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Gevrey Class Regularity and Stability for the Debye-Hückel System in Critical Fourier-Besov-Morrey Spaces

Achraf Azanzal, Chakir Allalou and Said Melliani

ABSTRACT: In this paper, we study the analyticity of mild solutions to the Debye-Hückel system with small initial data in critical Fourier-Besov-Morrey spaces. Specifically, by using the Fourier localization argument, the Littlewood-Paley theory and bilinear-type fixed point theory, we prove that global-in-time mild solutions are Gevrey regular. As a consequence of analyticity, we get time decay of mild solutions in Fourier-Besov-Morrey spaces. Finally, we show a blow-up criterion of the local-in-time mild solutions of the Debye-Hückel system.

Key Words: Debye-Hückel system, Space analyticity, Blow-up criterion, Littlewood-Paley theory, Fourier-Morrey-Besov spaces.

Contents

1. Introduction

In this paper, we consider the following Cauchy problem for the Debye-Hückel system in $\mathbb{R}^n \times \mathbb{R}^+$:

$$\begin{cases} \partial_t v - \Delta v = -\nabla \cdot (v \nabla \phi) & \text{in } \mathbb{R}^n \times (0,\infty), \\ \partial_t w - \Delta w = \nabla \cdot (w \nabla \phi) & \text{in } \mathbb{R}^n \times (0,\infty), \\ \Delta \phi = v - w & \text{in } \mathbb{R}^n \times (0,\infty), \\ v(x,0) = v_0(x), \quad w(x,0) = w_0(x) & \text{in } \mathbb{R}^n, \end{cases} \tag{1.1}$$

where the unknown functions $v = v(x,t)$ and $w = w(x,t)$ denote densities of the electron and the hole in electrolytes, respectively, $\phi = \phi(x,t)$ denotes the electric potential, $v_0(x)$ and $w_0(x)$ are the initial data. Throughout this paper, we assume that $n \geq 2$.

Notice that the function ϕ is determined by the Poison equation in the third equation of (1.1), and it is given by

$$\phi = (-\Delta)^{-1}(w - v) = \mathcal{F}^{-1}(|\xi|^{-2}\mathcal{F}(w - v)), \tag{1.2}$$

where $\mathcal{F}$ is the Fourier transform. So, the system (1.1) can be reduced to the following system:

$$\begin{cases} \partial_t v - \Delta v = -\nabla \cdot \big(v \nabla (-\Delta)^{-1}(w - v)\big) & \text{in } \mathbb{R}^n \times (0,\infty) \\ \partial_t w - \Delta w = \nabla \cdot \big(w \nabla (-\Delta)^{-1}(w - v)\big) & \text{in } \mathbb{R}^n \times (0,\infty) \\ v(x,0) = v_0(x), \quad w(x,0) = w_0(x) & \text{in } \mathbb{R}^n. \end{cases} \tag{1.3}$$

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Also the system (1.1) can be rewritten as the following equivalent integral system coming from the famous Duhamel principle:

$$\begin{cases} v(t) = e^{t\Delta}v_0 - B(v,\phi) \\ w(t) = e^{t\Delta}w_0 + B(w,\phi), \end{cases} \tag{1.4}$$

where

$$B(v,\phi) := -\int_0^t e^{(t-\tau)\Delta}\nabla\cdot\left(v\nabla(-\Delta)^{-1}(w-v)\right)(\tau)d\tau,$$

$$B(w,\phi) := \int_0^t e^{(t-\tau)\Delta}\nabla\cdot\left(w\nabla(-\Delta)^{-1}(w-v)\right)(\tau)d\tau,$$

Any solution that satisfies system (1.4) is called a mild solution of the system (1.1).

The Debye-Hückel system (1.1) is scaling invariant in the following sense: if (v,w) solves (1.1) with the initial data (v_0,w_0) (ϕ can be determined by (v,w)), then (v_γ,w_γ) with $(v_\gamma,w_\gamma)(x,t) := \left(\gamma^2 v,\gamma^2 w\right)\left(\gamma x,\gamma^2 t\right)$ is also a solution to (1.1) with the initial data

$$\left(v_{0,\gamma},w_{0,\gamma}\right)(x) := \left(\gamma^2 v_0,\gamma^2 w_0\right)(\gamma x) \tag{1.5}$$

(ϕ_γ can be determined by (v_γ,w_γ)).

Definition 1.1. *A critical space for the initial data of the system (1.1) is any Banach space $E\subset\mathcal{S}'(\mathbb{R}^n)$ whose norm is invariant under the scaling (1.5) for all $\gamma>0$, i.e.,*

$$\left\|\left(v_{0,\gamma},w_{0,\gamma}\right)(x)\right\|_E \approx \left\|\left(v_0,w_0\right)(x)\right\|_E .$$

In accordance with these scales, we can show that the space pairs $\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}\times\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}$ are critical for (1.1) (see Remark 2.5 for details).

There is a huge literature on well-posedness for fluid dynamics PDEs with singular data in different spaces, where the conditions are taken in norms of critical spaces. For instance, for Navier-Stokes equations and related models, we have well-posedness results in the critical case of the following spaces: Lebesgue space L^p [29], Marcinkiewicz space $L^{p,\infty}$ [40], Sobolev spaces H^s [19], Lei-Lin spaces $\mathcal{X}^s$ [10,9,11], Besov spaces $B^s_{p,q}$ [41], Triebel-Lizorkin spaces $F^s_{p,q}$ [15], Morrey spaces $\mathcal{M}^\lambda_p$ [30], Besov-Morrey spaces $\mathcal{N}^s_{p,\lambda,q}$ [32], Fourier-Besov spaces $\mathcal{F}B^s_{p,q}$ [17,25], Fourier-Herz spaces $\mathcal{F}B^s_{1,q}=\mathcal{B}^s_{1,q}$ [14], Fourier-Besov-Morrey spaces $\mathcal{FN}^s_{p,\lambda,q}$ [4,3,2,5,20,1], BMO^{-1} [31] and pseudomeasure spaces $\mathcal{PM}$ [35], among others.

The system (1.1) has been studied extensively in various function spaces. Karch in [28] established the proof of existence and uniqueness of global solutions of the system (1.1) for initial data in the Besov spaces $B^s_{p,\infty}$ with the condition of $-1<s<0$ and $p=\frac{n}{s+2}$. Later, Zhao et al. [43] established the global and local well-posedness for the system (1.1) in the critical Besov space $\dot{\mathcal{B}}^{-2+\frac{n}{p}}_{p,r}(\mathbb{R}^n)$ with $2\le p<2n$ and $1\le r\le\infty$ (which improved the corresponding results of Karch obtained in [28]). Kurokiba and Ogawa in [33] obtained similar results for the initial data in critical Lebesgue and Sobolev spaces. Very recently, Azanzal, Abbassi and Allalou [3] proved that small data global existence and large data local existence of mild solutions of the system (1.1) in critical Fourier-Morrey-Besov space $\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}\times\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}$, More related research can be consulted in [36,37]. It should be noted that for the Navier-Stokes equations, there is no existence result for initial data in a space with regularity index $s<-1$. In fact, the nonlinear term of (1.1) appears to be more closely related to the quadratic nonlinear heat equation ($\sim u^2$) than to the Navier-Stokes equations ($\sim u\cdot\nabla u$).Thus, the Debye-Hückel system has a better property than the Navier-Stokes equations in regard to the existence of solutions.

We mention here that if w vanishes ($w=0$), the system (1.1) becomes to the well-known Keller-Segel model of chemotaxis:

$$\begin{cases} \partial_t v = \Delta v - \nabla\cdot(v\nabla\phi) & \text{in } \mathbb{R}^n\times(0,\infty), \\ \Delta\phi = v & \text{in } \mathbb{R}^n\times(0,\infty), \\ v(x,0)=v_0(x), & \text{in } \mathbb{R}^n. \end{cases} \tag{1.6}$$

In the paper [12] the local well-posedness of the system (1.6) has been proved in the three-dimensional case. Iwabuchi and Nakamura [26,27] get the global well-posednes of (1.6) for small initial data in the critical space $\dot{\mathcal{B}}_{p,r}^{-2+\frac{n}{p}}(\mathbb{R}^n)$ with $1 \leq p < \infty$ and $1 \leq r \leq \infty$.

Our first aim in this paper is to show the analyticity of mild solutions to the system (1.1) by using the method of Gevrey estimate, which was first introduced by Foias and Temam [23]. Since then, the Gevrey class technique has become an important approach in the study of the space analyticity of solutions, which was later developed by several researchers, particularly with regard to the Navier-Stokes equations (NSE). Gruji and Kukavica [24] showed the Gevrey regularity in L^p for the NSE, Bae [6] proved the Gevrey estimate of solution for NSE in the critical Lie-Lin space $\mathcal{X}^{-1}$. More similar studies on the analyticity of solutions for NSE can be seen in Lemarie-Rieusset [34] and the references therein. Biswas [13] established Gevrey class regularity of solutions to a large class of dissipative equations in Besov type spaces defined via caloric extension. In 2016, Zhao [42] proved that the global mild solutions to the system (1.1) are Gevrey regular for all $2 \leq p < 2n$ and $1 \leq r \leq \infty$. Recently, Cui and Xiao [17] established the Gevrey class regularity for (1.1) in the Fourier-Besov space $\mathcal{F}B^s_{p,q}$. Inspired by this, we will establish the Gevrey class regularity for the system (1.1) in the Fourier-Besov-Morrey spaces $\mathcal{FN}^s_{p,\lambda,q}$ (larger than $\mathcal{F}B^s_{p,q}$-spaces, ie., $\mathcal{FN}^s_{p,0,q} = \mathcal{F}B^s_{p,q}$). The Gevrey class technique enables us to avoid cumbersome recursive estimation of higher-order derivatives (any order derivative of the solution (v, w) enjoys the same behavior with (v, w) in some sense), see [39]. The second aim of this paper is decay of Fourier-Besov-Morrey norms of mild solutions. In the end, we prove the blow-up criteria of the local mild solution of the Debye-Hückel system (1.1). In some suitable sense, our results extend/ complement some previous works such as [17,42,43,44,21,22].

Let us define our setting before we present our results. Denote the set of all polynomials by $\mathcal{P}$ and the Morrey spaces by $\mathcal{M}^\lambda_p$ with norm

$$\|f\|_{\mathcal{M}^\lambda_p} = \sup_{x_0\in\mathbb{R}^n} \sup_{r>0} r^{-\frac{\lambda}{p}} \|f\|_{L^p(B(x_0,r))} < \infty.$$

We define

$$\mathcal{FN}^s_{p,\lambda,q} = \left\{ f \in \mathcal{S}'\backslash\mathcal{P} \mid \|f\|_{\mathcal{FN}^s_{p,\lambda,q}} = \left(\sum_{j\in\mathbb{Z}} 2^{jsq} \left\| \varphi_j \hat{f} \right\|^q_{\mathcal{M}^\lambda_p} \right)^{\frac{1}{q}} < \infty \right\}, \tag{1.7}$$

where $\{\varphi_j\}$ is the Littlewood-Paley decomposition (see Section 2 for details). The Fourier-Besov-Morrey spaces $\mathcal{FN}^s_{p,\lambda,q}$ were introduced by [20] in the context of active scalar equations. Later, these spaces were employed to investigate the global well-posedness of the Navier-Stokes-Coriolis system in [18]. The space (1.7) belongs to a class whose definition of the norm is based on Fourier transform, but it is not contained in L^2, i.e., there is $f \in \mathcal{FN}^s_{p,\lambda,q}$ with infinite L^2-norm.

Throughout this paper, we use $\mathcal{FN}^s_{p,\lambda,q}$ to denote the homogenous Fourier-Besov-Morrey spaces and χ_A to denote the indicator function of a set A. Let X, Y be Banach spaces, we use $(v, w) \in X$ to denote $(v, w) \in X \times X$ and

$$\|v\|_{X\cap Y} := \|v\|_X + \|v\|_Y \quad ;; \quad \|(v, w)\|_X := \|v\|_X + \|w\|_X,$$

C will denote constants which can be different at different places, $A \sim B$ means that there are two constants $C_1, C_2 > 0$ such that

$$C_1 B \leq A \leq C_2 B,$$

$V \lesssim W$ denotes the estimate $V \leq CW$ for some constant $C \geq 1$, and p' is the conjugate of p satisfying $\frac{1}{p} + \frac{1}{p'} = 1$ for $1 \leq p \leq \infty$.

Since the argument employed in the proof of the global-in-time existence of mild solution of the system (1.1) with small initial data in $\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}$ is crucial in the proof of our first main result, let us recall the result obtained in [3].

Theorem 1.2. *Let $n \geq 2$, $\rho_0 > 2$, $\max\{n - (n - 1)p, 0\} \leq \lambda < n$, $1 \leq p < \infty$, $q \in [1, \infty]$, $(v_0, w_0) \in \mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}$ and $\frac{1}{\rho_0} + \frac{1}{\rho_0'} = 1$. Then there exists $T \geq 0$ such that the system (1.1) has a unique local mild*

solution
$(v,w)\in X$, *where*

$$X=\mathfrak{L}^{\rho_0}\left([0,T);\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}\right)\cap\mathfrak{L}^{\rho_0'}\left([0,T);\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}}\right),$$

and

$$(v,w)\in\mathcal{C}\left([0,T);\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}\right).$$

Besides, there exixts $K\geq 0$ *such that if* (v_0,w_0) *satisfies* $\|(v_0,w_0)\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}}\leq K$, *then the above assertion holds for* $T=\infty$; *i.e., the mild solution* (v,w) *is global.*

Now we give the first result of this paper, which shows that mild solutions obtained in Theorem 1.2 are analytic in the sense of the Gevrey class.

Theorem 1.3. *Let* $n\geq 2$, $\rho_0>2$, $\max\{n-(n-1)p,0\}\leq\lambda<n$, $1\leq p<\infty$, $q\in[1,\infty]$, $(v_0,w_0)\in\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}$ *and* $\frac{1}{\rho_0}+\frac{1}{\rho_0'}=1$.
Then, there exixts $K_0\geq 0$ *such that if* (v_0,w_0) *satisfies* $\|(v_0,w_0)\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}}\leq K_0$, *the global-in-time mild solution obtained in Theorem 1.2 is analytic in the sense that*

$$\|\left(e^{\sqrt{t}|D|}v,e^{\sqrt{t}|D|}w\right)\|_X\lesssim\|(v_0,w_0)\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}},$$

where $e^{\sqrt{t}|D|}$ *is the Fourier multiplier with symbol* $e^{\sqrt{t}|\xi|}$.

In the above theorem, we have proved analyticity of mild solutions, so that we can further obtain the time decay estimates of mild solutions.

Theorem 1.4. *Under the assumptions of Theorem 1.2. The global-in-time mild solution* $(v,w)\in X$ *and* $\left(e^{\sqrt{t}|D|}v,e^{\sqrt{t}|D|}w\right)\in X$ *obtained from Theorem 1.3 satisfies the following time decay estimate:*

$$\left\|\left((-\Delta)^{\frac{1}{2}}v(t),(-\Delta)^{\frac{1}{2}}w(t)\right)\right\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}}\lesssim t^{-\frac{1}{2}}\|(v_0,w_0)\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}},$$

where $(-\Delta)^{1/2}v=\mathcal{F}^{-1}\left(|\xi|\mathcal{F}(v)\right)$.

Lastly, if we suppose that the maximal time of existence is finite, the following theorem guarantees a blow-up criterion for solutions of the system (1.1).

Theorem 1.5. *Let* T^* *denote the maximal time of existence of a solution*

$$(v,w)\in\mathcal{L}^\infty\left([0,T^*);\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}\right)\cap\mathcal{L}^1\left([0,T^*),\mathcal{FN}_{p,\lambda,q}^{\frac{n}{p'}+\frac{\lambda}{p}}\right).$$

If $T^*<\infty$, *then*

$$\|(v,w)\|_{\mathcal{L}^1\left([0,T^*),\mathcal{FN}_{p,\lambda,q}^{\frac{n}{p'}+\frac{\lambda}{p}}\right)}=\infty.$$

2. Preliminaries

Existence and analyticity of Lei-Lin solution to the Debye-Hückel system
In this section, we recall some basic properties of Fourier-Besov-Morrey spaces and other analysis tools that we will employ throughout this study.
The function spaces $\mathcal{M}_p^\lambda$ are defined as follows.

Definition 2.1. *[30] Let $1 \leq p \leq \infty$ and $0 \leq \lambda < n$.*

- *The homogeneous Morrey space $\mathcal{M}_p^\lambda$ is the set of all functions $f \in L^p(B(x_0, r))$ such that*

$$\|f\|_{\mathcal{M}_p^\lambda} = \sup_{x_0 \in \mathbb{R}^n} \sup_{r>0} r^{-\frac{\lambda}{p}} \|f\|_{L^p(B(x_0,r))} < \infty, \tag{2.1}$$

where $B(x_0, r)$ is the open ball in $\mathbb{R}^n$ centered at x_0 and with radius $r > 0$. The space $\mathcal{M}_p^\lambda$ endowed with the norm $\|f\|_{\mathcal{M}_p^\lambda}$ is a Banach space. When $p = 1$, the L^1 -norm in (2.1) is understood as the total variation of the measure f on $B(x_0, r)$ and $\mathcal{M}_p^\lambda$ as a subspace of Radon measures. When $\lambda = 0$, we have $\mathcal{M}_p^0 = L^p$.

- *The mixed Morrey-sequence space $l^q(\mathcal{M}_p^\lambda)$ consists of all sequences $\{f_i\}_{i\in\mathbb{Z}}$ of measurable functions in $\mathbb{R}^n$ such that $\|\{f_i\}_{i\in\mathbb{Z}}\|_{l^q(\mathcal{M}_p^\lambda)} < \infty$. For $\{f_i\}_{i\in\mathbb{Z}} \in l^q(\mathcal{M}_p^\lambda)$ we define*

$$\|\{f_i\}_{i\in\mathbb{Z}}\|_{l^q(\mathcal{M}_p^\lambda)} := \left(\sum_{j\in\mathbb{Z}} \|f_i\|_{\mathcal{M}_p^\lambda}^q\right)^{\frac{1}{q}}.$$

The proofs of the results discussed in this work are based on a dyadic partition of unity in the Fourier variables, known as the homogeneous Littlewood-Paley decomposition. We present briefly this construction below. For more detail, we refer the reader to [8].
Let $f \in \mathcal{S}'(\mathbb{R}^n)$. Define the Fourier transform as

$$\hat{f}(\xi) = \mathcal{F}f(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) dx$$

and its inverse Fourier transform as

$$\check{f}(x) = \mathcal{F}^{-1}f(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi) d\xi.$$

Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$ be such that $0 \leq \varphi \leq 1$ and $\operatorname{supp}(\varphi) \subset \left\{\xi \in \mathbb{R}^d : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\right\}$ and

$$\sum_{j\in\mathbb{Z}} \varphi\left(2^{-j}\xi\right) = 1, \quad \text{for all } \xi \neq 0.$$

We denote

$$\varphi_j(\xi) = \varphi\left(2^{-j}\xi\right), \quad \delta_j(\xi) = \sum_{k\leq j-1} \varphi_k(\xi)$$

and

$$h(x) = \mathcal{F}^{-1}\varphi(x), \quad g(x) = \mathcal{F}^{-1}\delta(x).$$

We now present some frequency localization operators:

$$\dot{\Delta}_j f = \mathcal{F}^{-1}\varphi_j * f = 2^{dj} \int_{\mathbb{R}^d} h\left(2^j y\right) f(x-y) dy$$

and

$$\dot{S}_j f = \sum_{k\leq j-1} \dot{\Delta}_k f = \mathcal{F}^{-1}\delta_j * f = 2^{dj} \int_{\mathbb{R}^d} g\left(2^j y\right) f(x-y) dy.$$

where $\dot{\Delta}_j = \dot{S}_j - \dot{S}_{j-1}$ is a frequency projection to the annulus $\{|\xi| \sim 2^j\}$ and $\dot{S}_j$ is a frequency to the ball $\{|\xi| \lesssim 2^j\}$.
From the definition of $\dot{\Delta}_j$ and $\dot{S}_j$, one easily derives that

$$\dot{\Delta}_j \dot{\Delta}_k f = 0, \quad \text{if } |j-k| \geq 2$$
$$\dot{\Delta}_j\left(\dot{S}_{k-1} f \dot{\Delta}_k f\right) = 0, \quad \text{if } |j-k| \geq 5$$
$$\widehat{\dot{\Delta}_j f} = \varphi_j \widehat{f}.$$

The following Bony paraproduct decomposition will be applied throughout the paper.

$$uv = \dot{T}_u v + \dot{T}_v u + R(u,v)$$

where

$$\dot{T}_u v = \sum_{j\in\mathbb{Z}} \dot{S}_{j-1} u \dot{\Delta}_j v, \quad \dot{R}(u,v) = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j u \tilde{\Delta}_j v, \quad \tilde{\Delta}_j v = \sum_{|j'-j|\le 1} \dot{\Delta}_{j'} v.$$

Lemma 2.2. *[20] Let $1 \le p_1, p_2, p_3 < \infty$ and $0 \le \lambda_1,\ \lambda_2,\ \lambda_3 < n$.*
(i) (Hölder's inequality) Let $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{\lambda_3}{p_3} = \frac{\lambda_1}{p_1} + \frac{\lambda_2}{p_2}$, then we have

$$\|fg\|_{\mathcal{M}_{p_3}^{\lambda_3}} \le \|f\|_{\mathcal{M}_{p_1}^{\lambda_1}} \|g\|_{\mathcal{M}_{p_2}^{\lambda_2}}. \tag{2.2}$$

(ii) (Young's inequality) If $\varphi \in L^1$ and $g \in \mathcal{M}_{p_1}^{\lambda_1}$, then

$$\|\varphi * g\|_{\mathcal{M}_{p_1}^{\lambda_1}} \le \|\varphi\|_{L^1} \|g\|_{\mathcal{M}_{p_1}^{\lambda_1}}, \tag{2.3}$$

where $*$ denotes the standard convolution operator.

Now, we recall the Bernstein-type lemma in Fourier variables in Morrey spaces.

Lemma 2.3. *[20] Let $1 \le q \le p < \infty$, $0 \le \lambda_1, \lambda_2 < n$, $\frac{n-\lambda_1}{p} \le \frac{n-\lambda_2}{q}$ and let γ be a multi-index. If $supp(\widehat{f}) \subset \{|\xi| \le A2^j\}$, then there is a constant $C > 0$ independent of f and j such that*

$$\|(i\xi)^\gamma \widehat{f}\|_{\mathcal{M}_q^{\lambda_2}} \le C 2^{j|\gamma| + j(\frac{n-\lambda_2}{q} - \frac{n-\lambda_1}{p})} \|\widehat{f}\|_{\mathcal{M}_p^{\lambda_1}}. \tag{2.4}$$

Let us now recall the definition of Fourier-Besov-Morrey spaces $\mathcal{FN}_{p,\lambda,q}^s(\mathbb{R}^n)$, see [20].

Definition 2.4. *(Homogeneous Fourier-Besov-Morrey spaces)*
Let $1 \le p, q \le \infty$, $0 \le \lambda < n$ and $s \in \mathbb{R}$. The homogeneous Fourier-Besov-Morrey space $\mathcal{FN}_{p,\lambda,q}^s$ is defined as the set of all distributions $f \in \mathcal{S}'\backslash\mathcal{P}$, $\mathcal{P}$ is the set of all polynomials, such that the norm $\|f\|_{\mathcal{FN}_{p,\lambda,q}^s}$ is finite, where

$$\|f\|_{\mathcal{FN}_{p,\lambda,q}^s} \overset{\text{def}}{=} \begin{cases} \left(\sum_{j\in\mathbb{Z}} 2^{jsq} \left\|\varphi_j \hat{f}\right\|_{\mathcal{M}_p^\lambda}^q\right)^{\frac{1}{q}} & for \quad q < \infty \\ \sup_{j\in\mathbb{Z}} 2^{js} \left\|\varphi_j \hat{f}\right\|_{\mathcal{M}_p^\lambda} & for \quad q = \infty. \end{cases} \tag{2.5}$$

Note that the space $\mathcal{FN}_{p,\lambda,q}^s(\mathbb{R}^n)$ equipped with the norm (2.5) is a Banach space. Since $\mathcal{M}_p^0 = L^p$, we have $\mathcal{FN}_{p,0,q}^s = F\dot{B}_{p,q}^s$, and $\mathcal{FN}_{1,0,1}^s = \chi^s$ where χ^s is the Lei-Lin space [10].

Remark 2.5. *The space pair $\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}} \times \mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}$ is critical for (1.1). For this, set $v_{0,\gamma}(\xi) = \gamma^2 v_0(\gamma\xi)$, then its Fourier transform is $\widehat{v_{0,\gamma}}(\xi) = \gamma^{2-n}\hat{v}_0\left(\gamma^{-1}\xi\right)$.*
Let

$$f_j(\xi) \overset{def}{=} \varphi\left(2^{-j+[\log_2\gamma]-\log_2\gamma}\xi\right)\widehat{v_{0,\gamma}}(\xi) = \varphi\left(2^{-j+[\log_2\gamma]-\log_2\gamma}\xi\right)\gamma^{2-n}\hat{v}_0\left(\gamma^{-1}\xi\right)$$

By change of variable, we get:

$$\begin{aligned}
\|f_j\|_{\mathcal{M}_p^\lambda} &= \gamma^{2-n}\left\|\varphi\left(2^{-j+[\log_2\gamma]-\log_2\gamma}\xi\right)\hat{v}_0\left(\gamma^{-1}\xi\right)\right\|_{\mathcal{M}_p^\lambda} \\
&= \gamma^{2-n}\sup_{x_0\in\mathbb{R}^n}\sup_{r>0} r^{-\frac{\lambda}{p}}\left\|\varphi\left(2^{-j+[\log_2\gamma]}\gamma^{-1}\xi\right)\widehat{v_0}\left(\gamma^{-1}\xi\right)\right\|_{L^p(B(x_0,r))} \\
&= \gamma^{2-n}\gamma^{\frac{n}{p}}\gamma^{\frac{-\lambda}{p}}\sup_{x_0\in\mathbb{R}^n}\sup_{r>0}\left(\gamma^{-1}r\right)^{-\frac{\lambda}{p}}\left\|\varphi\left(2^{-j+[\log_2\gamma]}\eta\right)\widehat{v_0}(\eta)\right\|_{L^p(B(\gamma^{-1}x_0,\gamma^{-1}r))} \\
&= 2^{\left(2-\frac{n}{p'}-\frac{\lambda}{p}\right)\log_2\gamma}\left\|\varphi\left(2^{-j+[\log_2\gamma]}\eta\right)\widehat{v_0}(\eta)\right\|_{\mathcal{M}_p^\lambda},
\end{aligned}$$

which implies

$$\begin{aligned}
&\|\{2^{j(-2+\frac{n}{p'}+\frac{\lambda}{p})}\|f_j(\xi)\|_{\mathcal{M}_p^\lambda}\}\|_{l^q}\\
&=\|\{2^{j(2-\frac{n}{p'}-\frac{\lambda}{p})}2^{\log_2\gamma(-2+\frac{n}{p'}+\frac{\lambda}{p})}\|\varphi_{j-[\log_2\gamma]}\widehat{v_0}(\xi)\|_{\mathcal{M}_p^\lambda}\}\|_{l^q}\\
&=\|\{2^{(\log_2\gamma-[\log_2\gamma])(2-\frac{n}{p'}-\frac{\lambda}{p})}2^{(j-\log_2\gamma)(-2+\frac{n}{p'}+\frac{\lambda}{p})}\|\varphi_{j-[\log_2\gamma]}\widehat{v_0}(\xi)\|_{\mathcal{M}_p^\lambda}\}\|_{l^q}\\
&\approx\|v_0\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}}
\end{aligned}$$

and since

$$\varphi_j(\xi)\widehat{v_{0,\gamma}}(\xi)=\sum_{|k-j|\le 2}\varphi_j(\xi)f_k(\xi),$$

we can get

$$\|v_{0,\gamma}\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}}\approx\|v_0\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}}.$$

Similary, we have

$$\|w_{0,\gamma}\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}}\approx\|w_0\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}}.$$

Consequently,

$$\|(v_{0,\gamma},w_{0,\gamma})\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}}\approx\|(v_0,w_0)\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}}.$$

Lemma 2.6. *The derivation $\partial_\xi^\alpha:\mathcal{FN}_{p,\lambda,q}^{s+|\alpha|}\to\mathcal{FN}_{p,\lambda,q}^{s}$ is a bounded operator.*

Proof: We have

$$\begin{aligned}
\|\partial_\xi^\alpha v\|_{\mathcal{FN}_{p,\lambda,q}^s}&=\|\{2^{js}\varphi_j\widehat{\partial_\xi^\alpha v}\}_{j\in\mathbb{Z}}\|_{l^q(\mathcal{M}_p^\lambda)}\\
&=\|\{2^{js}\varphi_j|\xi|^\alpha\widehat{v}\}_{j\in\mathbb{Z}}\|_{l^q(\mathcal{M}_p^\lambda)}\\
&\lesssim\|\{2^{js}2^{j|\alpha|}\varphi_j\widehat{v}\}_{j\in\mathbb{Z}}\|_{l^q(\mathcal{M}_p^\lambda)} \qquad (2.6)\\
&\lesssim\|v\|_{\mathcal{FN}_{p,\lambda,q}^{s+|\alpha|}},
\end{aligned}$$

where in (2.6) we used the fact that $|\xi|\sim 2^j$ for all $j\in\mathbb{Z}$.

Remark 2.7. *As a consequence of Lemma 2.6, we have the following estimates:*

$$\begin{aligned}
\|\nabla v\|_{\mathcal{FN}_{p,\lambda,q}^s}&\lesssim\|v\|_{\mathcal{FN}_{p,\lambda,q}^{s+1}}\\
\|\nabla\cdot v\|_{\mathcal{FN}_{p,\lambda,q}^s}&\lesssim\|v\|_{\mathcal{FN}_{p,\lambda,q}^{s+1}},\\
\|\Delta v\|_{\mathcal{FN}_{p,\lambda,q}^s}&\lesssim\|v\|_{\mathcal{FN}_{p,\lambda,q}^{s+2}}.
\end{aligned}$$

The definition of mixed space-time spaces is given below.

Definition 2.8. *[21]*
Let $s\in\mathbb{R},\ 1\le p<\infty,\ 1\le q,\rho\le\infty,\ 0\le\lambda<n$, and $I=[0,T),\ T\in(0,\infty]$. The space-time norm is defined on $u(t,x)$ by

$$\|u(t,x)\|_{\mathcal{L}^\rho(I,\mathcal{FN}_{p,\lambda,q}^s)}=\Big\{\sum_{j\in\mathbb{Z}}2^{jqs}\|\widehat{\dot{\Delta}_j u}\|_{L^\rho(I,\mathcal{M}_p^\lambda)}^q\Big\}^{1/q},$$

and denote by $\mathcal{L}^\rho(I,\mathcal{FN}_{p,\lambda,q}^s)$ the set of distributions in $\mathcal{S}'(\mathbb{R}\times\mathbb{R}^n)/\mathcal{P}$ with finite $\|\cdot\|_{\mathcal{L}^\rho(I,\mathcal{FN}_{p,\lambda,q}^s)}$ norm.

According to Minkowski inequality, it is easy to verify that

$$L^\rho\left(I;\mathcal{FN}^s_{p,\lambda,q}\right)\hookrightarrow\mathcal{L}^\rho\left(I,\mathcal{FN}^s_{p,\lambda,q}\right),\quad \text{if } \rho\le q, \tag{2.7}$$

$$\mathcal{L}^\rho\left(I,\mathcal{FN}^s_{p,\lambda,q}\right)\hookrightarrow L^\rho\left(I;\mathcal{FN}^s_{p,\lambda,q}\right),\quad \text{if } \rho\ge q, \tag{2.8}$$

where $\|u(t,x)\|_{L^\rho\left(I;\mathcal{FN}^s_{p,\lambda,q}\right)}:=\left(\int_I\|u(\tau,\cdot)\|^\rho_{\mathcal{FN}^s_{p,\lambda,q}}d\tau\right)^{1/\rho}$.

At the end of this section, we will recall an existence and uniqueness result for an abstract operator equation in a Banach space that will be used to show Theorem 1.2 in the sequel. For the proof, we refer the reader to see [34,7].

Lemma 2.9. *Let X be a Banach space with norm $\|.\|_X$ and $B: X\times X\longmapsto X$ be a bounded bilinear operator satisfying*

$$\|B(u,v)\|_X\le C_0\|u\|_X\|v\|_X$$

for all $u,v\in X$ and a constant $C_0>0$. Then, if $0<\varepsilon<\frac{1}{4C_0}$ and if $y\in X$ such that $\|y\|_X\le\varepsilon$, the equation $x:=y+B(x,x)$ has a solution $\overline{x}$ in X such that $\|\overline{x}\|_X\le 2\varepsilon$. This solution is the only one in the ball $\overline{B}(0,2\varepsilon)$. Moreover, the solution depends continuously on y in the sense: if $\|y'\|_X<\varepsilon$, $x'=y'+B(x',x')$, and $\|x'\|_X\le 2\varepsilon$, then

$$\|\overline{x}-x'\|_X\le\frac{1}{1-4\varepsilon C_0}\|y-y'\|_X\,.$$

3. Gevrey Class Regularity

Let us sketch the proof of Theorem 1.2. Setting $V(t)=e^{\sqrt{t}|D|}v(t), W(t)=e^{\sqrt{t}|D|}w(t)$, and $\Phi(t)=e^{\sqrt{t}|D|}\phi(t)=W(t)-V(t)$. Then we see that $(V(t),W(t))$ satisfies the following integral system:

$$\begin{cases} V(t)=e^{\sqrt{t}|D|+t\Delta}v_0-\displaystyle\int_0^t e^{\left[(\sqrt{t}-\sqrt{s})|D|+(t-s)\Delta\right]}\nabla\cdot e^{\sqrt{s}|D|}\left(e^{-\sqrt{s}|D|}V(s)e^{-\sqrt{s}|D|}\nabla\Phi(s)\right)ds\\ \qquad :=e^{\sqrt{t}|D|+t\Delta}v_0-\mathcal{B}(V,\Phi),\\ W(t)=e^{\sqrt{t}|D|+t\Delta}w_0+\displaystyle\int_0^t e^{\left[(\sqrt{t}-\sqrt{s})|D|+(t-s)\Delta\right]}\nabla\cdot e^{\sqrt{s}|D|}\left(e^{-\sqrt{s}|D|}W(s)e^{-\sqrt{s}|D|}\nabla\Phi(s)\right)ds\\ \qquad :=e^{\sqrt{t}|D|+t\Delta}w_0+\mathcal{B}(W,\Phi).\end{cases}$$

We recall an auxiliary lemma that will help us to prove that the global-in-time mild solutions of the system (1.1) are Gevrey regular.

Lemma 3.1. *[38] Let $0<s\le t<\infty$. Then the following inequality holds*

$$t|a|-\frac{1}{2}(t^2-s^2)|a|^2-s|a-b|-s|b|\le\frac{1}{2} \tag{3.1}$$

for all $a,\ b\in\mathbb{R}^n$.

Now, we will establish some important linear estimates in Fourier-Besov-Morrey spaces.

Lemma 3.2. *Let $0\le\lambda<n$, $1\le p<\infty$, $1\le q\le\infty$ and $(v_0,w_0)\in\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}(\mathbb{R}^n)$. Then there exists a constant $C_1>0$ such that*

$$\|(e^{\sqrt{t}|D|}e^{t\Delta}v_0,e^{\sqrt{t}|D|}e^{t\Delta}w_0)\|_X\le C_1\|(v_0,w_0)\|_{\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}, \tag{3.2}$$

Proof: We have

$$e^{(\sqrt{t}|D|+\Delta t)}v_0=e^{-\frac{1}{2}(\sqrt{t}|D|-1)^2+\frac{1}{2}}e^{\frac{1}{2}t\Delta}v_0.$$

Using the Fourier transform, multiplying by φ_j and taking the $\mathcal{M}^\lambda_p$-norm we obtain

$$\|\varphi_j\widehat{e^{\sqrt{t}|D|}e^{t\Delta}v_0}\|_{\mathcal{M}^\lambda_p}\lesssim e^{-\frac{1}{2}t2^{2j}(\frac{3}{4})^2}\|\varphi_j\widehat{v_0}\|_{\mathcal{M}^\lambda_p}.$$

Multiplying by $2^{(-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0})j}$ and taking $l^q(\mathbb{Z})$-norm we get

$$\|e^{\sqrt{t}|D|}e^{t\Delta}v_0\|_{\mathfrak{L}^{\rho_0}\left([0,\infty[;\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}\right)} \lesssim \|v_0\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}}.$$

and

$$\|e^{\sqrt{t}|D|}e^{t\Delta}v_0\|_{\mathfrak{L}^{\rho_0'}\left([0,\infty[;\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}}\right)} \lesssim \|v_0\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}}.$$

Then

$$\|e^{\sqrt{t}|D|}e^{t\Delta}v_0\|_X \lesssim \|v_0\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}}.$$

Similary,

$$\|e^{\sqrt{t}|D|}e^{t\Delta}w_0\|_X \lesssim \|w_0\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}}.$$

We complete the proof of Lemma 3.2.

Lemma 3.3. *[16] Let $0 < T \leq \infty$, $s \in \mathbb{R}$, $0 \leq \lambda < n, 1 \leq p < \infty$, $1 \leq q, \rho, r \leq \infty$ and $1 \leq r \leq \rho$. There exists a constant $C > 0$ such that*

$$\left\|\int_0^t e^{\Delta(t-\tau)}h(\tau)d\tau\right\|_{\mathcal{L}^\rho([0,T),\mathcal{FN}_{p,\lambda,q}^s)} \leq C\,\|h\|_{\mathcal{L}^r([0,T),\mathcal{FN}_{p,\lambda,q}^{s-2-\frac{2}{\rho}+\frac{2}{r}})},$$

for all $h \in \mathcal{L}^r([0,T),\mathcal{FN}_{p,\lambda,q}^s)$.

The next lemma will be applied in the proof of Theorem 1.5.

Lemma 3.4. *Let $T > 0$, $s \in \mathbb{R}$, and $u, \theta \in S'(\mathbb{R}^n)$. Then,*

$$\int_0^t \left\|e^{(t-z)\Delta}\nabla\cdot(u\nabla\theta)\right\|_{\mathcal{FN}_{p,\lambda,q}^s} dz \lesssim \int_0^t \|u\nabla\theta\|_{\mathcal{FN}_{p,\lambda,q}^{s+1}}\, dz,$$

for all $t \in (0,T]$.

Proof: Observe that

$$\begin{aligned}\int_0^t \left\|e^{(t-z)\Delta}\nabla\cdot(u\nabla\theta)\right\|_{\mathcal{FN}_{p,\lambda,q}^s} dz &= \int_0^t \left(\sum_{j\in\mathbb{Z}} 2^{jsq}\left\|\varphi_j\mathcal{F}(e^{(t-z)\Delta}\nabla\cdot(u\nabla\theta))\right\|_{\mathcal{M}_p^\lambda}^q\right)^{\frac{1}{q}} dz\\ &= \int_0^t \left(\sum_{j\in\mathbb{Z}} 2^{jsq}\left\|\varphi_j e^{-(t-z)|\xi|^2}\mathcal{F}(\nabla\cdot(u\nabla\theta))\right\|_{\mathcal{M}_p^\lambda}^q\right)^{\frac{1}{q}} dz\\ &\lesssim \int_0^t \left(\sum_{j\in\mathbb{Z}} 2^{jsq}\left\|\varphi_j\mathcal{F}(\nabla\cdot(u\nabla\theta))\right\|_{\mathcal{M}_p^\lambda}^q\right)^{\frac{1}{q}} dz\\ &\lesssim \int_0^t \|\nabla\cdot(u\nabla\theta)\|_{\mathcal{FN}_{p,\lambda,q}^s}\, dz\\ &\lesssim \int_0^t \|u\nabla\theta\|_{\mathcal{FN}_{p,\lambda,q}^{s+1}}\, dz, \end{aligned} \tag{3.3}$$

where in (3.3) we have used Remark 2.7.

In the following proposition, we will establish the bilinear estimate which will be crucial in the proof of Theorem 1.2.

Proposition 3.5. *Under the hypothesis of Theorem 1.3, there exists a constant $C_0 > 0$ such that*

$$\|\mathcal{B}(V,\Phi)\|_X \le C_0 \|(V,W)\|_X^2$$

for all $V, W \in X$.

Proof: First, using Lemma 3.3 and Lemma 3.1 and Remark 2.7, we have

$$\begin{aligned}
&\|\mathcal{B}(V,\Phi)\|_{\mathfrak{L}^{\rho_0}\left([0,\infty[;\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}\right)} \\
&\lesssim \|e^{\sqrt{\tau}|D|}\nabla\cdot\left(e^{-\sqrt{s}|D|}V(s)e^{-\sqrt{s}|D|}\nabla\Phi(s)\right)\|_{\mathcal{L}^1\left([0,\infty[,\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}\right)} \\
&\lesssim \|e^{\sqrt{\tau}|D|}\left(e^{-\sqrt{s}|D|}V(s)e^{-\sqrt{s}|D|}\nabla\Phi(s)\right)\|_{\mathcal{L}^1\left([0,\infty[,\mathcal{FN}_{p,\lambda,q}^{-1+\frac{n}{p'}+\frac{\lambda}{p}}\right)}.
\end{aligned}$$

Then, it suffices to show that

$$\|e^{\sqrt{\tau}|D|}\left(e^{-\sqrt{s}|D|}V(s)e^{-\sqrt{s}|D|}\nabla\Phi(s)\right)\|_{\mathcal{L}^1\left([0,\infty[,\mathcal{FN}_{p,\lambda,q}^{-1+\frac{n}{p'}+\frac{\lambda}{p}}\right)} \lesssim \|(V,\Phi)\|_X^2. \tag{3.4}$$

Applying Bony para-product decomposition and quasi-orthogonality property for Littlewood-Paley decomposition, for fixed j, we obtain

$$\begin{aligned}
\dot{\Delta}_j e^{\sqrt{\tau}|D|}\left(e^{-\sqrt{s}|D|}Ve^{-\sqrt{s}|D|}\nabla\Phi\right) &= \sum_{|k-j|\le 4}\dot{\Delta}_j e^{\sqrt{\tau}|D|}(e^{-\sqrt{s}|D|}\dot{S}_{k-1}Ve^{-\sqrt{s}|D|}\dot{\Delta}_k\nabla\Phi) \\
&\quad + \sum_{|k-j|\le 4}\dot{\Delta}_j e^{\sqrt{\tau}|D|}(e^{-\sqrt{s}|D|}\dot{S}_{k-1}\nabla\Phi e^{-\sqrt{s}|D|}\dot{\Delta}_k V) \\
&\quad + \sum_{k\ge j-3}\dot{\Delta}_j e^{\sqrt{\tau}|D|}(e^{-\sqrt{s}|D|}\dot{\Delta}_k Ve^{-\sqrt{s}|D|}\widetilde{\dot{\Delta}}_k\nabla\Phi) \\
&:= R_j^1 + R_j^2 + R_j^3.
\end{aligned}$$

Then, by the triangle inequalities in $\mathcal{M}_p^\lambda$ and in $l^q(\mathbb{Z})$, one has

$$\begin{aligned}
&\|\mathcal{B}(V,\Phi)\|_{\mathfrak{L}^{\rho_0}\left([0,\infty[;\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}\right)} \\
&\lesssim \left\{\sum_{j\in\mathbb{Z}} 2^{j(-1+\frac{n}{p'}+\frac{\lambda}{p})q}\|\mathcal{F}\left(\dot{\Delta}_j e^{\sqrt{\tau}|D|}\left(e^{-\sqrt{s}|D|}Ve^{-\sqrt{s}|D|}\nabla\Phi\right)\right)\|^q_{L^1([0,\infty[,\mathcal{M}_p^\lambda)}\right\}^{\frac{1}{q}} \\
&\le \left\{\sum_{j\in\mathbb{Z}} 2^{j(-1+\frac{n}{p'}+\frac{\lambda}{p})q}\|\widehat{R_j^1}\|^q_{L^1([0,\infty[,\mathcal{M}_p^\lambda)}\right\}^{\frac{1}{q}} \\
&+ \left\{\sum_{j\in\mathbb{Z}} 2^{j(-1+\frac{n}{p'}+\frac{\lambda}{p})q}\|\widehat{R_j^2}\|^q_{L^1([0,\infty[,\mathcal{M}_p^\lambda)}\right\}^{\frac{1}{q}} \\
&+ \left\{\sum_{j\in\mathbb{Z}} 2^{j(-1+\frac{n}{p'}+\frac{\lambda}{p})q}\|\widehat{R_j^3}\|^q_{L^1([0,\infty[,\mathcal{M}_p^\lambda)}\right\}^{\frac{1}{q}} \\
&:= \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3.
\end{aligned}$$

By using the Young inequality in Morrey spaces and the Bernstein-type inequality with $|\gamma| = 0$, we have

$$\left\|\varphi_j \hat{V}\right\|_{L^1} \leq C 2^{j(\frac{n}{p'}+\frac{\lambda}{p})} \left\|\varphi_j \hat{V}\right\|_{\mathcal{M}_p^\lambda}.$$

Thus, using Young's inequality in Morrey spaces (2.3), the estimate (2.4) and Lemma 3.1, we get

$$\begin{aligned}
\|\widehat{R_j^1}\|_{L^1([0,\infty[,\mathcal{M}_p^\lambda)} &= \left\|\sum_{|k-j|\leq 4} \varphi_j e^{\sqrt{\tau}|\xi|} (\widehat{e^{-\sqrt{s}|D|}\dot{S}_{k-1}V e^{-\sqrt{s}|D|}\dot{\Delta}_k \nabla\Phi})\right\|_{L^1([0,\infty[,\mathcal{M}_p^\lambda)} \\
&= \left\|\sum_{|k-j|\leq 4} \varphi_j e^{\sqrt{\tau}|\xi|} \left[\left(\sum_{m\leq k-2} e^{-\sqrt{\tau}|\xi|}\varphi_m \widehat{V}\right) * e^{-\sqrt{\tau}|\xi|}\varphi_k \widehat{\nabla\Phi}\right]\right\|_{L^1([0,\infty[,\mathcal{M}_p^\lambda)} \\
&= \left\|\sum_{|k-j|\leq 4} \varphi_j \int_{\mathbb{R}^n} e^{\sqrt{\tau}(|\xi|-|\xi-y|-|y|)} \left(\sum_{m\leq k-2} \varphi_m \widehat{V}\right)(\xi-y)\varphi_k \widehat{\nabla\Phi}(y)dy\right\|_{L^1([0,\infty[,\mathcal{M}_p^\lambda)} \\
&\lesssim \sum_{|k-j|\leq 4} \|(\widehat{\dot{S}_{k-1}V\dot{\Delta}_k\nabla\Phi})\|_{L^1([0,\infty[,\mathcal{M}_p^\lambda)} \\
&\lesssim \sum_{|k-j|\leq 4} 2^k \|\varphi_k\widehat{\Phi}\|_{L^{\rho_0'}(I,\mathcal{M}_p^\lambda)} \sum_{l\leq k-2} \|\varphi_l \widehat{V}\|_{L^{\rho_0}(I,L^1)} \\
&\lesssim \sum_{|k-j|\leq 4} 2^k \|\varphi_k\widehat{\Phi}\|_{L^{\rho_0'}(I,\mathcal{M}_p^\lambda)} \sum_{l\leq k-2} 2^{(\frac{n}{p'}+\frac{\lambda}{p})l} \|\varphi_l \widehat{V}\|_{L^{\rho_0}(I,\mathcal{M}_p^\lambda)} \\
&\lesssim \sum_{|k-j|\leq 4} 2^k \|\varphi_k\widehat{\Phi}\|_{L^{\rho_0'}(I,\mathcal{M}_p^\lambda)} \sum_{l\leq k-2} 2^{(-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0})l} 2^{(2-\frac{2}{\rho_0})l} \|\varphi_l \widehat{V}\|_{L^{\rho_0}(I,\mathcal{M}_p^\lambda)} \\
&\lesssim \|V\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}})} \sum_{|k-j|\leq 4} 2^k \Big(\sum_{l\leq k-2} 2^{l(2-\frac{2}{\rho_0})q'}\Big)^{\frac{1}{q'}} \|\varphi_k\widehat{\Phi}\|_{L^{\rho_0'}(I,\mathcal{M}_p^\lambda)} \\
&\lesssim \|V\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}})} \sum_{|k-j|\leq 4} 2^{k(3-\frac{2}{\rho_0})} \|\varphi_k\widehat{\Phi}\|_{L^{\rho_0'}(I,\mathcal{M}_p^\lambda)} \\
&\lesssim \|V\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}})} 2^{j(-1-\frac{n}{p'}-\frac{\lambda}{p})} \times \\
&\qquad \sum_{k\in\mathbb{Z}} 2^{-(j-k)(1-\frac{n}{p'}-\frac{\lambda}{p})} \chi_{\{k';|k'|\leq 4\}}(j-k) 2^{k(\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'})} \|\varphi_k\widehat{\Phi}\|_{L^{\rho_0'}(I,\mathcal{M}_p^\lambda)} \\
&\lesssim \|V\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}})} 2^{j(1-\frac{n}{p'}-\frac{\lambda}{p})} (\mathcal{P}_{k'} * \mathcal{G}_k)_j,
\end{aligned}$$

with

$$\mathcal{P}_{k'} = 2^{-k'(1-\frac{n}{p'}-\frac{\lambda}{p})} \chi_{\{k';|k'|\leq 4\}} \quad \text{and} \quad \mathcal{G}_k = 2^{k(\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'})} \|\varphi_k\widehat{\Phi}\|_{L^{\rho_0'}(I,\mathcal{M}_p^\lambda)}.$$

Hence, by using the Young inequality for series, one has

$$\begin{aligned}
\mathrm{E}_1 &\lesssim \|V\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}})} \|\mathcal{P}_{k'}\|_{l^1(\mathbb{Z})} \|\mathcal{G}_k\|_{l^q(\mathbb{Z})} \\
&\lesssim \|V\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}})} \|\Phi\|_{\mathfrak{L}^{\rho_0'}(I,\mathcal{FN}_{p,\lambda,q}^{\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}})}.
\end{aligned}$$

Similary, we get

$$\mathrm{E}_2 \lesssim \|\Phi\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}})} \|V\|_{\mathfrak{L}^{\rho_0'}(I,\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}})}.$$

For I_3, first we use the Young inequality in Morrey spaces (2.3), the Bernstein-type inequality with $|\gamma| = 0$ together with the Hölder inequality, to get

$$\begin{aligned}
&\|\widehat{R_j^3}\|_{L^1([0,\infty[,\mathcal{M}_p^\lambda)} \\
&= \left\| \sum_{k\geq j-3} \dot{\Delta}_j e^{\sqrt{\tau}|D|} (e^{-\sqrt{s}|D|} \widehat{\dot{\Delta}_k V e^{-\sqrt{s}|D|}} \widetilde{\dot{\Delta}}_k \nabla\Phi) \right\|_{L^1([0,\infty[,\mathcal{M}_p^\lambda)} \\
&= \left\| \sum_{k\geq j-3} \varphi_j e^{\sqrt{\tau}|\xi|} (e^{-\sqrt{s}|D|} \dot{\Delta}_k \widehat{V e^{-\sqrt{s}|D|}} \widetilde{\dot{\Delta}}_k \nabla\Phi) \right\|_{L^1([0,\infty[,\mathcal{M}_p^\lambda)} \\
&= \left\| \sum_{k\geq j-3} \varphi_j e^{\sqrt{\tau}|\xi|} \left[\left(e^{-\sqrt{\tau}|\xi|} \varphi_k \widehat{V} \right) * \sum_{|m-k|\leq 1} e^{-\sqrt{\tau}|\xi|} \varphi_m \widehat{\nabla\Phi} \right] \right\|_{L^1([0,\infty[,\mathcal{M}_p^\lambda)} \\
&= \left\| \sum_{k\geq j-3} \varphi_j \int_{\mathbb{R}^n} e^{\sqrt{\tau}(|\xi|-|\xi-y|^2-|y|^2)} \varphi_k \widehat{V}(\xi-y) \sum_{|m-k|\leq 1} \varphi_m \widehat{\nabla\Phi}(y) dy \right\|_{L^1([0,\infty[,\mathcal{M}_p^\lambda)} \\
&\leq \sum_{k\geq j-3} \|(\widehat{\dot{\Delta}_k V} * \widehat{\widetilde{\dot{\Delta}}_k \nabla\Phi}\|_{L^1([0,\infty[,\mathcal{M}_p^\lambda)} \\
&\leq \sum_{k\geq j-3} \|\varphi_k \widehat{V}\|_{L^{\rho_0'}(I,\mathcal{M}_p^\lambda)} \sum_{|l-k|\leq 1} \|\varphi_l \widehat{\nabla\Phi}\|_{L^{\rho_0}(I,L^1)}
\end{aligned}$$

$$\begin{aligned}
&\lesssim \sum_{k\geq j-3} \|\varphi_k \widehat{V}\|_{L^{\rho_0'}(I,\mathcal{M}_p^\lambda)} \sum_{|l-k|\leq 1} 2^l 2^{l(\frac{n}{p'}+\frac{\lambda}{p})} \|\varphi_l \widehat{\Phi}\|_{L^{\rho_0}(I,\mathcal{M}_p^\lambda)} \\
&\lesssim \sum_{k\geq j-3} \|\varphi_k \widehat{V}\|_{L^{\rho_0'}(I,\mathcal{M}_p^\lambda)} \Big(\sum_{|l-k|\leq 1} 2^{l(1-\frac{2}{\rho_0})q'} \Big)^{\frac{1}{q'}} \|\Phi\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}})} \\
&\lesssim \|\Phi\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}})} \sum_{k\geq j-3} 2^{k(1-\frac{2}{\rho_0})} \|\varphi_k \widehat{V}\|_{L^{\rho_0'}(I,\mathcal{M}_p^\lambda)} \\
&\lesssim \|\Phi\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}})} 2^{j(-1-\frac{n}{p'}-\frac{\lambda}{p})} \times \\
&\qquad \sum_{k\in\mathbb{Z}} 2^{-(j-k)(1-\frac{n}{p'}-\frac{\lambda}{p})} \chi_{\{k';k'\leq 2\}}(j-k) 2^{k(\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'})} \|\varphi_k \widehat{V}\|_{L^{\rho_0'}(I,\mathcal{M}_p^\lambda)} \\
&\lesssim \|\Phi\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}})} 2^{j(1-\frac{n}{p'}-\frac{\lambda}{p})} (\mathcal{P}_{k'} * \mathcal{G}_k)_j,
\end{aligned}$$

with $\mathcal{P}_{k'} = 2^{-k'(1-\frac{n}{p'}-\frac{\lambda}{p})} \chi_{\{k';k'\leq 2\}}$ and $\mathcal{G}_k = 2^{k(\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'})} \|\varphi_k \widehat{V}\|_{L^{\rho_0'}(I,\mathcal{M}_p^\lambda)}$.

Then, applying the Young inequality for series, we obtain

$$\begin{aligned}
\mathrm{E}_3 &\lesssim \|\Phi\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}})} \|\mathcal{P}_{k'}\|_{l^1(\mathbb{Z})} \|\mathcal{G}_k\|_{l^q(\mathbb{Z})} \\
&\lesssim \|\Phi\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}})} \|V\|_{\mathfrak{L}^{\rho_0'}(I,\mathcal{FN}_{p,\lambda,q}^{\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}})}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\|\mathcal{B}(V,\Phi)\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}})}\\
&\lesssim \|V\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}})}\|\Phi\|_{\mathfrak{L}^{\rho_0'}(I,\mathcal{FN}_{p,\lambda,q}^{\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}})}\\
&\quad+\|\Phi\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}})}\|V\|_{\mathfrak{L}^{\rho_0'}(I,\mathcal{FN}_{p,\lambda,q}^{\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}})}.
\end{aligned}$$

Analogously,

$$\begin{aligned}
&\|\mathcal{B}(V,\Phi)\|_{\mathfrak{L}^{\rho_0'}(I,\mathcal{FN}_{p,\lambda,q}^{\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}})}\\
&\lesssim \|V\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}})}\|\Phi\|_{\mathfrak{L}^{\rho_0'}(I,\mathcal{FN}_{p,\lambda,q}^{\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}})}\\
&\quad+\|\Phi\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}})}\|V\|_{\mathfrak{L}^{\rho_0'}(I,\mathcal{FN}_{p,\lambda,q}^{\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}})}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\|\mathcal{B}(V,\Phi)\|_X\\
&\lesssim \|V\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}})}\|\Phi\|_{\mathfrak{L}^{\rho_0'}(I,\mathcal{FN}_{p,\lambda,q}^{\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}})}\\
&\quad+\|\Phi\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}})}\|V\|_{\mathfrak{L}^{\rho_0'}(I,\mathcal{FN}_{p,\lambda,q}^{\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}})}.
\end{aligned}$$

Since $\Phi=(-\Delta)^{-1}(W-V)=\mathcal{F}^{-1}(|\xi|^{-2}\mathcal{F}(W-V))$, then

$$\begin{aligned}
\|\mathcal{B}(V,\Phi)\|_X &\lesssim \|V\|_{\mathfrak{L}^{\rho_0}(I;\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}})}\times\|(-\Delta)^{-1}(W-V)\|_{\mathfrak{L}^{\rho_0'}(I;\mathcal{FN}_{p,\lambda,q}^{\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}})}\\
&\quad+\|(-\Delta)^{-1}(W-V)\|_{\mathfrak{L}^{\rho_0}(I;\mathcal{FN}_{p,\lambda,q}^{\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}})}\times\|V\|_{\mathfrak{L}^{\rho_0'}(I;\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}})}\\
&\lesssim \|V\|_{\mathfrak{L}^{\rho_0}(I;\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}})}\times\|W-V\|_{\mathfrak{L}^{\rho_0'}(I;\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}})}\\
&\quad+\|W-V\|_{\mathfrak{L}^{\rho_0}(I;\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}})}\times\|V\|_{\mathfrak{L}^{\rho_0'}(I;\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}})}\\
&\lesssim \|(V,W)\|_{\mathfrak{L}^{\rho_0}(I;\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}})}\times\|(V,W)\|_{\mathfrak{L}^{\rho_0'}(I;\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}})}\\
&\le C_0\|(V,W)\|_X^2.
\end{aligned}$$

Remark 3.6. *By following a similar argument to the one presented above, we get*

$$\|v\nabla\phi\|_{\mathcal{L}^1\left([0,\infty[,\mathcal{FN}_{p,\lambda,q}^{-1+\frac{n}{p'}+\frac{\lambda}{p}}\right)}\le\|(v,w)\|_{\mathfrak{L}^{\rho_0}(I;\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}})}\|(v,w)\|_{\mathfrak{L}^{\rho_0'}(I;\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}})}.$$

Particularly,

$$\|v\nabla\phi\|_{\mathcal{FN}_{p,\lambda,q}^{-1+\frac{n}{p'}+\frac{\lambda}{p}}}\lesssim\|(v,w)\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}}\|(v,w)\|_{\mathcal{FN}_{p,\lambda,q}^{\frac{n}{p'}+\frac{\lambda}{p}}}. \tag{3.5}$$

3.1. Proof of Theorem 1.3

From Proposition 3.5, we have

$$\|(\mathcal{B}(V,\Phi),\mathcal{B}(W,\Phi))\|_X\le C_0\|(V,W)\|_X^2. \tag{3.6}$$

By Lemma 2.9, we know that if $\|(e^{\sqrt{t}|D|}e^{t\Delta}v_0, e^{\sqrt{t}|D|}e^{t\Delta}w_0)\|_X \le \varepsilon$ with $\varepsilon = \frac{1}{4C_0}$, then the system (1.1) has a unique analytic solution in $\mathrm{B}(0,2\varepsilon) := \{y \in X : \|y\|_X \le 2\varepsilon\}$.
We explain that it is possible to choose the initial data such that $\|(e^{\sqrt{t}|D|}e^{t\Delta}v_0, e^{\sqrt{t}|D|}e^{t\Delta}w_0)\|_X \le \varepsilon$. According to Lemma 3.2, we have

$$\|(e^{\sqrt{t}|D|}e^{t\Delta}v_0, e^{\sqrt{t}|D|}e^{t\Delta}w_0)\|_X \le C_1\|(v_0,w_0)\|_{\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}. \tag{3.7}$$

So, if we choose $\|(v_0,w_0)\|_{\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} \le K_0$ with $K_0 = \frac{1}{4C_1C_0}$, then Lemma 2.9 together with the estimates (3.6) and (3.7) assures that the Debye-Hückel system (1.1) has a unique analytic solution so that

$$\|\left(e^{\sqrt{t}|D|}v, e^{\sqrt{t}|D|}w\right)\|_X \le C_1\|(v_0,w_0)\|_{\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}},$$

This completes the proof of Theorem 1.3.

4. Time decay of mild solutions

As an application of the analyticity of solutions discussed above, we shall prove Theorem 1.4

4.1. Proof of Theorem 1.4

By using the definition of the Fourier-Besov-Morrey spaces, we have

$$\begin{aligned}\left\|(-\Delta)^{\frac{1}{2}}v(t)\right\|_{\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} &= \left\|(-\Delta)^{\frac{1}{2}}e^{-\sqrt{t}|D|}e^{\sqrt{t}|D|}v(t)\right\|_{\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} \\ &= \left(\sum_{j\in\mathbb{Z}} 2^{j(-2+\frac{n}{p'}+\frac{\lambda}{p})q}\left\|\varphi_j\mathcal{F}\left((-\Delta)^{\frac{1}{2}}e^{-\sqrt{t}|D|}e^{\sqrt{t}|D|}v(t)\right)\right\|^q_{\mathcal{M}^\lambda_p}\right)^{\frac{1}{q}} \\ &= \left(\sum_{j\in\mathbb{Z}} 2^{j(-2+\frac{n}{p'}+\frac{\lambda}{p})q}\left\||\xi|e^{-\sqrt{t}|\xi|}\varphi_j\mathcal{F}\left(e^{\sqrt{t}|D|}v(t)\right)\right\|^q_{\mathcal{M}^\lambda_p}\right)^{\frac{1}{q}}\end{aligned}$$

Suppose the function $g(x) = xe^{-\sqrt{t}x}$, where $x \ge 0$. From the derivation of the function g, one can infer that $g(x) \le g((\frac{x}{\sqrt{t}})) \lesssim t^{-\frac{1}{2}}$. Therbey,

$$\begin{aligned}\left\|(-\Delta)^{\frac{1}{2}}v(t)\right\|_{\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} &\lesssim t^{-\frac{1}{2}}\left\|e^{\sqrt{t}|D|}v(t)\right\|_{\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} \\ &\lesssim t^{-\frac{1}{2}}\|v_0\|_{\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}.\end{aligned}$$

Similarly,

$$\begin{aligned}\left\|(-\Delta)^{\frac{1}{2}}w(t)\right\|_{\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} &\lesssim t^{-\frac{1}{2}}\left\|e^{\sqrt{t}|D|}w(t)\right\|_{\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} \\ &\lesssim t^{-\frac{1}{2}}\|w_0\|_{\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}.\end{aligned}$$

Thus,

$$\begin{aligned}\left\|\left((-\Delta)^{\frac{1}{2}}v(t),(-\Delta)^{\frac{1}{2}}w(t)\right)\right\|_{\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} &\lesssim t^{-\frac{1}{2}}\left\|\left(e^{\sqrt{t}|D|}v(t),e^{\sqrt{t}|D|}w(t)\right)\right\|_{\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} \\ &\lesssim t^{-\frac{1}{2}}\|(v_0,w_0)\|_{\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}.\end{aligned}$$

5. Blow-up criteria

By following the techniques described in [10], we will show the blow-up criteria of solutions if the maximal time of existence is finite.

5.1. Proof of Theorem 1.5

Let $T^* < \infty$ be the maximal existence time of solutions of the system (1.1) in

$$\mathcal{C}\Big([0,T^*), \mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}\Big)\mathbb{R}^n\Big)\Big) \cap \mathcal{L}^\infty\Big([0,T^*); \mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}\Big) \cap \mathcal{L}^1\Big([0,T^*), \mathcal{FN}_{p,\lambda,q}^{\frac{n}{p'}+\frac{\lambda}{p}}\Big).$$

By contradiction assume that $T^* < \infty$ and

$$\int_0^{T^*} \|(v,w)\|_{\mathcal{FN}_{p,\lambda,q}^{\frac{n}{p'}+\frac{\lambda}{p}}} < \infty. \tag{5.1}$$

Let $T_0 \in (0,T^*)$ so that

$$\|(v,w)\|_{\mathfrak{L}^1\Big([T_0,T^*);\mathcal{FN}_{p,\lambda,q}^{\frac{n}{p'}+\frac{\lambda}{p}}\Big)} < \frac{1}{4}.$$

For $t \in [T_0,T^*)$ and $s \in [T_0,t]$, we consider the following integral system:

$$\begin{cases} v(s) = e^{s\Delta}v_0 - \displaystyle\int_{T_0}^s e^{(s-\tau)\Delta}\nabla\cdot\big(v\nabla(-\Delta)^{-1}(w-v)\big)(\tau)d\tau \\ w(s) = e^{s\Delta}w_0 + \displaystyle\int_{T_0}^s e^{(s-\tau)\Delta}\nabla\cdot\big(w\nabla(-\Delta)^{-1}(w-v)\big)(\tau)d\tau. \end{cases} \tag{5.2}$$

The same calculus used to prove Proposition 3.5 yields

$$\begin{aligned}
&\|v(s)\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}} \\
&\lesssim \|v(T_0)\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}} + \|v\nabla\phi\|_{\mathcal{L}^1\Big([T_0,s),\mathcal{FN}_{p,\lambda,q}^{\frac{n}{p'}+\frac{\lambda}{p}}\Big)} \\
&\lesssim \|v(T_0)\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}} + \sup_{T_0\le s\le t}\|(v(s),w(s))\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}}\|(v,w)\|_{\mathcal{L}^1\Big([T_0,s),\mathcal{FN}_{p,\lambda,q}^{-1+\frac{n}{p'}+\frac{\lambda}{p}}\Big)} \\
&\lesssim \|v(T_0)\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}} + \frac{1}{4}\sup_{T_0\le s\le t}\|(v(s),w(s))\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}},
\end{aligned}$$

and

$$\begin{aligned}
&\|w(s)\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}} \\
&\lesssim \|w(T_0)\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}} + \|w\nabla\phi\|_{\mathcal{L}^1\Big([T_0,s),\mathcal{FN}_{p,\lambda,q}^{\frac{n}{p'}+\frac{\lambda}{p}}\Big)} \\
&\lesssim \|w(T_0)\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}} + \sup_{T_0\le s\le t}\|(v(s),w(s))\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}}\|(v,w)\|_{\mathcal{L}^1\Big([T_0,s),\mathcal{FN}_{p,\lambda,q}^{-1+\frac{n}{p'}+\frac{\lambda}{p}}\Big)} \\
&\lesssim \|w(T_0)\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}} + \frac{1}{4}\sup_{T_0\le s\le t}\|(v(s),w(s))\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\|(v(s),w(s))\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}} \\
&\lesssim \|(v(T_0),w(T_0))\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}} + \frac{1}{2}\sup_{T_0\le s\le t}\|(v(s),w(s))\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}}.
\end{aligned}$$

Consequently,

$$\sup_{T_0\le s\le t}\|(v(s),w(s))\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}} \lesssim 2\|(v(T_0),w(T_0))\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}}, \forall t \in [T_0,T^*).$$

Put

$$N = \max\left(2\left\|(v(T_0), w(T_0))\right\|_{\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} ; \max_{t\in[0,T_0]} \left\|(v(t), w(t))\right\|_{\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}\right).$$

Hence, we deduce

$$\|(v(t), w(t))\|_{\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} \lesssim N, \forall t \in [T_0, T^*). \tag{5.3}$$

Let $(\kappa_n)_{n\in\mathbb{N}}$ a sequence such that $\kappa_n \nearrow T^*$, where $\kappa_n \in (0, T^*)$, for all $n \in \mathbb{N}$. We want to prove that

$$\lim_{n,m\to\infty} \|(v, w)(\kappa_m) - (v, w)(\kappa_n)\|_{\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} = 0. \tag{5.4}$$

In order to achieve this aim, we use the integral form of (v, w) to obtain

$$\begin{aligned}
&(v, w)(\kappa_m) - (v, w)(\kappa_n)\\
&= \left([e^{\kappa_m\Delta} - e^{\kappa_n\Delta}]v_0, [e^{\kappa_m\Delta} - e^{\kappa_n\Delta}]w_0\right)\\
&- \left(\int_{\kappa_n}^{\kappa_m} e^{(\kappa_m-z)\Delta}\nabla\cdot(v\nabla\phi)\ dz, -\int_{\kappa_n}^{\kappa_m} e^{(\kappa_m-z)\Delta}\nabla\cdot(w\nabla\phi)\ dz\right)\\
&- \left(\int_0^{\kappa_n} e^{(\kappa_n-z)\Delta}(e^{(\kappa_m-\kappa_n)\Delta} - 1)\nabla\cdot(v\nabla\phi)\ dz, -\int_0^{\kappa_n} e^{(\kappa_n-z)\Delta}(e^{(\kappa_m-\kappa_n)\Delta} - 1)\nabla\cdot(w\nabla\phi)\ dz\right)\\
&:= \mathcal{A}_1(m, n) + \mathcal{A}_2(m, n) + \mathcal{A}_3(m, n).
\end{aligned}$$

First, we have

$$\begin{aligned}
\|[e^{\kappa_m\Delta} - e^{\kappa_n\Delta}]v_0\|_{\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} &= \left\{\sum_{j\in\mathbb{Z}} 2^{j(-2+\frac{n}{p'}+\frac{\lambda}{p})q}\|\varphi_j\left(e^{\kappa_m|\xi|^2} - e^{\kappa_n|\xi|^2}\right)\hat{v}_0\|^q_{\mathcal{M}^\lambda_p}\right\}^{\frac{1}{q}}\\
&\lesssim \left\{\sum_{j\in\mathbb{Z}} 2^{j(-2+\frac{n}{p'}+\frac{\lambda}{p})q}\|\varphi_j\left(e^{\kappa_m|\xi|^2} - e^{T^*|\xi|^2}\right)\hat{v}_0\|^q_{\mathcal{M}^\lambda_p}\right\}^{\frac{1}{q}},
\end{aligned}$$

provided that $\kappa_n < T^*$, for all $n \in \mathbb{N}$. As a result, by using the fact that $v_0 \in \mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}$, it derives from dominated convergence theorem that

$$\lim_{n,m\to\infty} \left\|[e^{\kappa_m\Delta} - e^{\kappa_n\Delta}]v_0\right\|_{\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} = 0.$$

By following a similar argument, one has

$$\lim_{n,m\to\infty} \left\|[e^{\kappa_m\Delta} - e^{\kappa_n\Delta}]w_0\right\|_{\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} = 0.$$

Consequently

$$\lim_{n,m\to\infty} \|\mathcal{A}_1(m, n)\|_{\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} = 0.$$

Furthermore, using Lemma 3.4 and the estimate (3.5), one reaches

$$\begin{aligned}
&\int_{\kappa_n}^{\kappa_m} \left\|e^{(\kappa_m-z)\Delta}\nabla\cdot(v\nabla\phi)\right\|_{\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} dz\\
&\lesssim \int_{\kappa_n}^{T^*} \|v\nabla\phi\|_{\mathcal{FN}^{-1+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} dz\\
&\lesssim \int_{\kappa_n}^{T^*} \|(v, w)\|_{\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} \|(v, w)\|_{\mathcal{FN}^{\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} dz\\
&\lesssim \left(\int_{\kappa_n}^{T^*} \|(v, w)\|^2_{\mathcal{FN}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} dz\right)^{\frac{1}{2}} \left(\int_{\kappa_n}^{T^*} \|(v, w)\|^2_{\mathcal{FN}^{\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} dz\right)^{\frac{1}{2}}.
\end{aligned}$$

By using (5.3), Hölder's inequality, and the estimate (5.1), one obtains

$$\int_{\kappa_n}^{\kappa_m} \left\| e^{(\kappa_m - z)\Delta} \nabla \cdot (v\nabla\phi) \right\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}} dz \lesssim \left(\int_{\kappa_n}^{T^*} \|(v,w)\|^2_{\mathcal{FN}_{p,\lambda,q}^{\frac{n}{p'}+\frac{\lambda}{p}}} dz \right)^{\frac{1}{2}} \leq C(T^* - \kappa_n)^{\frac{1}{2}}.$$

As a consequence,

$$\lim_{n,m\to\infty} \int_{\kappa_n}^{\kappa_m} \left\| e^{(\kappa_m - z)\Delta} \nabla \cdot (v\nabla\phi) \right\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}} dz = 0.$$

Analogously, one can infer

$$\lim_{n,m\to\infty} \int_{\kappa_n}^{\kappa_m} \left\| e^{(\kappa_m - z)\Delta} \nabla \cdot (w\nabla\phi) \right\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}} dz = 0.$$

Thereby, we have

$$\lim_{n,m\to\infty} \|\mathcal{A}_2(m,n)\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}} dz = 0.$$

Lastly,

$$\begin{aligned}
&\| \int_0^{\kappa_n} e^{(\kappa_n - z)\Delta} (e^{(\kappa_m-\kappa_n)\Delta} - 1) \nabla \cdot (v\nabla\phi) \ dz \|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}} \\
&\lesssim \left\{ \sum_{j\in\mathbb{Z}} 2^{j(-2+\frac{n}{p'}+\frac{\lambda}{p})q} \int_0^{\kappa_n} \|\varphi_j e^{-(\kappa_m - z)|\xi|^2} (1 - e^{-(\kappa_m-\kappa_n)|\xi|^2}) \widehat{\nabla \cdot (v\nabla\phi)} \|^q_{\mathcal{M}_p^\lambda} \right\}^{\frac{1}{q}} \\
&\lesssim \left\{ \sum_{j\in\mathbb{Z}} 2^{j(-1+\frac{n}{p'}+\frac{\lambda}{p})q} \int_0^{T^*} \|\varphi_j (1 - e^{-(T^*-\kappa_n)|\xi|^2}) \widehat{v\nabla\phi} \|^q_{\mathcal{M}_p^\lambda} \right\}^{\frac{1}{q}},
\end{aligned}$$

where $\kappa_n < T^*$, for all $n \in \mathbb{N}$. By using the assumption (5.1) and the estimate (5.3) , we get

$$\int_0^{T^*} \|v\nabla\phi\|_{\mathcal{FN}_{p,\lambda,q}^{-1+\frac{n}{p'}+\frac{\lambda}{p}}} dz < \infty.$$

Thus, by dominated convergence theorem, we deduce

$$\lim_{n,m\to\infty} \left\| \int_0^{\kappa_n} e^{(\kappa_n - z)\Delta} (e^{(\kappa_m-\kappa_n)\Delta} - 1) \nabla \cdot (v\nabla\phi) \ dz \right\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}} = 0.$$

Moreover, by applying an analogous process, we conclude that

$$\lim_{n,m\to\infty} \left\| \int_0^{\kappa_n} e^{(\kappa_n - z)\Delta} (e^{(\kappa_m-\kappa_n)\Delta} - 1) \nabla \cdot (w\nabla\phi) \ dz \right\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}} = 0.$$

Therefore,

$$\lim_{n,m\to\infty} \|\mathcal{A}_3(m,n)\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}} = 0.$$

Consequently,

$$\lim_{n,m\to\infty} \|(v,w)(\kappa_m) - (v,w)(\kappa_n)\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}} = 0.$$

This implies that $((v,w)(\kappa_n))_{n\in\mathbb{N}}$ satisfies the Cauchy criterion at T^* in the Banach space $\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}$. Then, there exists an element v^*, w^* in $\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}$ such that

$$\lim_{n\to\infty} \|(v,w)(\kappa_n) - (v^*,w^*)\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}} = 0.$$

We point out that the above limits are independent of $(\kappa_n)_{n\in\mathbb{N}}$. In other words,

$$\lim_{t\nearrow T^*}\|(v,w)(t)-(v^*,w^*)\|_{\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}}=0.$$

Now, consider the system (1.1) with the initial data (v^*,w^*), instead of (v_0,w_0)

$$\begin{cases}\partial_t v=\Delta v-\nabla\cdot(v\nabla\phi),\\ \partial_t w=\Delta w+\nabla\cdot(w\nabla\phi),\\ \Delta\phi=v-w,\\ v(x,0)=v^*,\quad w(x,0)=w^*.\end{cases}$$

We assure, by Theorem 1.2, the existence and uniqueness of $(\theta,\Omega)\in\mathcal{C}\Big([0,t_0),\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}(\mathbb{R}^n)\Big)$ $(t_0>0)$ for the Debye-Hückel system (1.1). Therefore

$$(\tilde{v},\tilde{w})(t)=\begin{cases}(v,w)(t), & \text{if } t\in[0,T^*)\\ (\theta,\Omega)(t-T^*) & \text{if } t\in[T^*,T^*+t_0]\end{cases}$$

is a solotion of (5.5) with initial data (v_0,w_0) on the interval $[0,T^*+t_0]$ which contradicts the maximality of T^*.

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Achraf Azanzal,
Laboratory LMACS,
FST, Sultan Moulay Slimane University,
Beni Mellal, Morocco.
E-mail address: achraf0665@gmail.com

and

Chakir Allalou,
Laboratory LMACS,
FST, Sultan Moulay Slimane University,
Beni Mellal, Morocco.
E-mail address: chakir.allalou@yahoo.fr

and

Said Melliani,
Laboratory LMACS,
FST, Sultan Moulay Slimane University,
Beni Mellal, Morocco.
E-mail address: saidmelliani@gmail.com