\mathcal{N}) \text{ for all } j \in J, n \in \mathcal{N}. \tag{4.11}$$

Suppose there exists $j_0 \in J$ such that $h(j_0) + j_0 \in Z(\mathcal{N})$. Taking j_0 instead of j in (4.10), we find that

$$(h(j_0) + j_0)(m + m) \in Z(\mathcal{N}) \text{ for all } m \in \mathcal{N}.$$

Using Lemma 2.4 (iii), we arrive at

$$h(j_0) + j_0 = 0 \text{ or } m + m \in Z(\mathcal{N}) \text{ for all } m \in \mathcal{N}.$$

Thus, (4.11) becomes

$$(h(j) + j) \circ n = 0 \text{ or } m + m \in Z(\mathcal{N}) \text{ for all } j \in J, n, m \in \mathcal{N}. \tag{4.12}$$

If $m + m \in Z(\mathcal{N})$ for all $m \in \mathcal{N}$, then we have

$$t(m + m) = tm + tm \in Z(\mathcal{N}) \text{ for all } m, t \in \mathcal{N}.$$

By Lemma 2.4 (iii), it follows that

$$t \in Z(\mathcal{N}) \text{ for all } t \in \mathcal{N} \text{ or } 2m = 0 \text{ for all } m \in \mathcal{N}.$$

Since $\mathcal{N}$ is 2-torsion free and $\mathcal{N} \neq \{0\}$, by Lemma 2.4 (iii), we get $\mathcal{N} \subseteq Z(\mathcal{N})$, which implies that $\mathcal{N}$ is a commutative ring by Lemma 2.3 (ii).
According to (4.12), we can assume that $(h(j)+j) \circ n = 0$ for all $j \in J, n \in \mathcal{N}$. It follows that

$$n(h(j)+j) = -(h(j)+j)n \text{ for all } j \in J, n \in \mathcal{N}.$$

Replacing n by nm, where $m \in \mathcal{N}$ in above equation, we can see

$$\begin{aligned} nm(h(j)+j) &= -(h(j)+j)nm \\ &= (h(j)+j)n(-m) \\ &= (-n(h(j)+j))(-m) \\ &= n(-(h(j)+j))(-m) \text{ for all } j \in J, n, m \in \mathcal{N}, \end{aligned}$$

which leads to

$$n(m(h(j)+j) - (-(h(j)+j))(-m)) = 0 \text{ for all } j \in J, n, m \in \mathcal{N}.$$

Replacing j by $-i$ in last equation, we may write

$$\mathcal{N}(-m(i+h(i)) + (i+h(i))m) = \{0\} \text{ for all } i \in J, m \in \mathcal{N}.$$

By the 3-primeness of $\mathcal{N}$, we conclude that $i + h(i) \in Z(\mathcal{N})$ for all $i \in J$, then $\mathcal{N}$ is a commutative ring by Theorem 4.1.

$(ii) \Rightarrow (iv)$. Suppose that

$$[h(j)+j, n] \in Z(\mathcal{N}) \text{ for all } j \in J, n \in \mathcal{N}. \tag{4.13}$$

Replacing n by $(h(j)+j)n$ in (4.13), we get

$$(h(j)+j)[h(j)+j, n] \in Z(\mathcal{N}) \text{ for all } j \in J, n \in \mathcal{N}. \tag{4.14}$$

By Lemma 2.4(iii), we obtain

$$h(j)+j \in Z(\mathcal{N}) \text{ or } [h(j)+j, n] = 0 \text{ for all } j \in J, n \in \mathcal{N}.$$

The both cases implies that $h(j) + j \in Z(\mathcal{N})$ for all $j \in J$. By Theorem 4.1, we conclude that $\mathcal{N}$ is a commutative ring.

$(iii) \Rightarrow (iv)$. Suppose that

$$h(j \circ n) + j \circ n \in Z(\mathcal{N}) \text{ for all } j \in J, n \in \mathcal{N}. \tag{4.15}$$

Replacing n by jn in (4.15), we obtain

$$h(j)h(j \circ n) + h(j)(j \circ n) + jh(j \circ n) + j(j \circ n) \in Z(\mathcal{N}) \quad \text{for all} \quad j \in J, n \in \mathcal{N}. \tag{4.16}$$

Thus

$$h(j)(h(j \circ n) + j \circ n) + j(h(j \circ n) + j \circ n) \in Z(\mathcal{N}) \quad \text{for all} \quad j \in J, n \in \mathcal{N}. \tag{4.17}$$

By (4.15), we get

$$(h(j \circ n) + j \circ n)(h(j)+j) \in Z(\mathcal{N}) \quad \text{for all} \quad j \in J, n \in \mathcal{N}. \tag{4.18}$$

According to Lemma 2.4 (iii), we have

$$h(j \circ n) + j \circ n = 0 \quad \text{or} \quad h(j)+j \in Z(\mathcal{N}) \quad \text{for all} \quad j \in J, n \in \mathcal{N}. \tag{4.19}$$

Suppose there exists $j_0 \in J$ such that $h(j_0 \circ n) + j_0 \circ n = 0$ for all $n \in \mathcal{N}$. Then by recurrence we prove that

$$h^k(j_0 \circ n) + (-1)^{k+1} j_0 \circ n = 0 \quad \text{for all} \quad n \in \mathcal{N}, k \in \mathbb{N}^*. \tag{4.20}$$

Since h is zero-power valued on $\mathcal{N}$, there exists an integer $k(j_0 \circ n) > 1$ such that $h^{k(j_0 \circ n)}(j_0 \circ n) = 0$. Replacing k by $k(j_0 \circ n)$ in (4.20), it follows that

$$j_0 \circ n = 0 \quad \text{for all} \quad n \in \mathcal{N}. \tag{4.21}$$

Taking j_0 in place of n in (4.21), we get $2j_0^2 = 0$. Using the 2-torsion freeness of $\mathcal{N}$, we obtain $j_0^2 = 0$. Putting $j_0 n$ in place of n in (4.21), we get $j_0 n j_0 = 0$ for all $n \in \mathcal{N}$, so $j_0 \mathcal{N} j_0 = \{0\}$. Thus $j_0 = 0$ by the 3-primeness of $\mathcal{N}$. In this case, (4.19) implies that $h(j) + j \in Z(\mathcal{N})$ for all $j \in J$. By Theorem 4.2 it follows that $\mathcal{N}$ is a commutative ring.

□

Theorem 4.4. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and J be a nonzero Jordan ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a nonzero homoderivation h which is zero-power valued on $\mathcal{N}$ and preserves J, then the following assertions are equivalent:*

(i) $[h(n), h(j) + j] = 0$ *for all* $j \in J, n \in \mathcal{N}$.

(ii) $h([j, n]) = [j, n]$ *for all* $j \in J, n \in \mathcal{N}$.

(iii) $\mathcal{N}$ *is a commutative ring.*

Proof. It is easy to see that $(iii) \Rightarrow (i)$ and $(iii) \Rightarrow (i)$.
$(i) \Rightarrow (iii)$ Suppose that

$$(h(j) + j)h(n) = h(n)(h(j) + j) \text{ for all } j \in J, n \in \mathcal{N}. \tag{4.22}$$

Taking $h(n)t$ instead of n and using Lemma 2.2, we arrive at

$$\begin{aligned} h(h(n)t)(h(j)+j) &= (h(j)+j)h^2(n)h(t) + (h(j)+j)h^2(n)t + (h(j)+j)h(n)h(t) \\ &= (h(j)+j)h(h(n)t) \\ &= h^2(n)h(t)(h(j)+j) + h^2(n)t(h(j)+j) + h(n)h(t)(h(j)+j) \end{aligned}$$

for all $j \in J, n, t \in \mathcal{N}$.
Using (4.22) in last expression, we obtain

$$(h(j) + j)h^2(n)t = h^2(n)t(h(j) + j) \text{ for all } j \in J, n, t \in \mathcal{N}. \tag{4.23}$$

Replacing t by tm in (4.23) and using it, we arrive at $h^2(n)\mathcal{N}[(h(j) + j), m] = \{0\}$ for all $j \in J, n, m \in \mathcal{N}$. Using the 3-primeness of $\mathcal{N}$ and the fact that $h \neq 0$, we obtain $h(j) + j \in Z(\mathcal{N})$ for all $j \in J$, which forces that $\mathcal{N}$ is a commutative ring by Theorem 4.2.
$(ii) \Rightarrow (iii)$ Suppose that

$$h([j, n]) = [j, n] \text{ for all } j \in J, n \in \mathcal{N}. \tag{4.24}$$

Replacing n by jn in (4.24), we arrive at

$$h(j)h([j, n]) + h(j)[j, n] + jh([j, n]) = j[j, n] \text{ for all } j \in J, n \in \mathcal{N}. \tag{4.25}$$

Using (4.24), then (4.25) implies $2h(j)[j, n] = 0$ for all $j \in J, n \in \mathcal{N}$. By the 2-torsion freeness of $\mathcal{N}$, we get $h(j)[j, n] = 0$ for all $j \in J, n \in \mathcal{N}$. Arguing as above we find that

$$h(j) = 0 \text{ or } j \in Z(\mathcal{N}) \text{ for all } j \in J. \tag{4.26}$$

Suppose there exists j_0 of J such that $h(j_0) = 0$. Then (4.24) becomes

$$[j_0, h(n)] = [j_0, n] \text{ for all } j \in J, n \in \mathcal{N}. \tag{4.27}$$

Invoking the definition of h and by recurrence, we arrive at

$$[j_0, h^k(n)] = [j_0, n] \quad \text{for all} \quad n \in \mathcal{N}, k \in \mathbb{N}^*. \tag{4.28}$$

Using the fact that h is zero-power valued on $\mathcal{N}$, there exists an integer $k(n) > 1$ such that $h^{k(n)}(n) = 0$. Replacing k by $k(n)$ in (4.28), we obviously get $j_0 \in Z(\mathcal{N})$. In this case, (4.26) becomes $J \subseteq Z(\mathcal{N})$, which forces that $\mathcal{N}$ is a commutative ring by Lemma 2.3 (ii). □

Theorem 4.5. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and J be a nonzero Jordan ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a nonzero homoderivation h which is zero-power valued on $\mathcal{N}$ and preserves J, then the following assertions are equivalent:*

(i) *$h(ij) + ij \in Z(\mathcal{N})$ for all $i, j \in J$.*

(ii) *$\mathcal{N}$ is a commutative ring.*

Proof. It is easy to see that $(ii) \Rightarrow (i)$.
$(i) \Rightarrow (ii)$. Suppose that

$$h(ij) + ij \in Z(\mathcal{N}) \text{ for all } i, j \in J. \tag{4.29}$$

Replacing j by $2j^2$ in (4.29), we get

$$\begin{aligned} h\left(i(2j^2)\right) + i(2j^2) &= h(i)h(2j^2) + h(i)(2j^2) + ih(2j^2) + i(2j^2) \\ &= h(i)(h(2j^2) + 2j^2) + i(h(2j^2) + 2j^2) \\ &= (h(2j^2) + 2j^2)(h(i) + i) \in Z(\mathcal{N}) \text{ for all } i, j \in J. \end{aligned}$$

Using Lemma 2.4(iii), it follows

$$h(2j^2) + 2j^2 = 0 \text{ for all } j \in J \text{ or } h(i) + i \in Z(\mathcal{N}) \text{ for all } i \in J.$$

If $h(2j^2) + 2j^2 = 0$ for all $j \in J$, by recurrence we have $h^k(2j^2) + (-1)^{k+1}(2j^2) = 0$ for all $j \in J, k \in \mathbb{N}^*$. Since h is zero-power valued on $\mathcal{N}$, there exists an integer $k(2j^2) > 1$ such that $h^{k(2j^2)}(2j^2) = 0$. Replacing k by $k(2j^2)$ in the above expression, we obtain $2j^2 = 0$ for all $j \in J$. Thus by the 2-torsion freeness of $\mathcal{N}$ we get $j^2 = 0$ for all $j \in J$. By Lemma 2.2 (i), we get $J = \{0\}$ a contradiction. Hence $h(i) + i \in Z(\mathcal{N})$ for all $i \in J$, and by Theorem 4.1 we deduce that $\mathcal{N}$ is a commutative ring. □

The following example shows that h is "zero-power valued on $\mathcal{N}$" cannot be omitted in the hypothesis of Theorems 4.1, 4.2, 4.3, 4.4 and 4.5.

Example 4.6. *Let $\mathcal{N} = J = M_2(\mathbb{Z})$ be the 2-torsion free 3-prime ring. We consider $h = -id_{\mathcal{N}}$, then it is clear that h is not "zero-power valued homoderivation on $\mathcal{N}$" which preserve J and satisfy the following conditions:*

(i) $h(j) + j \in Z(\mathcal{N})$,	(ii) $j + h(j) \in Z(\mathcal{N})$,
(iii) $[h(n), h(j) + j] = 0$,	(iv) $[h(j) + j, n] \in Z(\mathcal{N})$,
(v) $h([j, n]) + [j, n] = [h(j) + j, n]$,	(vi) $(h(j) + j) \circ n \in Z(\mathcal{N})$,
(vii) $h(ij) + ij \in Z(\mathcal{N})$,	$(viii)$ $h(j \circ n) + j \circ n \in Z(\mathcal{N})$,

for all $i, j \in J$, $n \in \mathcal{N}$ but $\mathcal{N}$ is not commutative.

The following example illustrates that the hypothesis "3-primeness of $\mathcal{N}$" is essential in Theorems 4.1, 4.2, 4.3, 4.4 and 4.5.

Example 4.7. *Let S be a zero-symmetric 2-torsion free left near-ring and*

$$\mathcal{N} = \left\{ \begin{pmatrix} a & b & 0 \\ 0 & 0 & 0 \\ 0 & c & 0 \end{pmatrix} \mid a, b, c, 0 \in S \right\}.$$

$$J = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & k & 0 \end{pmatrix} \mid k, 0 \in S \right\}.$$

Then $\mathcal{N}$ is a 2-torsion left near-ring which is not 3-prime, and J is a nonzero right Jordan ideal of $\mathcal{N}$. Let us defined $h : \mathcal{N} \to \mathcal{N}$ as follows:

$$h\begin{pmatrix} a & b & 0 \\ 0 & 0 & 0 \\ 0 & c & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is clear that h is a zero-power valued homoderivation on $\mathcal{N}$ which satisfy the following conditions:

(i) $h(j) + j \in Z(\mathcal{N})$ *for all* $j \in J$;

(ii) $j + h(j) \in Z(\mathcal{N})$ *for all* $j \in J$;

(iii) $h([j, n]) + [j, n] = [h(j) + j, n]$ *for all* $j \in J, n \in \mathcal{N}$;

(iv) $(h(j) + j) \circ n \in Z(\mathcal{N})$ *for all* $j \in J, n \in \mathcal{N}$.

(v) $[h(j) + j, n] \in Z(\mathcal{N})$ *for all* $j \in J, n \in \mathcal{N}$;

(vi) $h(j \circ n) + j \circ n \in Z(\mathcal{N})$ *for all* $j \in J, n \in \mathcal{N}$;

(vii) $[h(n), h(j) + j] = 0$ *for all* $j \in J, n \in \mathcal{N}$;

(viii) $h([j, n]) = [j, n]$ *for all* $j \in J, n \in \mathcal{N}$;

(ix) $h(ij) + ij \in Z(\mathcal{N})$ *for all* $i, j \in J$.

but $\mathcal{N}$ is not commutative.

References

1. E. F. Al harfie & N Muthana, *On homoderivations and commutativity of prime rings*, Bull. Soc. Math. Banja Luka, 9 (2019), 301-304.

2. A. Al-Kenani, A. Melaibari & N Muthana, *Homoderivations and commutativity of $*$-prime rings*, East-West J. Math., 2 (2015), 117-126.

3. H. E. Bell, *On derivations in near-rings*, II, Kluwer Acad. Publ., 426 (1997), 191-197.

4. A. Boua, *Homoderivations and Jordan right ideals in 3-prime near-rings*, AIP Conf. Proc. 2074, 020010-1-020010-8 (2019).

5. A. Boua, L. Oukhtite, A. Raji, *Jordan ideals and derivations in prime near-rings,* Comment. Math. Univ. Carolin., 55 (2) (2014), 131-139.

6. A. Boua, *Commutativity of near-rings with certain constrains on Jordan ideals*, Bol. Soc. Paran. Mat. 36 (4) (2018), 159-170.

7. A. Boua, *Commutativity of Jordan ideals in 3-prime near-rings with derivations*, Com. Korean Math. Soc. 33 (1) (2018), 37–44.

8. A. Boua and A. A. M. Kamal, *Lie ideals and Jordan ideals in 3-prime near rings with derivations*, JP J. Algebra Number Theory Appl. 37 (2015), 167-184.

9. A. Boua and H. E. Bell, *Jordan ideals and derivations satisfying algebraic identities,* Bull. Iranian Math. Soc., 44 (2018), 1543-1554.

10. A. Boua, A. Ali and Ul. Huque, *Several Algebraic Identities in 3-Prime Near-ring*, Kragujevac J. Math., 42 (2) 2018, 249-258.

11. M. M. El Sofy, *Rings with some kinds of mappings*, M. Sc. Thesis, Cairo University, Branch of Fayoum, (2000).

12. A. En-guady, A. Boua and E. Farhan, *Some functional identities on near-rings involving Jordan ideals and left generalized derivations.* Iraqi Journal of science, 62 (6) (2021), 1961-1967.

13. A. Melaibari, N. Muthana & A. Al-Kenani, *Homoderivations on Rings*, Gen. Math. Notes, 1 (2016), 1-8.

14. S. Mouhssine and A. Boua, *Homoderivations and semigroup ideals in 3-prime near-rings.* Alg. Struc. App., 8 (2) (2021), 177-194.

A. En-guady,

Department of Mathematics/Polydisciplinary Faculty,
Sidi Mohammed Ben Abdellah University,
Taza, Morocco.
E-mail address: `adel.enguady@usmba.ac.ma`

and

S. Mouhssine,
Department of Mathematics/Polydisciplinary Faculty,
Sidi Mohammed Ben Abdellah University,
Taza, Morocco.
E-mail address: `samir.mouhssine@usmba.ac.ma`

and

A. Boua,
Department of Mathematics/Polydisciplinary Faculty,
Sidi Mohammed Ben Abdellah University,
Taza, Morocco.
E-mail address: `abdelkarimboua@yahoo.fr`