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Weak Solutions for Double Phase Problem Driven by the $(p(x), q(x))$-Laplacian Operator Under Dirichlet Boundary Conditions

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ABSTRACT: In the present paper, in view of the topological degree methods and the theory of the variable exponent Sobolev spaces, we discuss a Dirichlet boundary value problem for elliptic equations involving the $(p(x), q(x))$-Laplacian operator with a reaction term depending on the gradient and on two real parameters. Under certain assumptions, we establish the existence of at least one weak solution to this problem. Our results extends some recent work in the literature.

Key Words: Double phase problem, $(p(x), q(x))$-Laplacian operators, Topological degree methods, Variable exponent Sobolev spaces.

Contents

1. Introduction and motivation

The study of differential equations with $p(x)$-Laplacian operator or $(p(x), q(x))$-Laplacian operator is an attractive topic and has been the object of considerable attention in recent years. Perhaps the impulse for this comes from the new search field that reflects a new type of physical phenomenon is a class of nonlinear problems with variable exponents. In the subject of fluid mechanics, for example, Rajagopal and M. Ruzicka recently developed a very interesting model for these fluids in [24] (see also [25]). Other applications relate to image processing [1,8], elasticity problems [20,21,22,23,28], the flow in porous media [4], and problems in the calculus of variations involving variational integrals with nonstandard growth [15,2].

Here and in the sequel, we will assume that Ω is a bounded domain in $\mathbb{R}^N (N > 1)$, with a Lipschitz boundary denoted by $\partial\Omega$, $\alpha(\cdot), \zeta(\cdot) \in C_+(\overline{\Omega})$, and let $\delta, R \in L^\infty(\Omega)$, μ and λ are two real parameters.

In this paper, we consider the following Dirichlet boundary value problem:

$$\begin{cases} -\Delta_{p,q}(u) + \delta(x)|u|^{\zeta(x)-2}u = R(x)|u|^{\alpha(x)-2}u + \mu\, g(x,u) + \lambda\, f(x,u,\nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where

$$\Delta_{p,q}(u) := \operatorname{div}\Big(|\nabla u|^{p(x)-2}\nabla u + a(x)|\nabla u|^{q(x)-2}\nabla u\Big). \tag{1.2}$$

In this problem, the coefficient $a : \overline{\Omega} \to \mathbb{R}^+$ is Lipschitz continuous function, $g : \Omega \times \mathbb{R} \to \mathbb{R}$ and $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are Carathéodory functions that satisfy the assumption of growth, and the variables exponents $p, q \in C_+(\overline{\Omega})$ are assumed to satisfy the following assumption:

$$1 < q^- \leq q \leq q^+ < p^- \leq p \leq p^+ < +\infty. \tag{1.3}$$

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The double phase operator has been used in the modelling of strongly anisotropic materials [28,30,31] and in Lavrentiev's phenomenon [32]. In the one hand, we have the physical motivation; since the double phase operator has been used to model the steady-state solutions of reaction diffusion problems, that arise in biophysic, plasma-physic and in the study of chemical reactions. In the other hand, these operators provide a useful paradigm for describing the behaviour of strongly anisotropic materials, whose hardening properties are linked to the exponent governing the growth of the gradient change radically with the point, where the coefficient $a(\cdot)$ determines the geometry of a composite made of two different materials (see [5,6,9,29] and the references given there).

Let us recall some known results on Problem (1.1). For example, Fan and Zhang [11], based on the theory of the spaces $L^{p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$, present several sufficient conditions for the existence of solutions for the problem (1.1) with $\zeta(x) = p(x)$, $\mu = 1$ and $a = \lambda = \delta = 0$.

R. Alsaedi [3] establishes sufficient conditions for the existence of nontrivial weak solutions for the problem (1.1) when $a = \lambda = 0$, $g(x, u) = |u|^{p(x)-2}u$.

Problems related to (1.1) in case $p(x) \equiv p$ and $q(x) \equiv q$ have been studied by many scholars, for example, Liu et al. [14] study the problem (1.1) when $\delta = R = \lambda = 0$ and $\mu = 1$, and Wang et al. [26] showed, by using the topological degree theory for a class of demicontinuous operators, the existence of at least one weak solution of (1.1) with $\delta = R = \mu = 0$ and $\lambda = 1$.

We would like to draw attention to the fact that the $p(x)$-laplacian operator has more complicated nonlinearity than the p-laplacian operator. For example, they are non-homogeneous, which prove that our problem is more difficult than the operators p-Laplacian type.

Motivated by the aforementioned works, in the present paper, we will generalize these works. By using a topological degree for a class of demicontinuous operators of generalized (S_+) type of [7] and the theory of the generalized Sobolev spaces, we establish the existence of weak solution u in $W_0^{1,p(x)}(\Omega)$ for the problem (1.1).

The remainder of the paper is organized as follows. In Section 2, we review some fundamental preliminaries about the functional framework where we will treat our problems. In Section 3, we introduce some classes of operators, as well as the topological degree methods for a class of demicontinuous operators of generalized (S_+). Finally, Section 4 is devoted to discussing the existence of weak solution to (1.1).

2. Preliminaries

In this section, we recall the most important and relevant properties and notations about generalized Sobolev spaces $W^{1,p(x)}(\Omega)$, that we will need in our analysis of the problem (1.1), by that, referring to [10,16,17,18,19] for more details.

Let Ω be a smooth bounded domain in $\mathbb{R}^N (N > 1)$, with a Lipschitz boundary denoted by $\partial\Omega$. Set

$$C_+(\overline{\Omega}) = \Big\{p : p \in C(\overline{\Omega}) \text{ such that } p(x) > 1 \text{ for any } x \in \overline{\Omega}\Big\}.$$

For each $p \in C_+(\overline{\Omega})$, we define

$$p^+ := \max\Big\{p(x),\ x \in \overline{\Omega}\Big\} \text{ and } p^- := \min\Big\{p(x),\ x \in \overline{\Omega}\Big\}.$$

For every $p \in C_+(\overline{\Omega})$, we define

$$L^{p(x)}(\Omega) = \Big\{u : \Omega \to \mathbb{R} \text{ is measurable such that } \int_\Omega |u(x)|^{p(x)} dx < +\infty\Big\},$$

equipped with the Luxemburg norm

$$|u|_{p(x)} = \inf\Big\{\lambda > 0 : \rho_{p(x)}\Big(\frac{u}{\lambda}\Big) \leq 1\Big\},$$

where

$$\rho_{p(x)}(u) = \int_\Omega |u(x)|^{p(x)} dx,\ \forall\ u \in L^{p(x)}(\Omega).$$

Proposition 2.1. *[10, Theorem 1.3 and Theorem 1.4] Let (u_n) be a sequence and $u \in L^{p(x)}(\Omega)$, then*

$$|u|_{p(x)} < 1(resp. = 1; > 1) \;\Leftrightarrow\; \rho_{p(x)}(u) < 1(resp. = 1; > 1), \tag{2.1}$$

$$|u|_{p(x)} > 1 \;\Rightarrow\; |u|^{p^-}_{p(x)} \leq \rho_{p(x)}(u) \leq |u|^{p^+}_{p(x)}, \tag{2.2}$$

$$|u|_{p(x)} < 1 \;\Rightarrow\; |u|^{p^+}_{p(x)} \leq \rho_{p(x)}(u) \leq |u|^{p^-}_{p(x)}, \tag{2.3}$$

$$\lim_{n\to\infty} |u_n - u|_{p(x)} = 0 \;\Leftrightarrow\; \lim_{n\to\infty} \rho_{p(x)}(u_n - u) = 0. \tag{2.4}$$

Remark 2.2. *According to (2.2) and (2.3), we have*

$$|u|_{p(x)} \leq \rho_{p(x)}(u) + 1, \tag{2.5}$$

$$\rho_{p(x)}(u) \leq |u|^{p^-}_{p(x)} + |u|^{p^+}_{p(x)}. \tag{2.6}$$

Proposition 2.3. *[13, Theorem 2.5 and Corollary 2.7] The space $\Big(L^{p(x)}(\Omega), |\cdot|_{p(x)}\Big)$ is a separable and reflexive Banach spaces.*

Proposition 2.4. *[13, Theorem 2.1] The conjugate space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ for all $x \in \Omega$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have the following Hölder-type inequality*

$$\Big|\int_\Omega uv\; dx\Big| \leq \Big(\frac{1}{p-} + \frac{1}{p'-}\Big)|u|_{p(x)}|v|_{p'(x)} \leq 2|u|_{p(x)}|v|_{p'(x)}. \tag{2.7}$$

Remark 2.5. *[10, Theorem 1.11] If $p_1,\ p_2 \in C_+(\overline{\Omega})$ with $p_1(x) \leq p_2(x)$ for any $x \in \overline{\Omega}$, then there exists the continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$.*

Now, let $p \in C_+(\overline{\Omega})$ and we define $W^{1,p(x)}(\Omega)$ as

$$W^{1,p(x)}(\Omega) = \Big\{u \in L^{p(x)}(\Omega) \text{ such that } |\nabla u| \in L^{p(x)}(\Omega)\Big\},$$

equipped with the norm

$$||u|| = |u|_{p(x)} + |\nabla u|_{p(x)}.$$

We also define $W^{1,p(x)}_0(\Omega)$ as the subspace of $W^{1,p(x)}(\Omega)$, which is the closure of $C^\infty_0(\Omega)$ with respect to the norm $||\cdot||$.

Proposition 2.6. *[13, Theorem 4.3] If the exponent $p(x)$ satisfies the log-Hölder continuity condition, i.e. there is a constant $b > 0$ such that for every $x,\ y \in \Omega,\ x \neq y$ with $|x - y| \leq \dfrac{1}{2}$ one has*

$$|p(x) - p(y)| \leq \frac{b}{-\log|x-y|}, \tag{2.8}$$

then we have the Poincaré inequality, i.e. there exists a constant $C > 0$ depending only on Ω and the function p such that

$$|u|_{p(x)} \leq C|\nabla u|_{p(x)},\ \forall\ u \in W^{1,p(x)}_0(\Omega). \tag{2.9}$$

In this paper we will use the following equivalent norm on $W^{1,p(x)}_0(\Omega)$

$$|u|_{1,p(x)} = |\nabla u|_{p(x)},$$

which is equivalent to $||\cdot||$.
Furthermore, we have the compact embedding $W^{1,p(x)}_0(\Omega) \hookrightarrow L^{p(x)}(\Omega)$(see [13]).

Proposition 2.7. *[13, Theorem 3.1] The spaces* $\Big(W^{1,p(x)}(\Omega), |\cdot|_{1,p(x)}\Big)$ *and* $\Big(W_0^{1,p(x)}(\Omega), |\cdot|_{1,p(x)}\Big)$ *are separable and reflexive Banach spaces.*

Remark 2.8. *The dual space of* $W_0^{1,p(x)}(\Omega)$ *denoted* $W^{-1,p'(x)}(\Omega)$*, is equipped with the norm*

$$|u|_{-1,p'(x)} = \inf\Big\{|u_0|_{p'(x)} + \sum_{i=1}^{N} |u_i|_{p'(x)}\Big\},$$

where the infinimum is taken on all possible decompositions $u = u_0 - \mathrm{div}F$ *with* $u_0 \in L^{p'(x)}(\Omega)$ *and* $F = (u_1, \dots, u_N) \in (L^{p'(x)}(\Omega))^N$.

3. A review on some classes of mappings and topological degree theory

We start by defining some classes of mappings. In what follows, let X be a real separable reflexive Banach space and X^* be its dual space with dual pairing $\langle \cdot, \cdot\rangle$ and given a nonempty subset Ω of X. Strong (weak) convergence is represented by the symbol $\to$ ($\rightharpoonup$).

Definition 3.1. *Let* Y *be another real Banach space. A operator* $F : \Omega \subset X \to Y$ *is said to be :*

1. *bounded, if it takes any bounded set into a bounded set.*
2. *demicontinuous, if for any sequence* $(u_n) \subset \Omega$, $u_n \to u$ *implies* $F(u_n) \rightharpoonup F(u)$.
3. *compact, if it is continuous and the image of any bounded set is relatively compact.*

Definition 3.2. *A mapping* $F : \Omega \subset X \to X^*$ *is said to be :*

1. *of class* (S_+)*, if for any sequence* $(u_n) \subset \Omega$ *with* $u_n \rightharpoonup u$ *and* $\limsup_{n\to\infty}\langle Fu_n, u_n - u\rangle \le 0$*, we have* $u_n \to u$.
2. *quasimonotone, if for any sequence* $(u_n) \subset \Omega$ *with* $u_n \rightharpoonup u$*, we have* $\limsup_{n\to\infty}\langle Fu_n,\ u_n - u\rangle \ge 0$.

Definition 3.3. *Let* $T : \Omega_1 \subset X \to X^*$ *be a bounded operator such that* $\Omega \subset \Omega_1$*. For any operator* $F : \Omega \subset X \to X$*, we say that*

1. F *of class* $(S_+)_T$*, if for any sequence* $(u_n) \subset \Omega$ *with* $u_n \rightharpoonup u$, $y_n := Tu_n \rightharpoonup y$ *and* $\limsup_{n\to\infty}\langle Fu_n,\ y_n - y\rangle \le 0$*, we have* $u_n \to u$.
2. F *has the property* $(QM)_T$*, if for any sequence* $(u_n) \subset \Omega$ *with* $u_n \rightharpoonup u$, $y_n := Tu_n \rightharpoonup y$*, we have* $\limsup_{n\to\infty}\langle Fu_n, y - y_n\rangle \ge 0$.

In the sequel, we consider the following classes of operators:

$$\mathcal{F}_1(\Omega) := \Big\{F : \Omega \to X^* : F \text{ is bounded, demicontinuous and of class } (S_+)\Big\},$$
$$\mathcal{F}_T(\Omega) := \Big\{F : \Omega \to X : F \text{ is demicontinuous and of class } (S_+)_T\Big\},$$
$$\mathcal{F}_{T,B}(\Omega) := \Big\{F \in \mathcal{F}_T(\Omega) : F \text{ is bounded }\Big\},$$

for any $\Omega \subset D(F)$, where $D(F)$ denotes the domain of F, and any $T \in \mathcal{F}_1(\Omega)$.
Now, let $\mathcal{O}$ be the collection of all bounded open sets in X and we define

$$\mathcal{F}(X) := \Big\{F \in \mathcal{F}_T(\overline{E}) : E \in \mathcal{O},\ \mathrm{T} \in \mathcal{F}_1(\overline{\mathrm{E}})\Big\},$$

where, $\mathrm{T} \in \mathcal{F}_1(\overline{\mathrm{E}})$ is called an essential inner map to F.

Lemma 3.4. *[12, Lemma 2.3] Let $T \in \mathcal{F}_1(\overline{E})$ be continuous and $S : D(S) \subset X^* \to X$ be demicontinuous such that $T(\overline{E}) \subset D(S)$, where E is a bounded open set in a real reflexive Banach space X. Then the following statements are true :*

1. *If S is quasimonotone, then $I + SoT \in \mathcal{F}_T(\overline{E})$, where I denotes the identity operator.*
2. *If S is of class (S_+), then $SoT \in \mathcal{F}_T(\overline{E})$.*

Definition 3.5. *Suppose that E is bounded open subset of a real reflexive Banach space X, $T \in \mathcal{F}_1(\overline{E})$ is continuous and $F, S \in \mathcal{F}_T(\overline{E})$. Then the affine homotopy $\mathcal{H} : [0,1] \times \overline{E} \to X$ defined by*

$$\mathcal{H}(t,u) := (1-t)Fu + tSu, \quad \textit{for} \quad (t,u) \in [0,1] \times \overline{E}$$

is called an admissible affine homotopy with the common continuous essential inner map T.

Remark 3.6. *[12, Lemma 2.5]* The above affine homotopy is of class $(S_+)_T$.

As in [12] we give the topological degree for the class $\mathcal{F}(X)$.

Theorem 3.7. *Let*

$$M = \Big\{(F, E, h) : E \in \mathcal{O},\ T \in \mathcal{F}_1(\overline{E}),\ F \in \mathcal{F}_{T,B}(\overline{E}),\ h \notin F(\partial E)\Big\}.$$

Then, there exists a unique degree function $d : M \longrightarrow \mathbb{Z}$ that satisfies the following properties:

1. *(*Normalization*) For any $h \in E$, we have*

$$d(I, E, h) = 1.$$

2. *(*Additivity*) Let $F \in \mathcal{F}_{T,B}(\overline{E})$. If E_1 and E_2 are two disjoint open subsets of E such that $h \notin F(\overline{E} \backslash (E_1 \cup E_2))$, then we have*

$$d(F, E, h) = d(F, E_1, h) + d(F, E_2, h).$$

3. *(*Homotopy invariance*) If $\mathcal{H} : [0,1] \times \overline{E} \to X$ is a bounded admissible affine homotopy with a common continuous essential inner map and h: $[0,1] \to X$ is a continuous path in X such that $h(t) \notin \mathcal{H}(t, \partial E)$ for all $t \in [0,1]$, then*

$$d(\mathcal{H}(t, \cdot), E, h(t)) = const \ for \ all \ t \in [0,1].$$

4. *(*Existence*) If $d(F, E, h) \neq 0$, then the equation $Fu = h$ has a solution in E.*
5. *(* Boundary dependence*) If $F,\ S \in \mathcal{F}_{\mathrm{T}}(\overline{\mathrm{E}})$ coincide on ∂E and $h \notin F(\partial E)$, then*

$$d(F, E, h) = d(S, E, h)$$

Definition 3.8. *[12, Definition 3.3] The above degree is defined as follows:*

$$d(F, E, h) \ := d_B(F|_{\overline{E}_0}, E_0, h),$$

where d_B is the Berkovits degree [7] and E_0 is any open subset of E with $F^{-1}(h) \subset E_0$ and F is bounded on $\overline{E}_0$.

4. Existence result

In this section, we will discuss the existence of weak solution of (1.1). For this, we list our assumptions associated with our problem to show the existence result.

From new on, we always assume that $\Omega \subset \mathbb{R}^N (N > 1)$ is a bounded domain with a Lipschitz boundary $\partial\Omega$, $\delta, R \in L^\infty(\Omega)$, $p \in C_+(\overline{\Omega})$ satisfy the log-Hölder continuity condition (2.8), $\zeta, \alpha \in C_+(\overline{\Omega})$ with $1 < \zeta^- \leq \zeta(x) \leq \zeta^+ < p^-$ and $1 < \alpha^- \leq \alpha(x) \leq \alpha^+ < p^-$, $g : \Omega \times \mathbb{R} \to \mathbb{R}$ and $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are functions such that:

(A_1) f is a Carathéodory function.

(A_2) There exists $\varrho > 0$ and $\gamma \in L^{p'(x)}(\Omega)$ such that

$$|f(x, \zeta, \xi)| \leq \varrho(\gamma(x) + |\zeta|^{k(x)-1} + |\xi|^{k(x)-1}).$$

(A_3) g is a Carathéodory function.

(A_4) There are $\sigma > 0$ and $\nu \in L^{p'(x)}(\Omega)$ such that

$$|g(x, \zeta)| \leq \sigma(\nu(x) + |\zeta|^{s(x)-1}),$$

for a.e. $x \in \Omega$ and all $(\zeta, \xi) \in \mathbb{R} \times \mathbb{R}^N$, where $k, s \in C_+(\overline{\Omega})$ with $1 < k^- \leq k(x) \leq k^+ < p^-$ and $1 < s^- \leq s(x) \leq s^+ < p^-$.

Remark 4.1.

- *Note that for all $\vartheta \in W_0^{1,p(x)}(\Omega)$*

$$\int_\Omega \Big(|\nabla u|^{p(x)-2} \nabla u \nabla \vartheta + a(x) |\nabla u|^{q(x)-2} \nabla u \nabla \vartheta \Big) dx$$

is well defined (see [11,14]).

- *$\delta(x)|u|^{\zeta(x)-2}u$, $R(x)|u|^{\alpha(x)-2}u$, $\mu\, g(x, u)$ and $\lambda\, f(x, u, \nabla u)$ are belongs to $L^{p'(x)}(\Omega)$ under $u \in W_0^{1,p(x)}(\Omega)$, the assumptions (A_2) and (A_4) and the given hypotheses about the exponents p, α, q and s because: $\gamma \in L^{p'(x)}(\Omega)$, $\nu \in L^{p'(x)}(\Omega)$, $r(x) = (q(x) - 1)p'(x) \in C_+(\overline{\Omega})$ with $r(x) < p(x)$, $\beta(x) = (\zeta(x) - 1)p'(x) \in C_+(\overline{\Omega})$ with $\beta(x) < p(x)$, $k(x) = (\alpha(x) - 1)p'(x) \in C_+(\overline{\Omega})$ with $k(x) < p(x)$ and $\kappa(x) = (s(x) - 1)p'(x) \in C_+(\overline{\Omega})$ with $\kappa(x) < p(x)$.*
  Then, by Remark 2.5 we can conclude that

$$L^{p(x)} \hookrightarrow L^{r(x)}, \ L^{p(x)} \hookrightarrow L^{\beta(x)}, \ L^{p(x)} \hookrightarrow L^{k(x)} \ \text{ and } \ L^{p(x)} \hookrightarrow L^{\kappa(x)}.$$

Hence, since $\vartheta \in L^{p(x)}(\Omega)$, we have

$$\Big(-\delta(x)|u|^{\zeta(x)-2}u + R(x)|u|^{\alpha(x)-2}u + \mu\, g(x, u) + \lambda\, f(x, u, \nabla u) \Big) \vartheta \in L^1(\Omega).$$

This implies that, the integral

$$\int_\Omega \Big(-\delta(x)|u|^{\zeta(x)-2}u + R(x)|u|^{\alpha(x)-2}u + \mu\, g(x, u) + \lambda\, f(x, u, \nabla u) \Big) \vartheta dx$$

exists.

Then, we shall use the definition of weak solution for (1.1) in the following sense:

Definition 4.2. *We say that a function $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution of (1.1), if for any $\vartheta \in W_0^{1,p(x)}(\Omega)$, it satisfies the following:*

$$\int_\Omega \Big(|\nabla u|^{p(x)-2} \nabla u \nabla \vartheta + a(x) |\nabla u|^{q(x)-2} \nabla u \nabla \vartheta \Big) dx = \int_\Omega \Big(-\delta(x)|u|^{\zeta(x)-2}u + R(x)|u|^{\alpha(x)-2}u + \mu\, g(x, u) + \lambda\, f(x, u, \nabla u) \Big) \vartheta dx.$$

Before giving the existence result for the problem (1.1), we first give two lemmas that will be used in the proof of this result.
Let us consider the following functional:

$$\mathcal{J}(u) := \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx + \int_\Omega \frac{a(x)}{q(x)} |\nabla u|^{q(x)} dx.$$

From [11,14], it is obvious that the derivative operator of the functional $\mathcal{J}$ in the weak sense at the point $u \in W_0^{1,p(x)}(\Omega)$ is the functional $\mathcal{T}(u) := \mathcal{J}'(u) \in W^{-1,p'(x)}(\Omega)$, given by

$$\langle \mathcal{T}u, \vartheta \rangle = \int_\Omega \Big(|\nabla u|^{p(x)-2} \nabla u \nabla \vartheta + a(x) |\nabla u|^{q(x)-2} \nabla u \nabla \vartheta \Big) dx,$$

for all $u, \vartheta \in W_0^{1,p(x)}(\Omega)$ where $\langle \cdot, \cdot \rangle$ the duality pairing between $W^{-1,p'(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$. Furthermore, the properties of the operator $\mathcal{T}$ are summarized in the following lemma (see [11, Theorem 3.1] and [14, Proposition 3.1]).

Lemma 4.3. *The mapping*

$$\begin{aligned} &\mathcal{T} : W_0^{1,p(x)}(\Omega) \longrightarrow W^{-1,p'(x)}(\Omega) \\ &\langle \mathcal{T}u, \vartheta \rangle = \int_\Omega \Big(|\nabla u|^{p(x)-2} \nabla u \nabla \vartheta + a(x) |\nabla u|^{q(x)-2} \nabla u \nabla \vartheta \Big) dx, \end{aligned} \tag{4.1}$$

is a continuous, bounded, strictly monotone operator, and is a mapping of class (S_+).

Lemma 4.4. *Assume that the assumptions $(A_1) - (A_2)$ hold, then the operator*

$$\begin{aligned} &\mathcal{S} : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega) \\ &\langle \mathcal{S}u, \vartheta \rangle = -\int_\Omega \Big(-\delta(x)|u|^{\zeta(x)-2}u + R(x)|u|^{\alpha(x)-2}u + \mu\, g(x,u) + \lambda\, f(x,u,\nabla u) \Big) \vartheta dx, \end{aligned} \tag{4.2}$$

for all $u, \vartheta \in W_0^{1,p(x)}(\Omega)$, is compact.

Proof. In order to prove this lemma, we proceed in five steps.
Step 1 : We define the operator $\Psi_1 : W_0^{1,p(x)}(\Omega) \to L^{p'(x)}(\Omega)$ by

$$\Psi_1 u(x) := \delta(x)|u(x)|^{\zeta(x)-2}u(x).$$

We will prove that Ψ_1 is bounded and continuous.
It is clear that Ψ_1 is continuous. Next we show that Ψ_1 is bounded.
Let $u \in W_0^{1,p(x)}(\Omega)$ and using (2.5) and (2.6), we obtain

$$\begin{aligned} |\Psi_1 u|_{p'(x)} &\le \rho_{p'(x)}(\Psi_1 u) + 1 \\ &= \int_\Omega |\delta(x)|^{p'(x)} |u|^{(\zeta(x)-1)p'(x)} dx + 1 \\ &\le ||\delta||^{p'}_{L^\infty(\Omega)} \int_\Omega |u|^{\beta(x)} dx + 1 \\ &= ||\delta||^{p'}_{L^\infty(\Omega)} \rho_{\beta(x)}(u) + 1 \\ &\le ||\delta||^{p'}_{L^\infty(\Omega)} \Big(|u|^{\beta^-}_{\beta(x)} + |u|^{\beta^+}_{\beta(x)} \Big) + 1. \end{aligned}$$

Hence, we deduce from $L^{p(x)} \hookrightarrow L^{\beta(x)}$ and (2.9) that

$$|\Psi_1 u|_{p'(x)} \le const \Big(|u|^{\beta^-}_{1,p(x)} + |u|^{\beta^+}_{1,p(x)} \Big) + 1,$$

and consequently, Ψ_1 is bounded on $W_0^{1,p(x)}(\Omega)$.
Step 2 : Let $\Psi_2 : W_0^{1,p(x)}(\Omega) \to L^{p'(x)}(\Omega)$ be an operator defined by

$$\Psi_2 u(x) := -R(x)|u(x)|^{\alpha(x)-2}u(x).$$

In this step, we will show that Ψ_2 is bounded and continuous.
First, it is obvious that Ψ_2 is continuous. Second, we show that Ψ_2 is bounded.
Let $u \in W_0^{1,p(x)}(\Omega)$ and using (2.5) and (2.6), we obtain

$$\begin{aligned}
|\Psi_2 u|_{p'(x)} &\leq \rho_{p'(x)}(\Psi_2 u) + 1 \\
&= \int_\Omega |R(x)|^{p'(x)}|u|^{(\alpha(x)-1)p'(x)}dx + 1 \\
&\leq ||R||^{p'}_{L^\infty(\Omega)} \int_\Omega |u|^{k(x)}dx + 1 \\
&\leq ||R||^{p'}_{L^\infty(\Omega)}\rho_{k(x)}(u) + 1 \\
&\leq ||R||^{p'}_{L^\infty(\Omega)}\Big(|u|^{k^-}_{k(x)} + |u|^{k^+}_{k(x)}\Big) + 1.
\end{aligned}$$

Thus, from $L^{p(x)} \hookrightarrow L^{k(x)}$ and (2.9), we deduce that

$$|\Psi_2 u|_{p'(x)} \leq const\Big(|u|^{k^-}_{1,p(x)} + |u|^{k^+}_{1,p(x)}\Big) + 1,$$

and then Ψ_2 is bounded on $W_0^{1,p(x)}(\Omega)$.
Step 3 : Let $\Psi_3 : W^{1,p(x)}(\Omega) \to L^{p'(x)}(\Omega)$ be an operator defined by

$$\Psi_3 u(x) := -\mu\, g(x,u).$$

In this step, we prove that the operator Ψ_3 is bounded and continuous.
First, let $u \in W^{1,p(x)}(\Omega)$, bearing (A_4) in mind and using (2.5) and (2.6), we infer

$$\begin{aligned}
|\Psi_3 u|_{p'(x)} &\leq \rho_{p'(x)}(\Psi_3 u) + 1 \\
&= \int_\Omega |\mu\, g(x,u(x))|^{p'(x)}dx + 1 \\
&= \int_\Omega |\mu|^{p'(x)}|g(x,u(x)|^{p'(x)}dx + 1 \\
&\leq \Big(|\mu|^{p'^-} + |\mu|^{p'^+}\Big)\int_\Omega |\sigma\Big(\nu(x) + |u|^{s(x)-1}\Big)|^{p'(x)}dx + 1 \\
&\leq const\Big(|\mu|^{p'^-} + |\mu|^{p'^+}\Big)\int_\Omega \Big(|\nu(x)|^{p'(x)} + |u|^{\kappa(x)}\Big)dx + 1 \\
&\leq const\Big(|\mu|^{p'^-} + |\mu|^{p'^+}\Big)\Big(\rho_{p'(x)}(\nu) + \rho_{\kappa(x)}(u)\Big) + 1 \\
&\leq const\Big(|\nu|^{p'^+}_{p(x)} + |u|^{\kappa^+}_{\kappa(x)} + |u|^{\kappa^-}_{\kappa(x)}\Big) + 1.
\end{aligned}$$

Then, we deduce from (2.9) and $L^{p(x)} \hookrightarrow L^{\kappa(x)}$, that

$$|\Psi_3 u|_{p'(x)} \leq const\Big(|\nu|^{p'^+}_{p(x)} + |u|^{\kappa^+}_{1,p(x)} + |u|^{\kappa^-}_{1,p(x)}\Big) + 1,$$

that means Ψ_3 is bounded on $W^{1,p(x)}(\Omega)$.
Second, we show that the operator Ψ_3 is continuous.
To this purpose let $u_n \to u$ in $W^{1,p(x)}(\Omega)$. We need to show that $\Psi_3 u_n \to \Psi_3 u$ in $L^{p'(x)}(\Omega)$. We will apply the Lebesgue's theorem.

Note that if $u_n \to u$ in $W^{1,p(x)}(\Omega)$, then $u_n \to u$ in $L^{p(x)}(\Omega)$. Hence there exist a subsequence (u_k) of (u_n) and ϕ in $L^{p(x)}(\Omega)$ such that

$$u_k(x) \to u(x) \quad \text{and} \quad |u_k(x)| \leq \phi(x), \tag{4.3}$$

for a.e. $x \in \Omega$ and all $k \in \mathbb{N}$.
Hence, from (A_2) and (4.3), we have

$$|g(x, u_k(x))| \leq \sigma(\nu(x) + |\phi(x)|^{s(x)-1}),$$

for a.e. $x \in \Omega$ and for all $k \in \mathbb{N}$.
On the other hand, thanks to (A_3) and (4.3), we get, as $k \longrightarrow \infty$

$$g(x, u_k(x)) \to g(x, u(x)) \quad \text{a.e.} \quad x \in \Omega.$$

Seeing that

$$\nu + |\phi|^{s(x)-1} \in L^{p'(x)}(\Omega) \quad \text{and} \quad \rho_{p'(x)}(\Psi_3 u_k - \Psi_3 u) = \int_\Omega |g(x, u_k(x)) - g(x, u(x))|^{p'(x)} dx,$$

then, from the Lebesgue's theorem and the equivalence (2.4), we have

$$\Psi_3 u_k \to \Psi_3 u \quad \text{in} \quad L^{p'(x)}(\Omega),$$

and consequently

$$\Psi_3 u_n \to \Psi_3 u \quad \text{in} \quad L^{p'(x)}(\Omega),$$

that is, Ψ_3 is continuous.
Step 4 : Let us define the operator $\Psi_4 : W^{1,p(x)}(\Omega) \to L^{p'(x)}(\Omega)$ by

$$\Psi_4 u(x) := -\lambda\, f(x, u(x), \nabla u(x)).$$

We will show that Ψ_4 is bounded and continuous.
Let $u \in W^{1,p(x)}(\Omega)$. According to (A_2) and the inequalities (2.5) and (2.6), we obtain

$$\begin{aligned}
|\Psi_4 u|_{p'(x)} &\leq \rho_{p'(x)}(\Psi_4 u) + 1 \\
&= \int_\Omega |\lambda\, f(x, u(x), \nabla u(x))|^{p'(x)} dx + 1 \\
&= \int_\Omega |\lambda|^{p'(x)} |f(x, u(x), \nabla u(x))|^{p'(x)} dx + 1 \\
&\leq \Big(|\lambda|^{p'^-} + |\lambda|^{p'^+}\Big) \int_\Omega |\varrho\Big(\gamma(x) + |u|^{q(x)-1} + |\nabla u|^{q(x)-1}\Big)|^{p'(x)} dx + 1 \\
&\leq const\Big(|\lambda|^{p'^-} + |\lambda|^{p'^+}\Big) \int_\Omega \Big(|\gamma(x)|^{p'(x)} + |u|^{r(x)} + |\nabla u|^{r(x)}\Big) dx + 1 \\
&\leq const\Big(|\lambda|^{p'^-} + |\lambda|^{p'^+}\Big)\Big(\rho_{p'(x)}(\gamma) + \rho_{r(x)}(u) + \rho_{r(x)}(\nabla u)\Big) + 1 \\
&\leq const\Big(|\gamma|^{p'^+}_{p(x)} + |u|^{r^+}_{r(x)} + |u|^{r^-}_{r(x)} + |\nabla u|^{r^+}_{r(x)} + |\nabla u|^{r^-}_{r(x)}\Big) + 1.
\end{aligned}$$

Taking into account that $L^{p(x)} \hookrightarrow L^{r(x)}$ and (2.9), we have then

$$|\Psi_4 u|_{p'(x)} \leq const\Big(|\gamma|^{p'^+}_{p(x)} + |u|^{r^+}_{1,p(x)} + |u|^{r^-}_{1,p(x)}\Big) + 1,$$

and consequently Ψ_4 is bounded on $W^{1,p(x)}(\Omega)$.
It remains to show that Ψ_4 is continuous. Let $u_n \to u$ in $W^{1,p(x)}(\Omega)$, we need to show that $\Psi_4 u_n \to \Psi_4 u$ in $L^{p'(x)}(\Omega)$. We will apply the Lebesgue's theorem.

Note that if $u_n \to u$ in $W^{1,p(x)}(\Omega)$, then $u_n \to u$ in $L^{p(x)}(\Omega)$ and $\nabla u_n \to \nabla u$ in $(L^{p(x)}(\Omega))^N$. Hence, there exist a subsequence (u_k) and ϕ in $L^{p(x)}(\Omega)$ and ψ in $(L^{p(x)}(\Omega))^N$ such that

$$u_k(x) \to u(x) \quad \text{and} \quad \nabla u_k(x) \to \nabla u(x), \tag{4.4}$$

$$|u_k(x)| \leq \phi(x) \quad \text{and} \quad |\nabla u_k(x)| \leq |\psi(x)|, \tag{4.5}$$

for a.e. $x \in \Omega$ and all $k \in \mathbb{N}$.
Hence, thanks to (A_1) and (4.4), we get, as $k \longrightarrow \infty$

$$f(x, u_k(x), \nabla u_k(x)) \to f(x, u(x), \nabla u(x)) \quad \text{a.e.} \quad x \in \Omega.$$

On the other hand, from (A_2) and (4.5), we can deduce the estimate

$$|f(x, u_k(x), \nabla u_k(x))| \leq \varrho(\gamma(x) + |\phi(x)|^{q(x)-1} + |\psi(x)|^{q(x)-1}),$$

for a.e. $x \in \Omega$ and for all $k \in \mathbb{N}$.
Seeing that

$$\gamma + |\phi|^{q(x)-1} + |\psi(x)|^{q(x)-1} \in L^{p'(x)}(\Omega),$$

and taking into account the equality

$$\rho_{p'(x)}(\Psi_4 u_k - \Psi_4 u) = \int_\Omega |f(x, u_k(x), \nabla u_k(x)) - f(x, u(x), \nabla u(x))|^{p'(x)} dx,$$

then, we conclude from the Lebesgue's theorem and (2.4) that

$$\Psi_4 u_k \to \Psi_4 u \quad \text{in} \quad L^{p'(x)}(\Omega),$$

and consequently

$$\Psi_4 u_n \to \Psi_4 u \quad \text{in} \quad L^{p'(x)}(\Omega),$$

and then Ψ_4 is continuous.
Step 4: Let $I^* : L^{p'(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ be the adjoint operator of the operator $I : W^{1,p(x)}(\Omega) \to L^{p(x)}(\Omega)$.
We then define

$$I^* o\Psi_1 : W^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega),$$

$$I^* o\Psi_2 : W^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega),$$

$$I^* o\Psi_3 : W^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega),$$

and

$$I^* o\Psi_4 : W^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega).$$

On another side, taking into account that I is compact, then I^* is compact. Thus, the compositions $I^* o\Psi_1$, $I^* o\Psi_2$, $I^* o\Psi_3$ and $I^* o\Psi_4$ are compact, that means $\mathcal{S} = I^* o\Psi_1 + I^* o\Psi_2 + I^* o\Psi_3 + I^* o\Psi_4$ is compact. With this last step the proof of Lemma 4.4 is completed. □

We are now in the position to get the existence result of weak solution for (1.1).

Theorem 4.5. *Assume that the assumptions $(A_1) - (A_4)$ hold, then the problem (1.1) possesses at least one weak solution u in $W_0^{1,p(x)}(\Omega)$.*

Proof. The basic idea of our proof is to reduce the problem (1.1) to a new one governed by a Hammerstein equation, and apply the theory of topological degree introduced in Section 3 to show the existence of a weak solutions to the state problem.

First, for all $u, \vartheta \in W_0^{1,p(x)}(\Omega)$, we define the operators $\mathcal{T}$ and $\mathcal{S}$, as defined in (4.1) and (4.2) respectively,

$$\begin{aligned}
&\mathcal{T} : W_0^{1,p(x)}(\Omega) \longrightarrow W^{-1,p'(x)}(\Omega)\\
&\langle \mathcal{T}u, \vartheta\rangle = \int_\Omega \Big(|\nabla u|^{p(x)-2}\nabla u \nabla\vartheta + a(x)|\nabla u|^{q(x)-2}\nabla u\nabla\vartheta\Big)dx,
\end{aligned}$$

$$\begin{aligned}
&\mathcal{S} : W_0^{1,p(x)}(\Omega) \longrightarrow W^{-1,p'(x)}(\Omega)\\
&\langle \mathcal{S}u, \vartheta\rangle = -\int_\Omega \Big(-\delta(x)|u|^{\zeta(x)-2}u + R(x)|u|^{\alpha(x)-2}u + \mu\, g(x,u) + \lambda\, f(x,u,\nabla u)\Big)\vartheta dx.
\end{aligned}$$

Consequently, the problem (1.1) is equivalent to the equation

$$\mathcal{T}u = -\mathcal{S}u, \quad u \in W_0^{1,p(x)}(\Omega). \tag{4.6}$$

Taking into account that, by Lemma 4.3, the operator $\mathcal{T}$ is a continuous, bounded, strictly monotone and of class (S_+), then, by [27, Theorem 26 A], the inverse operator

$$\mathcal{L} := \mathcal{T}^{-1} : W^{-1,p'(x)}(\Omega) \to W_0^{1,p(x)}(\Omega),$$

is also bounded, continuous, strictly monotone and of class (S_+).
On another side, according to Lemma 4.4, we have that the operator $\mathcal{S}$ is bounded, continuous and quasi-monotone.
Consequently, following Zeidler's terminology [27], the equation (4.6) is equivalent to the following abstract Hammerstein equation

$$u = \mathcal{L}\vartheta \ \text{ and } \ \vartheta + \mathcal{S}o\mathcal{L}\vartheta = 0, \quad u \in W_0^{1,p(x)}(\Omega) \text{ and } \vartheta \in W^{-1,p'(x)}(\Omega). \tag{4.7}$$

Seeing that (4.6) is equivalent to (4.7), then to solve (4.6) it is thus enough to solve (4.7). In order to solve (4.7), we will apply the Berkovits topological degree introducing in Section 3.
First, let us set

$$\mathcal{B} := \Big\{ \vartheta \in W^{-1,p'(x)}(\Omega) : \exists\, t \in [0,1] \ \text{ such that } \ \vartheta + t\mathcal{S}o\mathcal{L}\vartheta = 0 \Big\}.$$

Next, we show that $\mathcal{B}$ is bounded in $\in W^{-1,p'(x)}(\Omega)$.
Let us put $u := \mathcal{L}\vartheta$ for all $\vartheta \in \mathcal{B}$. Taking into account that $|\mathcal{L}\vartheta|_{1,p(x)} = |\nabla u|_{p(x)}$, then we have the following two cases:
First case : If $|\nabla u|_{p(x)} \leq 1$, then $|\mathcal{L}\vartheta|_{1,p(x)} \leq 1$, that means $\Big\{\mathcal{L}\vartheta : \vartheta \in \mathcal{B}\Big\}$ is bounded.
Second case : If $|\nabla u|_{p(x)} > 1$, then, we deduce from (2.2), (A_2) and (A_4), the inequalities (2.7) and

(2.6) and the Young's inequality that

$$
\begin{aligned}
|\mathcal{L}\vartheta|^{p^-}_{1,p(x)} &= |\nabla u|^{p-}_{p(x)}\\
&\leq \rho_{p(x)}(\nabla u)\\
&= \langle \mathcal{T}u,\ u\rangle\\
&= \langle \vartheta,\ \mathcal{L}\vartheta\rangle\\
&= -t\langle \mathcal{S}o\mathcal{L}\vartheta,\ \mathcal{L}\vartheta\rangle\\
&= t\int_\Omega \Big(-\delta(x)|u|^{\zeta(x)-2}u + R(x)|u|^{\alpha(x)-2}u + \mu\, g(x,u) + \lambda\, f(x,u,\nabla u)\Big)u dx\\
&\leq const\Big(\int_\Omega |u|^{\zeta(x)}dx + \int_\Omega |u|^{\alpha(x)}dx + \int_\Omega |\nu(x)u(x)|dx + \int_\Omega |u(x)|^{s(x)}dx\\
&\qquad + \int_\Omega |\gamma(x)u(x)|dx + \int_\Omega |u(x)|^{q(x)}dx + \int_\Omega |\nabla u|^{q(x)-1}|u|dx\Big)\\
&\leq const\Big(\rho_{\zeta(x)}(u) + \rho_{\alpha(x)}(u) + \int_\Omega |\nu(x)u(x)|dx + \int_\Omega |\gamma(x)u(x)|dx\\
&\qquad + \rho_{s(x)}(u) + \rho_{q(x)}(u) + \int_\Omega |\nabla u|^{q(x)-1}|u|dx\Big)\\
&\leq const\Big(|u|^{\zeta^-}_{\zeta(x)} + |u|^{\zeta^+}_{\zeta(x)} + |u|^{\alpha^-}_{\alpha(x)} + |u|^{\alpha^+}_{\alpha(x)} + |\nu|_{p'(x)}|u|_{p(x)} + |\gamma|_{p'(x)}|u|_{p(x)}\\
&\qquad + |u|^{s^+}_{s(x)} + |u|^{s^-}_{s(x)} + |u|^{q^+}_{q(x)} + |u|^{q^-}_{q(x)} + \frac{1}{q'^-}\rho_{q(x)}(\nabla u) + \frac{1}{q^-}\rho_{q(x)}(u)\Big)\\
&\leq const\Big(|u|^{\zeta^-}_{\zeta(x)} + |u|^{\zeta^+}_{\zeta(x)} + |u|^{\alpha^-}_{\alpha(x)} + |u|^{\alpha^+}_{\alpha(x)} + |u|_{p(x)} + |u|^{s^+}_{s(x)} + |u|^{s^-}_{s(x)}\\
&\qquad + |u|^{q^+}_{q(x)} + |u|^{q^-}_{q(x)} + |\nabla u|^{q^+}_{q(x)}\Big),
\end{aligned}
$$

then, according to $L^{p(x)} \hookrightarrow L^{\zeta(x)}$, $L^{p(x)} \hookrightarrow L^{\alpha(x)}$, $L^{p(x)} \hookrightarrow L^{s(x)}$ and $L^{p(x)} \hookrightarrow L^{q(x)}$, we get

$$|\mathcal{L}\vartheta|^{p^-}_{1,p(x)} \leq const\Big(|\mathcal{L}\vartheta|^{\zeta^+}_{1,p(x)} + |\mathcal{L}\vartheta|^{\alpha^+}_{1,p(x)} + |\mathcal{L}\vartheta|_{1,p(x)} + |\mathcal{L}\vartheta|^{s^+}_{1,p(x)} + |\mathcal{L}\vartheta|^{q^+}_{1,p(x)}\Big),$$

what implies that $\Big\{\mathcal{L}\vartheta : \vartheta \in \mathcal{B}\Big\}$ is bounded.
On the other hand, we have that the operator is $\mathcal{S}$ is bounded, then $\mathcal{S}o\mathcal{L}\vartheta$ is bounded. Thus, thanks to (4.7), we have that $\mathcal{B}$ is bounded in $W^{-1,p'(x)}(\Omega)$.
However, $\exists\ r > 0$ such that

$$|\vartheta|_{-1,p'(x)} < r \quad \text{for all} \quad \vartheta \in \mathcal{B},$$

which leads to

$$\vartheta + t\mathcal{S}o\mathcal{L}\vartheta \neq 0, \quad \vartheta \in \partial\mathcal{B}_r(0) \quad \text{and} \quad t \in [0,1],$$

where $\mathcal{B}_r(0)$ is the ball of center 0 and radius r in $W^{-1,p'(x)}(\Omega)$.
Moreover, by Lemma 3.4, we conclude that

$$I + \mathcal{S}o\mathcal{L} \in \mathcal{F}_{\mathcal{L}}(\overline{\mathcal{B}_r(0)}) \quad \text{and} \quad I = \mathcal{T}o\mathcal{L} \in \mathcal{F}_{\mathcal{L}}(\overline{\mathcal{B}_r(0)}).$$

On another side, taking into account that I, $\mathcal{S}$ and $\mathcal{L}$ are bounded, then $I + \mathcal{S}o\mathcal{L}$ is bounded. Hence, we infer that

$$I + \mathcal{S}o\mathcal{L} \in \mathcal{F}_{\mathcal{L},B}(\overline{\mathcal{B}_r(0)}) \quad \text{and} \quad I = \mathcal{T}o\mathcal{L} \in \mathcal{F}_{\mathcal{L},B}(\overline{\mathcal{B}_r(0)}).$$

Next, we define the homotopy

$$
\begin{aligned}
\mathcal{H} : [0,1] \times \overline{\mathcal{B}_r(0)} &\to W^{-1,p'(x)}(\Omega)\\
(t,\vartheta) &\mapsto \mathcal{H}(t,\vartheta) := \vartheta + t\mathcal{S}o\mathcal{L}\vartheta.
\end{aligned}
$$

Hence, thanks to the properties of the degree d seen in Theorem 3.7, we obtain

$$d(I + \mathcal{S}o\mathcal{L}, \mathcal{B}_r(0), 0) = d(I, \mathcal{B}_r(0),\ 0) = 1 \neq 0,$$

what implies that there exists $\vartheta \in \mathcal{B}_r(0)$ which verifies

$$\vartheta + \mathcal{S}o\mathcal{L}\vartheta = 0.$$

Finally, we infer that $u = \mathcal{L}\vartheta$ is a weak solutions of (1.1). The proof is completed. $\square$

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