Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 2023 (41)** : 1–11.
ISSN-0037-8712 in press
doi:10.5269/bspm.62706

The Multiplicity of Solutions for a Critical Problem Involving the Fracional p-Laplacian Operator

Djamel Abid, Kamel Akrout and Abdeljabbar Ghanmi

ABSTRACT: This paper deals with the existence of multiple solutions for the following critical fractional p-Laplacian problem

$$\begin{cases} (-\Delta)_p^s u(x) = \lambda \left|u\right|^{p-2} u + f(x,u) + \mu g(x,u) \text{ in } \Omega, u > 0, \\ u = 0 \text{ on } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where $p > 1$, $s \in (0,1)$, $\Omega \subset \mathbb{R}^n (n > ps)$, be a bounded smooth domain, λ, μ are positive parameters and the functions $f, g : \overline{\Omega} \times [0,\infty) \longrightarrow [0,\infty)$, are continuous and differentiable with respect to the second variable. Our main tools are based on variational methods combined with a classical concentration compacteness method.

Key Words: Nehari manifold, Fibering maps, Ekland's variational principle, Multiplicity of solutions.

Contents

1. Introduction

Let $p > 1$, $s \in (0,1)$ and let Ω be a smooth bounded domain in $\mathbb{R}^n (n > ps)$. In this paper, we study the existence of at least three nontrivial solutions for the following fractional p-Laplacian problem

$$\begin{cases} (-\Delta)_p^s u(x) = \lambda a(x) \left|u\right|^{p-2} u + f(x,u) + \mu g(x,u) \text{ in } \Omega, u > 0, \\ u = 0 \text{ on } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1.1}$$

where λ and μ are positive parameters, the functions $f, g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}_+$, and $a : \Omega \to \mathbb{R}$ are continuous, $(-\Delta)_p^s$ is a nonlocal operator which is defined', up to normalization factors, by the Riesz potential as

$$(-\Delta)_p^s u(x) := 2 \lim_{\epsilon \to 0} \int_{\Omega \setminus B_\epsilon(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+ps}} \mathrm{d}y, \quad x \in \Omega,$$

where $B_\epsilon(x) := \{y \in \Omega : |x - y| < \epsilon\}$. Note that, in [3,10], the eigenvalue problem associated with the fractional nonlinear operator $(-\Delta)_p^s$ was studied, and particularly some properties of the first eigenvalue λ_1 were obtained. We refer to [3,10,20,21] for more details on nonlocal fractional operators.

Problems like (1.1) are naturally arise in many different contexts, such as, among the others, the thin obstacle problem, optimization, finance, conservation laws, ultra relativistic limits of quantum mechanics, quasi-geostrophic flows, materials science and water waves. For more details, we can see [20,21].
When $p = 2$, (1.1) becomes an elliptic problem involving a linear fractional Laplacian, and when s gets close to 1, problem (1.1), becomes an elliptic problem involving the p-Laplace operator $div(|\nabla u|^{p-2}\nabla u))$. Recently, a great deal of attention has been focused on studying problems involving these type of operators

2010 *Mathematics Subject Classification:* 35P30, 35J35, 35J60.
Submitted February 28, 2022. Published July 28, 2022

see for example [5,10,11,12,15,18,19]. Precisely, in [12], Ghanmi studied the following fractional p-Laplacian problem

$$\begin{cases} (-\Delta)_p^s u(x) = f(x,u) + \lambda a(x)\,|u|^{q-2}\,u, \text{ in } \Omega, u > 0, \\ u = 0 \text{ on } \mathbb{R}^n \setminus \Omega. \end{cases}$$

Under appropriate conditions, and using the decomposition of the Nehari manifold, the author proved that the above non-local elliptic problem has at least two nontrivial solutions.
In [16], using the Nehari manifold method, Kratou, studied the following elliptic problem

$$\begin{cases} -(\Delta)u(x) = f(x)\,|u|^{p-2}\,u + \lambda g(x)\,|u|^{q-2}\,u, \text{ in } \Omega, \\ u = 0 \text{ on } \mathbb{R}^n \setminus \Omega. \end{cases}$$

Under adequate assumptions on the sources terms f and g, the author established the existence of three solutions: one is positive, one is negative and the other changes sign.
In order to precisely state our result, we introduce the assumptions on the functions f and g. We suppose that, there exist positive constants α_i and β_i for $i = 1, 2, 3, 4$ such that

$$\min(\alpha_1, \beta_1) \leq \max(\alpha_1, \beta_1) < \frac{1}{p-1} < p < \min(\alpha_2, \beta_2) \leq \max(\alpha_2, \beta_2) < \min(p_s^*, \alpha_4, \beta_4).$$

Moreover, for any $u \in L^{p_s^*}(\Omega)$, we have

$$\alpha_3 \|u\|^{p_s^*}_{L^{p_s^*}(\Omega)} \leq \alpha_2 \int_\Omega F(x,u)dx \leq \int_\Omega f(x,u)udx \leq \alpha_1 \int_\Omega f_u(x,u)u^2 dx \leq \alpha_4 \|u\|^{p_s^*}_{L^{p_s^*}(\Omega)} \tag{1.2}$$

and

$$\beta_3 \|u\|^q_{L^q(\Omega)} \leq \beta_2 \int_\Omega G(x,u)dx \leq \int_\Omega g(x,u)udx \leq \beta_1 \int_\Omega g_u(x,u)u^2 dx \leq \beta_4 \|u\|^q_{L^q(\Omega)}, \tag{1.3}$$

for some q with $p < q < p_s^*$. Where

$$p_s^* = \frac{np}{n-sp}.$$

and F, G are defined by

$$\begin{cases} F(x,u) = \int_0^u f(x,s)ds, \\ G(x,u) = \int_0^u g(x,s)ds. \end{cases}$$

Our main result of this paper is the following theorem.

Theorem 1.1. *If Equations (1.2), and (1.3) hold, then there exists $\mu^* > 0$, such that for every $\lambda \in (0, \lambda_1)$ and $\mu > \mu^*$, problem (1.1) admits three different nontrivial solutions. Moreover, these solutions are, one negative, one positive and the other has non-constant sign.*

The rest of this paper is organized as follows: in Section 2, we introduce the functional settings of problem (1.1), and we study the Nehari manifold and fibering map analysis. Section 3, is devoted to the proof of Theorem 1.1.

2. Functional analytic settings and Nehari manifold analysis

In this section, we briefly recall some definitions and basic properties of the fractional Sobolev spaces. After that, we define the Nehari manifold sets and we discuss the relationship between fibering maps and the Nehari manifold.
For any $1 \geq q \geq \infty$, We denote the usual norm of $L^q(\Omega)$ by $|u|_q$. Moreover, for each $0 < s < 1 < p < \infty$, and for all measurable function $u : \mathbb{R}^N \to \mathbb{R}$, we define the Gagliardo seminorm which is defined by

$$[u]_{s,p} = \left(\int_Q \frac{|u(x) - u(y)|^p}{|x-y|^{n+ps}} dxdy \right)^{\frac{1}{p}}. \tag{2.1}$$

The fractional Sobolev space

$$W^{s,p}(\mathbb{R}^n) = \left\{u \text{ is measurable in } \ L^P(\mathbb{R}^n) : [u]_{s,p} < \infty\right\},$$

is equipped with the norm

$$\|u\|_{s,p} = \left(|u|_p^p + \int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} dxdy\right)^{\frac{1}{p}}.$$

In the rest of this paper, we shall work in the following space

$$X_0 = \{u \in W^{s,p}(\mathbb{R}^n) : u = 0 \text{ for a.e. } \ x \in \mathbb{R}^n \setminus \Omega\}. \tag{2.2}$$

As we can see in ([21], Theorem 7.1), X_0 can be equivalently equipped by setting

$$\|u\| = [u]_{s,p}.$$

It is well known(see [21,23]), that X_0 is a uniformly convex and separable Banach space. Moreover, the embedding $X_0 \hookrightarrow L^\sigma(\Omega)$ is compact for any $1 \le \sigma < p_s^*$ and continuous for each $1 \le \sigma \le p_s^*$. Moreover, from [14], the fractional p-Laplacian is redefined variationally as the nonlinear operator from X_0 into its dual $X_0^\star$, which is defined, for all $u, v \in X_0$, by

$$\langle \mathcal{T}(u), v\rangle = \int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+ps}} dxdy. \tag{2.3}$$

Definition 2.1. *We say that $u \in X_0$, is a weak solution of problem (1.1), if for all $v \in X_0$, we have the following weak formulation*

$$\langle \mathcal{T}(u), v\rangle = \lambda \int_\Omega a(x)|u|^{p-2}uvdx + \int_\Omega f(x,u)v(x)dx + \mu \int_\Omega g(x,u)v(x)dx.$$

Associated to the problem (1.1), we define the functional $\Phi_{\lambda,\mu} : X_0 \to \mathbb{R}$, as

$$\Phi_{\lambda,\mu}(u) = \frac{1}{p} A(u) - B(u) - \mu C(u). \tag{2.4}$$

where

$$A(u) = \|u\|^p - \lambda \int_\Omega a(x)|u|^p dx; \quad B(u) = \int_\Omega F(x,u)dx, \text{ and } \ C(u) = \int_\Omega G(x,u)dx.$$

It is not difficult to prove that Φ is of class C^1, moreover, for any $u, v \in X_0$, we have

$$\langle \Phi'_{\lambda,\mu}(u), v\rangle = \langle \mathcal{T}(u), v\rangle - \lambda \int_\Omega a(x)|u|^{p-2}uvdx - \int_\Omega f(x,u)v(x)dx - \mu \int_\Omega g(x,u)v(x)dx.$$

So, according to Definition 2.1, we can see that, critical points of the functional Φ correspond to solutions for the problem (1.1).

To prove the main result of this paper, we will use the same aproach as in [24]. That is, we will constact three disjoint sets K_1, K_2 and K_3 not containing 0 such that $\Phi_{\lambda,\mu}$ has a critical point in K_i. These sets will be subsets of the following C^1 manifolds

$$M_1 = \left\{u \in X_0 : \int_\Omega u_+ > 0 \text{ and } A(u_+) - \int_\Omega f(x,u)u_+ - \mu \int_\Omega g(x,u)u_+ = 0\right\},$$

$$M_2 = \left\{u \in X_0 : \int_\Omega u_- > 0 \text{ and } A(u_-) - \int_\Omega f(x,u)u_- - \mu \int_\Omega g(x,u)u_- = 0\right\},$$

and

$$M_3 = M_1 \cap M_2,$$

where $u_+ = \max\{u, o\}$, $u_- = \max\{-u, 0\}$ are the negative and positive parts of u.

Lemma 2.2. *For every $w_0 \in X_0, w_0 > 0, (w_0 < 0)$, there exists $t_\mu > 0$ such that $t_\mu w_0 \in M_1, (t_\mu w_0 \in M_2)$. Moreover, $\lim_{\mu\to\infty} t_\mu = 0$. In particular, if $w_0 > 0$ and $w_1 < 0$, are two functions in X_0 with disjoint supports, then, there exist $t'_\mu, t_\mu > 0$ such that $t'_\mu w_0 + t_\mu w_1 \in M_3$. Moreover t'_μ and t_μ tend to zero as μ tends to infinity.*

Proof. For $w \in X_0$, we put

$$\phi(w) = A(w) - \int_\Omega f(x,w)wdx - \mu \int_\Omega g(x,w)wdx.$$

Let $w_0 \geq 0$. We will prove that $\phi(t_\mu w_0) = 0$ for some $t_\mu > 0$.
Using conditions (1.2) and (1.3), for $t > 0$, we get

$$\begin{aligned}\phi(tw_0) &= A(tw_0) - \int_\Omega f(x,tw_0)tw_0dx - \mu\int_\Omega g(x,tw_0)tw_0dx \\ &\geq t^pA(w_0) - \alpha_4 t^{p_s^*}|w_0|_{p_s^*}^{p_s^*} - \mu\beta_4 t^q|w_0|_q^q,\end{aligned}$$

and

$$\phi(tw_0) \leq t^pA(w_0) - \alpha_3 t^{p_s^*}|w_0|_{p_s^*}^{p_s^*} - \mu\beta_3 t^q|w_0|_q^q.$$

We can easily see that for all $u \in X_0$. If $\lambda < \lambda_1$, then

$$\left(1 - \frac{\lambda}{\lambda_1}\right)\|u\|^p \leq A(u) \leq \|u\|^p. \tag{2.5}$$

Therefore, the fact that $p < q < p_s^*$, implies that $\phi(tw_0)$ is negative for t large enough, and positive for t small enough . Consequently, by Bolzano's theorem, there exists $t_\mu > 0$, such that $\phi(t_\mu w_0) = 0$. On the other hand, we have

$$\begin{aligned}\phi(tw_0) &\leq t^pA(w_0) - \mu\beta_3 t^q|w_0|_q^q \\ &\leq t^p\mu\beta_3|w_0|_q^q\left(\frac{A(w_0)}{\mu\beta_3|w_0|_q^q} - t^{q-p}\right).\end{aligned}$$

So, we can choose t_μ, such that

$$0 < t_\mu < \left(\frac{A(v_0)}{\mu\beta_3|w_0|_q^q}\right)^{\frac{1}{q-p}}. \tag{2.6}$$

Finally, from (2.6), we can see that $t_\mu \to 0$ as $\mu \to +\infty$. This completes the proof of Lemma 2.2. □

Put

$$K_1 = \{u \in M_1 : u \geq 0\},\ K_2 = \{u \in M_2 : u \leq 0\},\ \text{ and }\ K_3 = M_3.$$

Lemma 2.3. *For every $u \in K_i, i = 1,2,3$, we have*

$$\|u\|^p \leq \left(1 - \frac{\lambda}{\lambda_1}\right)^{-1}\left(\int_\Omega f(x,u)udx + \mu\int_\Omega g(x,u)udx\right), \tag{2.7}$$

and

$$\left(\frac{1}{p} - \frac{1}{\min(\alpha_2,\beta_2)}\right)\left(\int_\Omega f(x,u)udx + \mu\int_\Omega g(x,u)udx\right) \leq \Phi_{\lambda,\mu}(u) \leq \left(\frac{1}{p} + \frac{1}{\min(\alpha_2,\beta_2)}\right)\|u\|^p \tag{2.8}$$

Proof. Let $u \in K_i$. Then, from the definition of K_i, we have

$$A(u) = \int_\Omega f(x,u)udx + \mu\int_\Omega g(x,u)udx. \tag{2.9}$$

So, from (2.5), we get

$$\|u\|^p \leq \left(1-\frac{\lambda}{\lambda_1}\right)^{-1}\left(\int_\Omega f(x,u)udx+\mu\int_\Omega g(x,u)udx\right).$$

This establishes inequality (2.7). On the other hand, Now, by combining Equations (1.2) and (1.3) with (2.9), we obtain

$$B(u)+\mu C(u) \leq \frac{1}{\min(\alpha_2,\beta_2)}\left(\int_\Omega f(x,u)udx+\mu\int_\Omega g(x,u)udx\right).$$

So

$$\begin{aligned}\Phi_{\lambda,\mu}(u) &= \frac{1}{p}A(u)-(B(u)+\mu C(u))\\ &\geq \left(\frac{1}{p}-\frac{1}{\min(\alpha_2,\beta_2)}\right)\left(\int_\Omega f(x,u)udx+\mu\int_\Omega g(x,u)udx\right).\end{aligned}$$

This proves the first inequality in (2.8). On the other hand, from (1.2), (1.3), we obtain

$$\begin{aligned}\Phi_{\lambda,\mu}(u) &\leq \frac{1}{p}A(u)+\frac{1}{\alpha_2}\int_\Omega f(x,u)udx+\frac{\mu}{\beta_2}\int_\Omega g(x,u)udx\\ &\leq \frac{1}{p}A(u)+\max\left(\frac{1}{\alpha_2},\frac{1}{\beta_2}\right)\left(\int_\Omega f(x,u)udx+\mu\int_\Omega g(x,u)udx\right)\\ &\leq \frac{1}{p}A(u)+\frac{1}{\min(\alpha_2,\beta_2)}\left(\int_\Omega f(x,u)udx+\mu\int_\Omega g(x,u)udx\right).\end{aligned}$$

Therefore, from (2.5) and (2.9), we get

$$\Phi(u) \leq \left(\frac{1}{p}+\frac{1}{\min(\alpha_2,\beta_2)}\right)A(u) \leq \left(\frac{1}{p}+\frac{1}{\min(\alpha_2,\beta_2)}\right)\|u\|^p.$$

This finishes the proof □

Lemma 2.4. *There exists $c>0$ such that,*

$$\begin{aligned}\|u_-\| &\geq c, \text{ for all } u\in K_2\\ \|u_+\| &\geq c, \text{ for all } u\in K_1\\ \min(\|u_-\|,\|u_+\|) &\geq c, \text{ for all } u\in K_3\end{aligned}$$

Proof. Let $u\in K_i$, then, from Equations (1.2), (1.3) and the Sobolev embedding, we have

$$\begin{aligned}A(u_\pm) &= \int_\Omega f(x,u_\pm)u_\pm dx+\mu\int_\Omega g(x,u_\pm)u_\pm dx\\ &\leq \alpha_4|u_\pm|_{p_s^*}^{p_s^*}+\beta_4\mu|u_\pm|_q^q\\ &\leq \alpha_4c_1\|u_\pm\|^{p_s^*}+\beta_4c_2\|u_\pm\|^q.\end{aligned}$$

for some positive constants c_1 and c_2.
Now, from (2.5), one has

$$\left(1-\frac{\lambda}{\lambda_1}\right)\|u_\pm\|^p \leq \left(\alpha_4c_1\|u_\pm\|^{p_s^*}+\beta_4c_2\|u_\pm\|^q\right). \tag{2.10}$$

Since $0<\lambda<\lambda_1$ and $p<q<p_s^*$, then, the result of Lemma 2.4 follows immediately from Equation (2.10). □

Lemma 2.5. *There exists $l > 0$ such that, for every $u \in X_0$, we have*

$$\Phi_{\lambda,\mu}(u) \geq l\|u\|^p,$$

provided that $\|u\|$ is small enough.

Proof. Let $u \in X_0$, then, by combining Equations (1.2) and (1.3) with (2.5), we get

$$\begin{aligned}
\Phi_{\lambda,\mu}(u) &= \frac{1}{p}A(u) - B(u) - \mu C(u) \\
&\geq \frac{1}{p}\left(1 - \frac{\lambda}{\lambda_1}\right)\|u\|^p - \frac{\alpha_4}{\alpha_2}|u|_{p_s^*}^{p_s^*} - \frac{\beta_4 \mu}{\beta_2}|u|_q^q \\
&\geq \frac{1}{p}\left(1 - \frac{\lambda}{\lambda_1}\right)\|u\|^p - \frac{c_1\alpha_4}{\alpha_2}\|u\|^{p_s^*} - \frac{c_2\beta_4 \mu}{\beta_2}\|u\|^q \\
&\geq \|u\|^p\left(\frac{1}{p}\left(1 - \frac{\lambda}{\lambda_1}\right) - \frac{c_1\alpha_4}{\alpha_2}\|u\|^{p_s^*-p} - \frac{c_2\beta_4 \mu}{\beta_2}\|u\|^{q-p}\right).
\end{aligned}$$

Since $0 < \lambda < \lambda_1$ and $p < q < p_s^*$. Then, from the above inequality, we see that for $\|u\|$ small enough, we have

$$\Phi(u) \geq l\|u\|^p,$$

for some positive constant l. □

Now we introduce lemma for describing the properties of the manifolds M_i

Lemma 2.6. *M_i, is a C^1 sub-manifold of X_0 of co-dimension 1 ($i = 1, 2$) and of co-dimension 2 for $i = 3$. The sets K_i are complete. Moreover, for every $u \in M_i$ we have the direct decomposition*

$$T_u X_0 = T_u M_i \bigoplus \texttt{span}\left\langle u_-, u_+ \right\rangle,$$

where $T_u M$ is the tangent space at u of the banach manifold M. Finally, the projection onto the first component in this decomposition is uniformly continuous on bounded sets of M_i.

Proof. Let us denote

$$\bar{M}_1 = \left\{u \in X_0 : \int_\Omega u_+ dx > 0\right\}.$$

$$\bar{M}_2 = \left\{u \in X_0 : \int_\Omega u_- dx > 0\right\}.$$

$$\bar{M}_3 = \bar{M}_1 \cap \bar{M}_2.$$

We see that $M_i \subset \bar{M}_i$. The set $\bar{M}_i$ is open in X_0, than it will be enough to prove that M_i is C^1 sub-manifold of $\bar{M}_i$. In order to do this, we have to construct a C^1-functions $\phi_i : \bar{M}_i \to R^d$ with $d = 1$ for $i = 1, 2$ and $d = 2$ for $i = 3$ and we will get $M_i = \phi_i^{-1}(0)$, where 0 is regular value of ϕ_i. First we define

$$\begin{aligned}
\phi_1(u) &= A(u_+) - \int_\Omega f(x,u)u_+ dx - \mu \int_\Omega g(x,u)u_+ dx \texttt{ for } u \in \bar{M}_1, \\
\phi_2(u) &= A(u_-) - \int_\Omega f(x,u)u_- dx - \mu \int_\Omega g(x,u)u_- dx \texttt{ for } u \in \bar{M}_2, \\
\phi_3(u) &= (\phi_1(u), \phi_2(u)) \texttt{ for } u \in \bar{M}_3.
\end{aligned}$$

We can easly see that $M_i = \phi_i^{-1}(0)$. From standard arguments see [4], ϕ_i is of class C^1. Therefore, we just need to prove that 0 is a regular value for ϕ_i. Let $u \in M_1$, then we have

$$\begin{aligned}\langle \phi_1'(u), u_+\rangle &= pA(u_+) - \int_\Omega f(x,u)u_+dx - \int_\Omega f_u(x,u)u_+^2dx - \mu\int_\Omega g(x,u)u_+dx - \mu\int_\Omega g_u(x,u)u_+^2dx\\ &\le pA(u_+) - \int_\Omega\left(1+\frac{1}{\alpha_1}\right)f(x,u)u_+dx + \mu\left(1+\frac{1}{\beta_1}\right)\int_\Omega g(x,u)u_+dx\\ &\le pA(u_+) - \left(1+\frac{1}{\max(\alpha_1,\beta_1)}\right)\left(\int_\Omega f(x,u)u_+dx + \mu\int_\Omega g(x,u)u_+dx\right)\\ &= \left(p-1-\frac{1}{\max(\alpha_1,\beta_1)}\right)A(u_+)\end{aligned}$$

The fact that $\max(\alpha_1,\beta_1) < \frac{1}{p-1}$, implies that $\langle \phi_1'(u), u_+\rangle < 0$. Therefore, M_1 is a C^1 sub-manfold of X. The same arguments are used to prove that M_2 and M_3, are C^1 sub-manfold of X.
Now, we will prove that K_i is complete,
Let u_k be a Cauchy sequence in K_i, then $u_k \to u$ in X. Moreover $(u_k)_\mp \to (u)_\mp$ in X. Moreover, from Lemma 2.4, we can deduce that $u \in K_i$. Finally, we have the decomposition

$$T_uX = T_uM_1 \bigoplus \mathtt{span}\,\langle u_+\rangle,$$

where

$$M_1 = \{u : \phi_1(u) = 0\} \text{ and } T_uM_1 = \left\{v : \langle \phi_1'(u), v\rangle = 0\right\}.$$

Put

$$\gamma = \frac{\langle \phi_1'(u), v\rangle}{\langle \phi_1'(u), u_+\rangle},$$

and let $v \in T_uX_0$ be unit tangential vector, then, we can write $v = v_1 + v_2$ where $v_2 = \gamma u_+$ and $v_1 = v - v_2$. Moreover, it is clear that $v_1 \in T_uM_1$, and $\langle \phi_1'(u), v_1\rangle = 0$. The same arguments are used to show that

$$T_uX = T_uM_2 \bigoplus \mathtt{span}\,\langle u_-\rangle, \text{ and } T_uX = T_uM_3 \bigoplus \mathtt{span}\,\langle u_-, u_+\rangle.$$

The proof of Lemma 2.6 is now completed. □

Lemma 2.7. *The unrestricted functional $\Phi_{\lambda,\mu}$ verifies the palais-Smale condition for energy level*

$$c < \frac{s}{n}\left(\frac{\alpha_4}{\alpha_2}p_s^*\right)^{\frac{-n}{sp_s^*}}\left(1-\frac{\lambda}{\lambda_1}\right)^{\frac{n}{ps}} S_p^{\frac{n}{ps}}, \tag{2.11}$$

where S_p is the best Sobolev constant which is given by

$$S_p = \inf_{v\in X_0\setminus\{0\}} \frac{\|v\|_{X_0}^p}{\|v\|_{L^{p_s^*}}^p}. \tag{2.12}$$

Proof. Let $\{u_k\} \subset X_0$ be such that

$$\Phi_{\lambda,\mu}(u_k) \to c, \text{ and } \Phi_{\lambda,\mu}'(u_k) \to 0, \text{ as } k \to \infty.$$

We need to prove that $\{u_k\}$ have a convergent sub-sequence.
From Lemma 2.3, we see that $\{u_k\}$ is bounded in X_0. Then, up to a sequence, still denoted by $\{u_k\}$, there exists $u_* \in X_0$ such that

$$u_k \rightharpoonup u_*, \text{ weakly in } X_0.$$

So immediately, we have

$$A(u_k) \to A(u_*), \text{ as } k \to \infty.$$

Moreover, by [23], [lemma 8], as $k\to\infty$, we get

$$\begin{cases} u_k \rightharpoonup u_*, & \text{weakly in } L^{p_s^*}(\mathbb{R}^n), \\ u_k \to u_*, & \text{strongly in } L^{r+1}(\mathbb{R}^n), \\ u_k \to u_*, & \text{a.e. in } \mathbb{R}^n. \end{cases}$$

On the other hand, from Theorem IV-9 in [4], there exists $l\in L^{r+1}(\mathbb{R}^n)$, such that:

$$|u_k(x)| \le l(x) \text{ in } \mathbb{R}^n.$$

Therefore,the dominated convergence theorem, implies that

$$C(u_k) \longrightarrow C(u_*), \text{ as } k\to\infty.$$

Now, by using Brezis-Lieb lemma [26], we obtain

$$\begin{aligned} A(u_k) &= A(u_k-u_*)+A(u_*)+o(1), \\ B(u_k) &= B(u_k-u_*)+B(u_*)+o(1). \end{aligned}$$

From the above equations, one has

$$\begin{aligned} \left\langle \Phi'_{\lambda,\mu}(u_k),u_k\right\rangle_{X_0} &= A(u_k)-p_s^*B(u_k)-\mu qC(u_k) \\ &= A(u_k-u_*)+A(u_*)-p_s^*(B(u_k-u_*)+B(u_*))-\mu qC(u_k)+o(1) \\ &= \left\langle J'_{\lambda,\mu}(u_*),u_*\right\rangle_{X_0}+A(u_k-u_*)-p_s^*B(u_k-u_*). \end{aligned}$$

Since $\left\langle \Phi'_{\lambda,\mu}(u_*),u_*\right\rangle_{X_0}=0$, and $\left\langle \Phi'_{\lambda,\mu}(u_k),u_k\right\rangle_{X_0}\longrightarrow 0$ as $k\longrightarrow\infty$, then, we obtain

$$\lim_{k\to\infty} A(u_k-u_*)=\lim_{k\to\infty} p_s^*B(u_k-u_*):=b. \tag{2.13}$$

If $b=0$, then the proof is completed. So we assume that $b>0$. From (2.5), we have

$$p_s^*B(u_k-u_*) \le \frac{\alpha_4}{\alpha_2}p_s^*S_p^{-\frac{p_s^*}{p}}\left(1-\frac{\lambda}{\lambda_1}\right)^{-\frac{p_s^*}{p}}(A(u_k-u_*))^{\frac{p_s^*}{p}}.$$

By letting k tends to infinity, we get

$$b \ge \left(\frac{\alpha_4}{\alpha_2}p_s^*\right)^{\frac{-n}{sp_s^*}}\left(1-\frac{\lambda}{\lambda_1}\right)^{\frac{n}{ps}} S_p^{\frac{n}{ps}}.$$

On the other hand, we have

$$\begin{aligned} c &= \lim_{k\longrightarrow\infty}\left(\frac{1}{p}A(u_k)-B(u_k)-\mu C(u_k)\right) \\ &= \lim_{k\longrightarrow\infty}\left(\frac{1}{p}A(u_k-u_*)-B(u_k-u_*)-\frac{1}{p}A(u_*)-B(u_*)-\mu C(u_k)\right)+o(1) \\ &= \Phi_{\lambda,\mu}(u_*)+b\left(\frac{1}{p}-\frac{1}{p_s^*}\right) \\ &\ge \Phi_{\lambda,\mu}(u_*)+\frac{s}{n}\left(\frac{\alpha_4}{\alpha_2}p_s^*\right)^{\frac{-n}{sp_s^*}}\left(1-\frac{\lambda}{\lambda_1}\right)^{\frac{n}{ps}} S_p^{\frac{n}{ps}}. \end{aligned}$$

By the assumption that $c<\frac{s}{n}\left(\frac{\alpha_4}{\alpha_2}p_s^*\right)^{\frac{-n}{sp_s^*}}\left(1-\frac{\lambda}{\lambda_1}\right)^{\frac{n}{ps}} S_p^{\frac{n}{ps}}$, we obtain $\Phi_{\lambda,\mu}(u_*)<0$. In particular, $u_*\neq 0$, and

$$B(u_*) > \tfrac{1}{p}A(u_*)-\mu C(u_*). \tag{2.14}$$

So

$$
\begin{aligned}
c &= \lim_{k\longrightarrow\infty}\Phi_{\lambda,\mu}(u_k) = \lim_{k\longrightarrow\infty}\left(\Phi_{\lambda,\mu}(u_k) - \frac{1}{p}\left\langle \Phi'_{\lambda,\mu}(u_k), u_k\right\rangle_{X_0}\right)\\
&= \lim_{k\longrightarrow\infty}(\frac{p_s^*}{p}-1)(B(u_k-u_*)) + B(u_*) - \mu(\frac{p-q}{p})C(u_k)\\
&= \frac{sp_s^*}{n}(B(u_k-u_*)+B(u_*)) - \mu(\frac{p-q}{p})C(u_*)\\
&\geq \frac{s}{n}\left(\frac{p_s^*\alpha_4}{\alpha_2}\right)^{\frac{-n}{sp_s^*}}\left(1-\frac{\lambda}{\lambda_1}\right)^{\frac{n}{ps}} S_p^{\frac{n}{ps}} + \frac{sp_s^*}{n}B(u_*) + \mu\left(\frac{q-p}{p}\right)C(u_*)\\
&\geq \frac{s}{n}\left(\frac{\alpha_4}{\alpha_2}p_s^*\right)^{\frac{-n}{sp_s^*}}\left(1-\frac{\lambda}{\lambda_1}\right)^{\frac{n}{ps}} S_p^{\frac{n}{ps}},
\end{aligned}
$$

which is a contradiction. Hence, $b = 0$, and $u_k \to u_*$ strongly in X_0. This completes the proof. □

3. Proof of Theorem 1.1

In this section, we will prove the main result of this paper (Theorem 1.1). First of all, we begin by remark that if $u \in K_i$ is a critical point of the restricted functional $\Phi_{\lambda,\mu}|_{K_i}$. Then u is also a critical point of the unrestricted functional $\Phi_{\lambda,\mu}$. Which implies that u is a weak solution for problem (1.1).

Lemma 3.1. *If c satisfies (2.11), then the functional $\Phi_{\lambda,\mu}$ defined on K_i satifies the Plais-Smale condition at level c.*

Proof. Let $(u_k) \in K_i$ be a sequence such that $\Phi_{\lambda,\mu}(u_k)$ is uniformly bounded and $\Phi'_{\lambda,\mu}(u_k) \to 0$. Let $v_j \in T_{u_j}X_0$, be a unit tangenttial vector such that

$$\langle \Phi'_{\lambda,\mu}(u_j), v_j\rangle = \|\Phi'_{\lambda,\mu}(u_j)\|_{X'}.$$

By lemma 2.6, we have that $v_j = w_j + y_j$, for some $w_j \in T_{u_j}M_i$ and $y_j \in span\,\langle (u_j)_+, (u_j)_-\rangle$.
Since $\Phi_{\lambda,\mu}(u_j)$ is uniformly bounded, then, by lemma 2.3, u_j is also uniformly bounded in X_0. So, w_j is uniformly bounded in X_0. Therefore, as j tends to infinity, we get

$$\|\Phi'_{\lambda,\mu}(u_j)\|_{X'} = \langle \Phi'_{\lambda,\mu}(u_j), v_j\rangle = \langle \Phi'_{\lambda,\mu}|_{K_i}(u_j), v_j\rangle \to 0.$$

as a consequences we get

$$\Phi'_{\lambda,\mu}|_{K_i}(u_k) \to 0.$$

Finally, the result follows immediately from Lemma 2.7. □

Now we need to show that the functional $\Phi_{\lambda,\mu}|_{K_i}$ satifies the hypothesis of the Ekcland's Variational Principle [8]. We have as a direct consequence of the construction of the manifold K_i that $\Phi_{\lambda,\mu}$ is bounded below over K_i.
Hence, by Ekeland's Variational Principle, there existe $v_k^i \in K_i$ such that

$$\Phi_{\lambda,\mu}(v_k^i) \to c_i := \inf_{K_i}\Phi_{\lambda,\mu} \quad \text{and} \quad (\Phi_{\lambda,\mu}|_{K_i})'(v_k^i) \to 0.$$

On the other hand, from Lemma 2.2, if μ is large, then we have

$$c_i < \frac{s}{n}\left(\frac{\alpha_4}{\alpha_2}p_s^*\right)^{\frac{-n}{sp_s^*}}\left(1-\frac{\lambda}{\lambda_1}\right)^{\frac{n}{ps}} S_p^{\frac{n}{ps}}.$$

Hence, Lemma 2.7 implies that $\Phi_{\lambda,\mu}$ satisfies the palais smail condition in K_i for $i = 1, 2, 3$. Therefore, up to sub-sequences, there exist $u \in K_1, v \in K_2, w \in K_3$ such that, as k tends to infinity, we have $v_k^1 \longrightarrow u$,

$v_k^2 \longrightarrow v$, and $v_k^3 \longrightarrow w$.
The fact that $K_i \subset M_i$ implies that

$$\int_\Omega u_+ dx > 0, \int_\Omega v_- dx > 0, \text{ and } \int_\Omega w_+ dx > 0.$$

So, u, v and w are nontrivial. On the other hand, since K_1, K_2 and K_3 are disjoint, then, u, v and w are distinct. Now, since the convergence of v_k^i is strongly, then, u, v and w are critical points of the functional $\Phi_{\lambda,\mu}$. Finally, the fact that $u \in K_1, v \in K_2, w \in K_3$, implies that problem (1.1) admits three nontrivial solutions, moreover these solutions are one positive, one negative and the other change sign. The proof of Theorem 1.1 is now completed.

References

1. J. G. Azorero, I. P. Alonso, *Multiplicity of solutions for elliptic problems with critical exponent or with a nonsymmetric term*, Trans. Am. Math. Soc., 323(2)(1991), 877–895.
2. T. Bartsch, Z. Liu, *On a superlinear elliptic p-laplacian equation*, J. Differ. Equ. 198(2004), 149–175.
3. L. Brasco, E. Parini, *The second eigenvalue of the fractional p-Laplacian*, Adv. Calc. Var. 9(2016), 323–355.
4. H. Brezis, P. G. Ciarlet,J. L. Lions, *Analyse fonctionnelle: théorie et applications*, Paris: Dunod, 1999. Print.
5. A. D. Castro, T. Kuusi, G. Palatucci, *Local behavior of fractional p-minimizers*, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 33(5) (2016) 1279–1299.
6. S. Cingolani, G. Vannella, *Multiple positive solutions for a critical quasilinear equation via morse theory*, Ann. I. H. Poincarrré 26(2009)397–413.
7. G. Dinca, P. Jebelean, J. Mawhin, *Variational and topological methods for dirichlet problems with p-laplacian* Port. Math. (N. S.), 58 (2001), 339–378.
8. I. Ekeland, *On the variational principle*, J. Math. Anal. Appl., 47(2) (1974), 324–353.
9. J. F. Escobar, *Uniqueness theorems on conformal deformation of metrics, Sobolev inequalities, and an eigenvalue estimate*, Commun. Pure Appl. Math. 43(7)(1990), 857–883.
10. G. Franzina, G. Palatucci, *Fractional p-eigenvalues*, Riv. Mat. Univ. Parma (N. S.) 5 (2014) 373–386.
11. A. Ghanmi, K. Saoudi, *The Nehari manifold for a singular elliptic equation involving the fractional Laplace operator*, Frac. Differ. Calculus, 6 (2016), 201-217.
12. A. Ghanmi, *Multiplicity of nontrivial solutions of a class of fractional p-Laplacian problem*, Z. Anal. Anwend., 34 (2015), 309-319.
13. M. Guedda,L. Véron, *Quasilinear elliptic equations involving critical sobolev exponents*, Nonlinear Anal. Theory Methods Appl. 13(8) (1989), 879–902.
14. A. Iannizzotto, S. B. Liu, K. Perera, M. Squassina, *Existence results for fractional p-Laplacian problem via morse theory*, Adv. Calc. Var. 9(2) (2016), 101–125.
15. S. Jarohs, *Strong comparison principle for the fractional p-Laplacian and applications to starshaped rings*, Adv. Nonlinear Stud. 18(4) (2018) 691–704.
16. M. Kratou, *Three solutions for a semilinear elliptic boundary value problem*,Proc Math Sci 129, 22 (2019). https://doi.org/10.1007/s12044-019-0465-0.
17. P. L. Lions, *The concentration-compactness principle in the calculus of variations. the limit case, part 1.* Rev. Mat. Iberoam. 1 (1985), 145–201.
18. E. Lindgren, *Hölder estimates for viscosity solutions of equations of fractional p-Laplace type*, Nonlinear Differ. Equ. Appl. 23(5) (2016), 23–55.
19. X. Mingqi, B. Zhang, V.D. Ràdulescu, *Superlinear Schrödinger-Kirchhoff type problems involving the fractional p-Laplacian and critical exponent*, Adv. Nonlinear Anal. 9(1) (2020) 690–709.
20. G. Molica Bisci, V.D. Ràdulescu, R. Servadei, *Variational Methods for Nonlocal Fractional Problems*, vol.162, Cambridge University Press, Cambridge, 2016.
21. E. Di Nezza, G. Palatucci, E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. 136 (2012), 521–573.
22. J. T. Schwartz, *Generalizing the lusternik-schnirelman theory of critical points*, Commun. Pure Appl. Math. 17(3)(1964), 307–315.
23. R. Servadei, E. Valdinoci, *Mountain pass solutions for non-local elliptic operators*, J. Math. Anal. Appl. 389(2) (2012), 887–898.

24. M. Struwe, *Three non-trival solutions of anticoercive boundary value problems for the pseudo-laplace-operator*, Journal fr die reine und angewandte Mathematik, 325(1991), 68–74.

25. P. Tolksdorf, *Regularity for a more general class of quasilinear elliptic equations*, J. Differ. equ. 51(1)(1984), 126–150.

26. M. Willem, *Minimax theorems*, PNLDE 24, Birkhäuser, Boston-Basel-Berlin 1996.

27. Z. Zhang, J. Chen, S. Li, *Construction of pseudo-gradient vector field and sign- changing multiple solutions involving p-laplacian*, J. Diffe. Equ. 201(2) (2004), 287–303.

Djamel Abid,
LAMIS Laboratory,
Larbie Tebessi University, Tebessa,
Algeria.
E-mail address: djamelabid0312@gmail.com

and

Kamel Akrout,
LAMIS Laboratory,
Larbie Tebessi University, Tebessa,
Algeria.
E-mail address: kamel.akrout@univ-tebessa.dz

and

Abdeljabbar Ghanmi,
Faculté des Sciences de Tunis, LR10ES09 Modélisation matématique, analyse harmonique et téorie du potentiel,
Université de Tunis El Manar, Tunis 2092,
Tunisie.
E-mail address: abdeljabbar.ghanmi@lamsin.rnu.tn.