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On the Existence of Almost Automorphic Generalized Solutions to Some Differential Equations

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ABSTRACT: This paper is devoted to study some regularity of almost automorphic and asymptotic almost automorphic generalized solution of the differential equation $\frac{d}{dt}u(t) = A\,u(t) + f(t)$, in the framework of the Colombeau algebras. Under certain assumptions about the second member we showed that the generalized solution is an asmptotically almost automorphic in the sense of genaralized functions.

Key Words: Generalized function, almost automorphic generalized function, Colombeau algebra.

Contents

1. Introduction

We have explored new properties of the almost automorphic functions in the framework of the generalized functions. In 1982,J.F.Colombeau introduced an algebra $\mathcal{G}$ of generalized functions to deal with the multiplication problem for distributions, see [5],[4]. This algebra $\mathcal{G}$ is differential which contains the space $\mathcal{D}^{'}$ of distributions. Furthermore, nonlinear operations more general than the multiplication make sense in the algebra $\mathcal{G}$. Therefore the algebra $\mathcal{G}$ is a very convenient one to find and study solutions of nonlinear differential equations with singular data and coefficients. After Schwartz' "impossibility result" [11] for algebras of generalized functions with a prescribed list of (natural) assumptions, several new approaches have appeared with the aim of applications in nonlinear problems. We refer to the recent monograph [7] for the historical background as well as for the list of relevant references, mainly for algebras of generalized functions today called Colombeau type algebras. (see [1], [4]). Colombeau and all other successors introduce algebras of generalized functions through various analytical methods. By now, these algebras have become an important tool in the theory of PDEs, stochastic analysis, differential geometry and general relativity. We show that such algebras fit in the general theory of well-known sequence spaces forming appropriate algebras. Several interesting works deal with the concept of almost automorphy, namely the book by Diagana [6] and that of N'Gu´er´ekata [10] and the series of papers by Shen and Yi [12]. Within the framework of Colombeau's algebra, work on almost periodic, almost automorphic and asymptotically almost automorphic functions remains somewhat rare, except for the work of C.Bouzar et al see [3] noting that C.Bouzar who was constructed for the first time the classes of Colombeau algebra compatible with almost periodic, almost automorphic and asymptotically almost automorphic functions. Our paper is inspired by his work in order to add some properties missing from them, namely the properties of the solution of the equation presented in the abstract above. We study

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here the following system of singular differential equations, The aim of this work is to studies the generalized differential equation $\frac{d}{dt}u(t) = A\,u(t) + f(t)$ in two cases: the first we assume that A is a generalized complex number where we showed that if f has a representative $(f_\varepsilon)_{\varepsilon\in(0,1]}$is an almost automorphic function, then it's the case of the generalized solution of this differential equation. The second case we assume that A is an infinitesimal generator of a strongly continuous one-parameter group, where we proved also that if any representative of the generalized one parameter group is continuous for every real number x and $(f_\varepsilon)_\varepsilon$ is Bouchner integrable. then the generaized solution is is asymptotically almost automorphic. This work is organized in the following way. In the section two we recall various basics definitions and important properties about the algebras of generalized functions,to get some working knowledge. The section three is reserved to the recal the notions of almost automorphic functions, and the asymptotically almost automorphic functions. In section four we recal the definition and basics properties of the almost automorphic functions, and the asymptotically almost automorphic functions in the framework of the Colmbeau's algebra, (see [3]). Finally in the setion five we present our main results.

2. Preliminaries

Let $\mathcal{C}_b$ denote the space of bounded continuous complex-valued functions defined on $\mathbf{R}$, endowed with the norm of uniform convergence on $\mathbf{R}$, it is well known that$(\mathcal{C}_b, \|\,.\,\|_\infty)$ is a Banach algebra.

Definition 2.1. *Let $\omega \in \mathbf{R}$, the translation operator τ_ω is defined by $\tau_\omega f(.) = f(. + \omega)$.*

We recall some properties of almost automorphic functions, see [2], [13], and [14].

Definition 2.2. *A complex-valued function g defined and continuous on $\mathbf{R}$ is called almost automorphic if for any sequence $(s_m)_{m\in\mathbf{N}} \subset \mathbf{R}$; one can extract a subsequence $(s_{m_k})_{k\in\mathbf{N}} \subset \mathbf{R}$ such that*

$$\tilde{g}(x) = \lim_{k\longrightarrow\infty} g(x + s_{m_k}) \ \ exists \ \ for \ \ every \ \ x \in \mathbf{R}. \tag{2.1}$$

And

$$\lim_{k\longrightarrow\infty} \tilde{g}(x - s_{m_k}) = g(x) \ \ exists \ \ for \ \ every \ \ x \in \mathbf{R}. \tag{2.2}$$

The space of almost automorphic functions on $\mathbf{R}$ is denoted by $\mathcal{C}_{aa}$.
We note that the function $\tilde{g}$ is not continuous, but $g \in L^\infty$.
we cite some important properties of the space $\mathcal{C}_{aa}$.

Proposition 2.3. *1. The space $(\mathcal{C}_{aa}, \|\,.\,\|_\infty)$ is a Banach subalgebra of $\mathcal{C}_b$.*
2. $\tau_\omega \mathcal{C}_{aa} \subset \mathcal{C}_{aa} \quad \forall \omega \in \mathbf{R}$.
3. $\mathcal{C}_{aa} * L^1 \subset \mathcal{C}_{aa}$.
4. A primitive of an almost automorphic function is almost automorphic if and only if it is bounded.

Proof For 1. it is enough to show taht $\mathcal{C}_{aa}$ is a closed subspace of $\mathcal{C}_b$, and 3. is based on the definition of the convolution product, And the proof of 4. is given in [13], it remains to show 2.
2. Let f be an element of $\mathcal{C}_{aa}$, and ω a real number, we have to prove that $\tau_\omega f$ is an element of $\mathcal{C}_{aa}$. we have for any sequence $(s_m)_{m\in\mathbf{N}}$, there exists asubsequence $(s_{m_k})_{k\in\mathbf{N}}$, such that $\tilde{f}(x) := \lim_{k\longrightarrow\infty} f(x + s_{m_k}) \ \ exists \ \ for \ \ every \ \ x \in \mathbf{R}$, and $\lim_{k\longrightarrow\infty} \tilde{f}(x - s_{m_k}) = f(x) \ \ exists \ \ for \ \ every \ \ x \in \mathbf{R}$ hold, then it is the same for
$\tilde{g}(x) := \lim_{k\longrightarrow\infty} f(x+\omega+s_{m_k})$, and $\lim_{k\longrightarrow\infty} \tilde{g}(x+\omega-s_{m_k}) = f(x) \ \ exist$, which means that for the some subsequence $(s_{m_k})_k$, we have $\tilde{g}(x) = \lim_{k\longrightarrow\infty} \tau_\omega f(x+\omega+s_{m_k})$, and $\lim_{k\longrightarrow\infty} \tilde{g}(x+\omega-s_{m_k}) = \tau_\omega f(x) \ \ exist$, and therefore $\tau_\omega f \in \mathcal{C}_{aa}$.

2.1. Smooth almost automorphic functions

Let $\mathbf{J} = [0, +\infty)$ be the half real line, and let $\mathcal{E}(\mathbf{I})$ be the space of infinitely derivable functions on $\mathbf{I} = \mathbf{R}$ or $\mathbf{J}$, define the space

$$\mathcal{D}_{L^p(\mathbf{I})} := \left\{ \varphi \in \mathcal{E}(\mathbf{I}) : \forall j \in \mathbf{N}, \varphi^{(j)} \in L^p(\mathbf{I}) \right\}, \quad p \in [1, +\infty],$$

that we endow with the topology defined by the family of norms

$$|\varphi|_{k,p,\mathbf{I}} := \sum_{j\le k} \|_{\varphi^{(i)}\|_{L^p(\mathbf{I})}}, \qquad k\in\mathbf{N}, \varphi\in\mathcal{D}_{L^p(\mathbf{I})}.$$

So,$\mathcal{D}_{L^p(\mathbf{I})}$ is a Frechet subalgebra of $\mathcal{E}(\mathbf{I})$. We denote $\mathcal{B}(\mathbf{I})=\mathcal{D}_{L^\infty(\mathbf{I})}$. Let $\mathcal{B}$ be the closure in $\mathcal{B}$=$\mathcal{B}(\mathbf{R})$ of the space $\mathcal{D}$ of smooth functions with compact support.

Definition 2.4. *We define the space of smooth almost automorphic functions bay:*

$$\mathcal{B}_{aa} = \left\{\varphi\in\mathcal{E} : \forall j\in\mathbf{Z}+, \quad \varphi^{(j)}\in\mathcal{C}_{aa}\right\}.$$

we give the important properties in the following

Proposition 2.5. *1.The space $\mathcal{B}_{aa}$ is a Frechet subalgebra of $\mathcal{B}$.*
2.$\tau_\omega\mathcal{B}_{aa}\subset\mathcal{B}_{aa}, \forall\omega\in\mathbf{R}$.
3.$\mathcal{B}_{aa} * L^1\subset\mathcal{B}_{aa}$.

Proof. 1. As the topology of $\mathcal{B}_{aa}\subset\mathcal{D}_{L^\infty}$ is given by the countable family of submultiplicative norms $|.|_{k,\infty}$, $k\in\mathbf{N}$, it remains to show the completeness of $\mathcal{B}_{aa}$. Let $(f_m)_{m\in\mathbf{N}}\subset\mathcal{B}_{aa}$ be a Cauchy sequence, it is clear that, $\forall i\in\mathbf{N}$, $(f_m^{(i)})_{m\in\mathbf{N}}$ is a Cauchy sequence in $\mathcal{C}_{aa}$ the space of almost continuous functions, which is complete with respect to the norm of unform convergence, then $\forall i\in\mathbf{N}$, $f_m^{(i)}m$ converges uniformly to $f_i\in\mathcal{C}aa$, setting $f_0=f$, we obtain, due to the uniform convergence, that $f\in\mathcal{C}^\infty$, $\forall i\in\mathbf{N}$, $f^{(i)}=f_i\in\mathcal{C}_{aa}$, i.e. $(f_m)_{m\in\mathbf{N}}$ converges to f in the topology of $\mathcal{B}_{aa}$, which means that $\mathcal{B}_{aa}$ is complete.
2. Let $f\in\mathcal{B}_{aa}$, and let ω be a real number, we have to prove that $\tau_\omega f$ is an element of $\mathcal{B}_{aa}$, because the translation of any smooth function is also a smooth function, it remains to show that $\tau_\omega f$ is an almost automorphic function, which is obvious by the definition.
3. If $h\in L^1$ and $f\in\mathcal{B}_{aa}$, then $(f*h)\in\mathcal{C}^\infty$ and $\forall i\in\mathbf{N}$, $(f*h)^{(i)}=f^{(i)}*h\in\mathcal{C}_{aa}$. The following result is a consequence of the last proposition. ☐

Corollary 2.6. *Let $f\in\mathcal{D}_{L^\infty}$, then $f\in\mathcal{B}_{aa}$ if and only if $\forall\varphi\in\mathcal{D}$, $f*\varphi\in\mathcal{C}_{aa}$.*

Before presenting our main results, we recall here some notations and formulas to be used later. The elements of Colombeau algebras $\mathcal{G}$ are equivalence classes of regularizations, i.e., sequences of smooth functions satisfying asymptotic conditions in the regularization parameter. Therefore, for any set X, the family of sequences $(u_\varepsilon)_{\varepsilon\in(0,1)}$ of elements of a set X will be denoted by $X^{(0,1)}$, such sequences will also be called nets and simply written as u_ε. see [7].
Let Ω be an open subset of $\mathbf{R}^n$. The basic objects of the theory as we use it are families $(u_\varepsilon)_{\varepsilon\in(0;1)}$ of smooth functions $u_\varepsilon\in\mathcal{C}^\infty(\Omega)$ for $0<\varepsilon<1$. We single out the following subalgebras of
Moderate families, denoted by $\mathcal{E}_M(\Omega)$, are defined by the property

$$\forall K\subset\Omega, \forall\alpha\in\mathbf{N}_0^n, \exists p\ge 0 : \sup_{x\in K}|\partial^\alpha u_\varepsilon(x)| = \mathcal{O}_{\varepsilon\to 0}(\varepsilon^{-p}).$$

Null families, denoted by $\mathcal{E}_M(\Omega)$, are defned by the property

$$\forall K\subset\Omega, \forall\alpha\in\mathbf{N}_0^n, \forall q\ge 0 : \sup_{x\in K}|\partial^\alpha u_\varepsilon(x)| = \mathcal{O}_{\varepsilon\to 0}(\varepsilon^{q}).$$

Thus moderate families satisfy a locally uniform polynomial estimate as $\varepsilon\to 0$, together with all derivatives, while null functionals vanish faster than any power of ε in the same situation. The null families from a differential ideal in the collection of moderate families.
The Colombeau algebra is the factor algebra

$$\mathcal{G}(\Omega) = \mathcal{E}_M(\Omega)/\mathcal{N}(\Omega).$$

The algebra $\mathcal{G}(\Omega)$ just defined coincides with the special Colombeau algebra in [7], where the notation $\mathcal{G}^s(\Omega)$ has been employed. It was called the simplified Colombeau algebra in [7]. The Colombeau algebra on a closed half space $\mathbf{R}^n \times [0,1)$ is defined in a similary way. The restriction of an element $u \in \mathcal{G}(\mathbf{R}^n \times [0,1))$ to the line $\{t=0\}$ is defined on representatives by

$$u/\{t=0\} = Class\ of\ (u_\varepsilon(\cdot,0))_{\varepsilon\in(0,1)}.$$

Similary, restrictions of the elements of $\mathcal{G}(\Omega)$ to open subsets of Ω are defined on representatives. One can see that $\Omega \longrightarrow \mathcal{G}(\Omega)$ is a sheaf of differential algebras on $\mathbf{R}^n$. The space of compactly supported distributions is imbedded in $\mathcal{G}(\Omega)$ by convolution :

$$i : \mathcal{E}^{'}(\Omega) \longrightarrow \mathcal{G}(\Omega), \quad i(\omega) = \ class\ of\ (\omega * (\phi_\varepsilon)/\Omega)_{\varepsilon\in(0,1)}.$$

where

$$\phi_\varepsilon(x) = \varepsilon^{-n}\phi(\frac{x}{\varepsilon}).$$

is obtained by scaling a fixed test function $\mathcal{S}(\mathbf{R}^n)$ of integral one with all moments vanishing. By the sheaf property, this can be extended in a unique way to an imbedding of the space of distributions $\mathcal{D}(\Omega)$. One of the main features of the Colombeau construction is the fact that this imbedding renders $\mathcal{C}^\infty(\Omega)$ a faithful subalgebra. In fact, given $f \in \mathcal{C}^\infty(\Omega)$, one can define a corresponding element of $\mathcal{G}(\Omega)$ by the constant imbedding

$$\sigma(f) = \ class\ of\ [(\varepsilon, x) \longrightarrow f(x)].$$

Then the important equality $i(f) = \sigma(f)$ holds in $\mathcal{G}(\Omega)$.

If $u \in \mathcal{G}(\Omega)$ and f is a smooth function which is of at most polynomial growth at infinity, together with all its derivatives, the superposition $f(u)$ is a well-defined element of $\mathcal{G}(\Omega)$.

In the literature of Colombeau algebras the regularity theory is based on the subalgebra $\mathcal{G}^\infty(\Omega)$ of regular generalized functions in $\mathcal{G}(\Omega)$. It is defined by those elements which have a representative satisfying

$$\forall K \subset \Omega, \forall\alpha \in \mathbf{N}_0^n, \exists p \geq 0 : \sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| = \mathcal{O}_{\varepsilon\longrightarrow 0}(\varepsilon^{-p}).$$

Observe the change of quantifiers with respect to the last formula, locally, all derivatives of a regular generalized function have the same order of growth in $\varepsilon > 0$. One has that (see [8]).

$$\mathcal{G}^\infty(\Omega) \cap \mathcal{D}^{'}(\Omega) = \mathcal{C}^\infty(\Omega).$$

For the purpose of describing the regularity of Colombeau generalized functions, $\mathcal{G}^\infty(\Omega)$ plays the same role as $\mathcal{C}^\infty(\Omega)$ does in the setting of distributions.
A net $(r_\varepsilon)_{\varepsilon\in(0,1]}$ of complex numbers is called a slow scale net if

$$|r_\varepsilon|^p = \mathcal{O}(\varepsilon^{-1}) \qquad as \quad \varepsilon \to 0.$$

for every $p \geq 0$. We refer to [8] for a detailed discussion of slow scale nets.

3. Almost automorphic generalized functions

We start this section by presenting a class of Colombeau algebras, which will be compatible with the theory of almost automorphic functions. For more information about this class see for example [3].

Definition 3.1. *We define the space of moderate elements by*

$$\mathcal{M}_{aa} = \left\{(u_\varepsilon)_\varepsilon \in (\mathcal{B}_{aa})^I : \forall k \in \mathbf{N}, \exists m \in \mathbf{N}, |u_\varepsilon|_{k,\infty} = \mathcal{O}(\varepsilon^{-m}), \varepsilon \to 0\right\},$$

and the space of null elements by

$$\mathcal{N}_{aa} = \left\{(u_\varepsilon)_\varepsilon \in (\mathcal{B}_{aa})^I : \forall k \in \mathbf{N}, \forall m \in \mathbf{N}, |u_\varepsilon|_{k,\infty} = \mathcal{O}(\varepsilon^{m}), \varepsilon \to 0\right\}.$$

The following result gives a null characterization of $\mathcal{N}_{aa}$.

Proposition 3.2. *We have*

$$\mathcal{N}_{aa} = \{(u_\varepsilon)_\varepsilon \in \mathcal{M}_{aa} \ : \forall m \in \mathbf{N}, |u_\varepsilon|_{0,\infty} = \mathcal{O}(\varepsilon^m) \ as \ \varepsilon \to 0\}.$$

The proof of this result you can see [7] which it's given in the general case.

Definition 3.3. *The algebra of almost automorphic generalized functions is defined as the quotient algebra*

$$\mathcal{G}aa = \mathcal{M}aa/\mathcal{N}aa.$$

3.1. Asymptotically almost automorphic generalized functions

We introduce the algebra of asymptotically almost automorphic generalized functions as in the case of $\mathcal{G}_{aa}$ where we replace the algebra of moderate functions $\mathcal{M}_{aa}$, and the algebra of negligeable functions $\mathcal{N}_{aa}$ by $\mathcal{M}_{aaa}$, and $\mathcal{N}_{aa}$, respectively.

Definition 3.4. *We define the space of moderate elements by*

$$\mathcal{M}_{aaa} = \left\{(u_\varepsilon)_\varepsilon \in (\mathcal{B}_{aaa})^I : \forall k \in \mathbf{N}, \exists m \in \mathbf{N}, |u_\varepsilon|_{k,\infty} = O(\varepsilon^{-m}), \varepsilon \to 0\right\},$$

where $\mathcal{B}_{aaa}$ is the algebra of asumptotically almost automorphic smooth functions, and the space of null elements by

$$\mathcal{N}_{aaa} = \left\{(u_\varepsilon)_\varepsilon \in (\mathcal{B}_{aaa})^I : \forall k \in \mathbf{N}, \forall m \in \mathbf{N}, |u_\varepsilon|_{k,\infty} = O(\varepsilon^{m}), \varepsilon \to 0\right\}$$

The algebra of asymptotically almost automorphic generalized functions is defined as the quotient algebra

$$\mathcal{G}aaa = \mathcal{M}aaa/\mathcal{N}aaa,$$

for more details on this algebra we can see [3].

4. Main results

We consider in the algebra of almost automorphic generalized functions the differential equation

$$\frac{d}{dt}u(t) = A\,u(t) + f(t). \tag{4.1}$$

We present the various conditions for ensuring almost automorphy in the frame of colombeau's algebra of the strong and mild solutions.

4.1. The case $A \in \tilde{\mathbf{C}}$ (A is a generalized comlex number):

We consider linear system of ordinary differential equation 4.1
where $A = [(\lambda_\varepsilon)_\varepsilon] \in \tilde{\mathbf{C}}$ ($\lambda_\varepsilon = \lambda \in \mathbf{C}$) is a generalized comolex number, $u \in \mathcal{G}_{aa}$, and $f = [(f_\varepsilon)_\varepsilon] \in \mathcal{G}_{aa}$.

Definition 4.1. *A generalized function $u \in \mathcal{G}_{aa}$ is called solution of 4.1 on $\mathbf{J} = [0, \infty)$ if it satisfies*

$$\left(u_\varepsilon^{'}(t) - \lambda u_\varepsilon(t) - f_\varepsilon(t)\right)_\varepsilon \in \mathcal{N}_{aa}.$$

where $\lambda = \lambda_\varepsilon \ \forall \varepsilon \in I$, and $f = [(f_\varepsilon(t))]$.

Theorem 4.2. *With the notation of the abouve definition, and if $\lambda \in \mathbf{C}$, $f_\varepsilon : \mathbf{R} \longrightarrow \mathbf{C}$ be an almost automorphic generalized function, then the solution of 4.1 is given by the almost automorphic generalized function*

$$u_1(t) = \big(-\int_t^{+\infty} e^{\lambda(t-s)} f_\varepsilon(s)ds\big)_\varepsilon + \mathcal{N}_{aa} \qquad if \quad Re\lambda > 0.$$

$$u_2(t) = \big(\int_{-\infty}^{t} e^{\lambda(t-s)} f_\varepsilon(s)ds\big)_\varepsilon + \mathcal{N}_{aa} \qquad if \quad Re\lambda < 0.$$

Proof. we have to show first that if $u(t)$ is a well defined element of $\mathcal{G}_{aa}$, then this is the case for $\frac{d}{dt}u(t)$, indeed: A representative of $u_j(t),\ \ j=1,2$ is given by

$$u_{j,\varepsilon}(t) = -\int_t^{\infty} e^{\lambda(t-r)}\, f_\varepsilon(r)dr \quad if\ \ Re\,(\lambda) > 0\, and\ j=1,2,$$

we have $(u_{j,\varepsilon})_\varepsilon(t) \in \mathcal{M}_{aa}$, which means that

$$\forall k \in \mathbf{N}\,,\ \forall m>0,\ \exists c>0,\ \exists \varepsilon_0 \in I\ \ |u_{j,\varepsilon}|_{k,\infty} \leq c^{-\varepsilon}\ \ \forall \varepsilon < \varepsilon_0,$$

we have for $j=1,2$

$$\begin{aligned} |u'_{j,\varepsilon}|_{k,\infty} &= \sum_{j\leq k} \parallel (u'_{j,\varepsilon})^{(j)} \parallel_\infty \\ &= \sum_{j\leq k} \parallel (u'_{j,\varepsilon})^{(j+1)} \parallel_\infty \\ &= \sum_{l\leq k+1} \parallel (u'_{j,\varepsilon})^{(l)} \parallel_\infty \\ &= \sum_{l\leq k+1} \parallel (u'_{j,\varepsilon})^{(l)} \parallel_\infty \\ &= |u_{j,\varepsilon}|_{k+1,\infty}. \end{aligned}$$

Sicne k is arbitrary in the definition of $\mathcal{M}_{aa}$, we obtain $|u'_{j,\varepsilon}|_{k,\infty} = O(\varepsilon^{-m})$ for some $m>0$. which prove that $u'_j\quad j=1,2$ is also an element of $\mathcal{M}_{aa}$. It is oubvious that $u_1(t)$ and $u_2(t)$ are solutions of 4.1 in the sense of the previous definition. It remains to prove that they are almost automorphic generalized functions.
Let $s=t-r$, then we can write

$$u_{j,\varepsilon}(t) = -\int_{-\infty}^{0} e^{\lambda s} f_\varepsilon(t-s)ds. \qquad j=1,2.$$

Let $(s_n)_n$ be a sequence of real numbers. Since for all $\varepsilon \in I$ where I is a set of index, f_ε is an almost automorphic function, there exists a subsequence $(s_{n_{k(\varepsilon)}})_k$ of $(s_n)_n$ such that

$$\lim_{k\to+\infty} f_\varepsilon(t+s_{n_{k(\varepsilon)}}) = g_\varepsilon(t) \quad and \quad \lim_{k\to+\infty} g_\varepsilon(t-s_{n_{k(\varepsilon)}}) = f_\varepsilon(t)\ pointwise\ \ on\ \mathbf{R}.$$

We have $u_{1,\varepsilon}(t) = -\int_{-\infty}^{0} e^{\lambda s} f(t-s+s_{n_k})ds$, let us prove that there exists $c>0$ such that $|e^{\lambda s} f(t - . + s_{n_k})|_{k,\infty} \leq c\, e^{Re\,\lambda\, s}$ for each $s \in \mathbf{R}$. Indeed, we have for any $k \in \mathbf{N}$

$$|e^{\lambda s} f_\varepsilon(t-s+s_{n_k})|_{k,\infty} = \sum_{j\leq k} \parallel f_\varepsilon^{(j)} \parallel_\infty .$$

by the classical Landau-Kolomogorov inequality: $\parallel f^{(p)} \parallel_\infty \leq 2\pi \parallel f \parallel_\infty^{1-p/n} \parallel f^{(n)} \parallel_\infty^{p/n}$ where $0<p<n\in\mathbf{N}$, and f is of class $\mathcal{C}^n$. In particular for $p=j$, and $n=2j$, we obtain for all $k\in\mathbf{N}$

$$\begin{aligned} |e^{\lambda\, s} f(t-s+s_{n_k})|_{k,\infty} &\leq e^{(Re\,\lambda)} \sum_{j\leq k} (2\pi) \parallel f_\varepsilon \parallel_\infty^{1-1/2} \parallel f_\varepsilon^{(2j)} \parallel_\infty^{1/2} \\ &\leq e^{(Re\lambda)\,s}\, 2\pi(|f_\varepsilon|_{0,\infty})^{1/2} \sum_{j\leq k} \parallel f_\varepsilon^{(2j)} \parallel_\infty^{1/2} \\ &= \Big(2\pi(|f_\varepsilon|_{0,\infty})^{1/2} \sum_{j\leq k} \parallel f_\varepsilon^{(2j)} \parallel_\infty^{1/2} \Big)\, e^{(Re\lambda)} \\ &\leq \big(2\pi c^{1/2}\, c_2^{1/2}\, \varepsilon^{-m_1/2}\varepsilon^{m_2/2}\big)\, e^{(Re\lambda)}. \end{aligned}$$

Since the term between bracket in the right-hand side of the last inequality is a constant which does not depend on the variable s, so the right term is in $L^1(-\infty, 0)$, furthermore $Re\,\lambda > 0$, and the cost we can apply the Lebesque dominated convergence theorem, sicnce g_ε is bounded and measurable function on $\mathbf{R}$, we get

$$\lim_{k\to+\infty} u_{1,\varepsilon}(t+s_{n_k}) = -\int_{-\infty}^{0} e^{\lambda\, s}\, g_\varepsilon(t-s)ds := y_\varepsilon(t) \quad for\ each\ t\in\mathbf{R},$$

defining a function $y_\varepsilon(t)$, and we apply the same reasoning to obtain

$$\lim_{k\to+\infty} y_\varepsilon(t-s_{n_k}) = -\int_{-\infty}^{0} e^{\lambda\, s}\, g_\varepsilon(t-s)ds := u_{1,\varepsilon}(t) \quad for\ each\ t\in\mathbf{R},$$

which prove that $u_1(t) = [(u_{1,\varepsilon}(t))_\varepsilon]$ is an almost automorphic generalized function. □

4.2. The case A is the infinitesimal generator of a C_0-group of operators

In this paragraph we will need the notion of generalized semigroup, which is defined in [9] by

Definition 4.3. *$\mathcal{SX}_M([0,\infty):\mathcal{L}(X))$ is the space of nets $(T_\varepsilon)_\varepsilon$ of strongly continuous mappings $T_\varepsilon:[0,\infty)\to\mathcal{L}(X), \varepsilon\in(0,1)$ with the property that for every $T>0$ there exists $a\in\mathbf{R}$ such that*

$$\sup_{t\in[0,T)} \|T_\varepsilon(t)\| = \mathcal{O}\left(\varepsilon^a\right), \quad as\ \varepsilon\to 0. \tag{4.2}$$

where X is a Banach algebra.

$\mathcal{S}N([0,\infty):\mathcal{L}(X))$ is the space of nets $(N_\varepsilon)_\varepsilon$ of strongly continuous mappings $N_\varepsilon:[0,\infty)\to\mathcal{L}(X), \varepsilon\in(0,1)$, with the properties: For every $b\in\mathbf{R}$ and $T>0$

$$\sup_{t\in[0,T)} \|N_\varepsilon(t)\| = \mathcal{O}\left(\varepsilon^b\right),\ as\ \varepsilon\to 0. \tag{4.3}$$

There exist $t_0>0$ and $a\in\mathbf{R}$ such that

$$\sup_{t<t_0} \left\|\frac{N_\varepsilon(t)}{t}\right\| = \mathcal{O}\left(\varepsilon^a\right). \tag{4.4}$$

There exists a net $(H_\varepsilon)_\varepsilon$ in $\mathcal{L}(X)$ and $\varepsilon_0\in(0,1)$ such that

$$\lim_{t\to 0}\frac{N_\varepsilon(t)}{t}x = H_\varepsilon x, x\in E, \varepsilon<\varepsilon_0. \tag{4.5}$$

For every $b>0$,

$$\|H_\varepsilon\| = \mathcal{O}\left(\varepsilon^b\right), \quad as \quad \varepsilon\to 0. \tag{4.6}$$

Now we define Colombeau type algebra as the factor algebra

$$\mathcal{SG}([0,\infty):\mathcal{L}(X)) = \mathcal{SX}_M([0,\infty):\mathcal{L}(X))/\mathcal{SN}([0,\infty):\mathcal{L}(X)).$$

$\mathcal{SG}([0,\infty):\mathcal{L}(X))$ called the algebra of Colombeau C_0 semigroup for more details see ([9]).

We assume that $f\in\mathcal{G}_{aa}$ and A generates a generalized C_0-group of linear operators $T(t) = [(T_\varepsilon(t))_{t\in\mathbf{R}}]$. Let us first recall the following definition

Definition 4.4. *A function $u(t) = [(u_\varepsilon(t))]\in\mathcal{G}_{aa}$ with the integral representation*

$$u_\varepsilon(t) = T_\varepsilon(t)u(0) + \int_0^t T_\varepsilon(t-s)f_\varepsilon(s)ds \quad \forall\varepsilon\in I. \tag{4.7}$$

will be said mild solution of the diferential equation (4.1).

Theorem 4.5. *Assume that any representative* $(T_\varepsilon(t))$ *of the generalized* C_0-*group* $(T(t))_{t\in\mathbf{R}}$ *satisfy* $T_\varepsilon(t)\,x$ *is an almost automorphic function from* $\mathbf{R}$ *to a Banach algebra* X*. and that* $f=[(f_\varepsilon)_\varepsilon]\in\mathcal{G}(\mathbf{R})$*, and whose representative* $(f_\varepsilon)_{\varepsilon\in I}$ *in* $\mathcal{G}(\mathbf{R})$ *is an element of* $L^1(\mathbf{J};X)$*.*
Then every mild solution of (4.1) restricted to $\mathbf{J}$ *is asymptotically almost automorphic.*

Proof. Let $x_\varepsilon(t)=T_\varepsilon(t)u(0)+\int_0^t T_\varepsilon(t-s)f_\varepsilon(s)ds \quad \forall\varepsilon\in I$ be a representative of the mild solution of (4.1) and consider $v_\varepsilon(t):\mathbf{R}\longrightarrow X$ defined by

$$v_\varepsilon(t)=\int_t^{+\infty}T_\varepsilon(t-s)f_\varepsilon(s)ds.$$

$v_\varepsilon(t)$ is well-defined and continuous on $\mathbf{R}^+$, and $\lim\limits_{t\to\infty}\parallel v_\varepsilon(t)\parallel=0$. Indeed, since $(T(t))_{t\in\mathbf{R}}$ is a C_0-group we have

$$\sup_{t\in\mathbf{R}}\parallel T_\varepsilon(t)\parallel<\infty,$$

using the uniform boundedness principle. Hence,

$$\parallel v_\varepsilon(t)\parallel\leq M\int_t^{+\infty}\parallel f_\varepsilon(s)\parallel ds\longrightarrow 0,\quad as\ t\longrightarrow\infty.$$

And it is oubvious that v_ε is continous on $\mathbf{R}$.
on the other hand the function $u(t)$

$$\begin{aligned}u_\varepsilon(t) &= T_\varepsilon(t)\,x(0)+\int_0^\infty T_\varepsilon(t-s)f_\varepsilon(s)ds\\ &= T_\varepsilon(t)\big(x(0)+\int_0^\infty T_\varepsilon(-s)f_\varepsilon(s)ds.\end{aligned}$$

is almost automorphic since $T_\varepsilon(-t)f(t):\mathbf{R}\longrightarrow X$ is a continuous function and

$$\int_0^\infty\parallel T_\varepsilon(-s)\,f_\varepsilon(s)\parallel ds\leq M\int_0^\infty\parallel f(s)\parallel ds.$$

Therefore $\int_0^\infty T_\varepsilon(-s)\,f(s)ds$ exists in X. Now we observe that $x(t)=u(t)+v(t)\quad\forall t\in\mathbf{J}$, so that $x(t)$ is asymptotically almost automorphic. □

Corollary 4.6. *In last theorem, if in addition we assume that the* A *is an invertible operator, then we have the uniqueness of the almost automorphic generalized solution.*

Definition 4.7. *we say in the classical case that a* C_0*-semigroup* $T(t)_{t\in\mathbf{R}^+}$ *is exponentially stable if there exist* $M\geq 1$ *and* $\alpha>0$ *such that*

$$\parallel T(t)\parallel M\,e^{-\alpha\,t}\ \ for\ every\ t\geq 0. \tag{4.8}$$

Theorem 4.8. *Assume that* A *is the infinitesimal generator of a generalized* C_0*-semigroup exponentially stable and* f *is an almost automorphic generalized function Then 4.1 has a unique almost automorphic generalized mild solution.*

Proof. Let $u_\varepsilon^{'}(t)=A_\varepsilon u_\varepsilon(t)-f_\varepsilon(t)$ be the regularization of the equation 4.1 in terms of representatives, in the algebra $\mathcal{M}_{aa}$. We know that this equation admits a mild solution defined by

$$x_\varepsilon(t)=T_\varepsilon(t-s)\,x_\varepsilon(t_0)+\int_{t_0}^t T_\varepsilon(t-s)\,f_\varepsilon(s)ds\qquad t\geq t_0.$$

We have to prove that its a well defined element of the algebra $\mathcal{M}_{aa}$, and if we take another solution y_ε we will prove that $(x_\varepsilon(t)-y_\varepsilon(t))_\varepsilon$ is an element of $\mathcal{N}_{aa}$.

First we will prove that x_ε is almost automorphic. Indeed
Let us consider the function $u_\varepsilon(t)=\int_{-\infty}^{t}T_\varepsilon(t-s)\,f_\varepsilon(s)ds$ Thank's to the property 4.8, we obtain

$$\begin{aligned}\| u_\varepsilon(t) \| &= \| \int_{-\infty}^{t} T_\varepsilon(t-s)f_\varepsilon(s)ds \| \\ &\leq M \| \int_{-\infty}^{t} exp(\alpha(t-s))f_\varepsilon(s)ds \| \\ &\leq M \| f_\varepsilon \|_\infty C_\alpha,\end{aligned}$$

where C_α is a positive constant depends on α, which proves that the integral

$$\int_{-\infty}^{t} T_\varepsilon(t-s)f_\varepsilon(s)ds.$$

is absolutely convergent, therefore it converges. And thus $u(t)$ exists and

$$\| u_\varepsilon(t) \| \leq M \| f_\varepsilon \|_\infty C_\alpha \qquad t\geq 0.$$

Now, let $(\tau_n^{'})_n$ be a sequence of real numbers. Since $f\in\mathcal{G}_{aa}$, there is a subsequence $(\tau_n)_n$ of $(\tau_n^{'})_n$ such that

$$\lim_{n\to\infty} f_\varepsilon\left(t+\tau_n\right)=\bar{f}_\varepsilon(t).$$

is well defined for every $t\in\mathbf{R}$, and we have

$$\lim_{t\to\infty} \bar{f}_\varepsilon\left(t-\tau_n\right)=f_\varepsilon(t), \quad \forall t\in\mathbf{R}.$$

We consider

$$\begin{aligned}u_\varepsilon\left(t+\tau_n\right) &= \int_{-\infty}^{t+\tau_n} T_\varepsilon\left(t+\tau_n-(s)\right)f_\varepsilon(s)ds \\ &= \int_{-\infty}^{t+\tau_n} T_\varepsilon\left(t+\tau_n-(r+\tau_n)\right)f_\varepsilon\left(r+\tau_n\right)dr \\ &= \int_{-\infty}^{t} T_\varepsilon(t-r)f_\varepsilon\left(r+\tau_n\right)dr \\ &= \int_{-\infty}^{t} T_\varepsilon(t-r)f_{\varepsilon,n}(r)dr,\end{aligned}$$

where $f_{\varepsilon,n}(r)=f_\varepsilon\left(r+\tau_n\right), n=0,1,\cdots$. We have also,

$$\|u_\varepsilon\left(t+\tau_n\right)\| \leq K\|f_\varepsilon\|_\infty C_\alpha, \quad \forall n\in\mathbf{N},$$

by the continuity of the semigroup,

$$\lim_{n\to\infty} T_\varepsilon(t-r)f_{\varepsilon,n}(r)=T_\varepsilon(t-r)\bar{f}_\varepsilon(r), \quad \forall r\in\mathbf{R} \ and \ r\leq t.$$

Let

$$\bar{u}_\varepsilon(t)=\int_{-\infty}^{t} T_\varepsilon(t-s)\,\bar{u}_\varepsilon(s)ds,$$

we observe that the integral is absolutely convergent for each $t\in\mathbf{R}$. Therefore, according to Lebesgue's dominated convergence theorem, we have

$$\lim_{n\longrightarrow\infty} u_\varepsilon(t+\tau)=\bar{u}_\varepsilon(t) \qquad \forall t\in\mathbf{R}.$$

Similarly, we can prove that

$$\lim_{n\longrightarrow\infty} \bar{u}_\varepsilon(t-\tau) = u_\varepsilon(t) \qquad \forall t \in \mathbf{R}.$$

This proves that $u_\varepsilon(t)$ is almost automorphic function.
Now let $a \in \mathbf{R}$ be given. So we have

$$u_\varepsilon(a) = \int_{-\infty}^{a} T_\varepsilon(a-s)\, f_\varepsilon(s)ds.$$

Then, for every $t \geq$, we by using the property of continuity of semigroup,

$$T_\varepsilon(t-a)\, u_\varepsilon(a) = \int_{-\infty}^{a} T_\varepsilon(a-s)\, f_\varepsilon(s)ds,$$

and thus

$$\begin{aligned}\int_a^t T\varepsilon(t-s)\, f_\varepsilon(s)ds &= \int_{-\infty}^{t} T\varepsilon(t-s)\, f_\varepsilon(s)ds - \int_{-\infty}^{a} T\varepsilon(t-s)\, f_\varepsilon(s)ds \\ &= u_\varepsilon(t) - T_\varepsilon(t-a)\, u_\varepsilon(a).\end{aligned}$$

Hence

$$u_\varepsilon(t) = T_\varepsilon(t-a)\, u_\varepsilon(a) + \int_a^t T_\varepsilon(t-s)\, f_\varepsilon(s)ds.$$

If we take $x(a) = u(a)$, we get $x(t) = u(t)$, then $(x_\varepsilon)_\varepsilon \in \mathcal{M}_{aa}$. It remains to prove the uniqueness of the generalized mild solution in $\mathcal{G}_{aa}$. Let $x = (x_\varepsilon)_\varepsilon + \mathcal{N}_{aa}$ and $y = (y_\varepsilon)_\varepsilon + \mathcal{N}_{aa}$ are two mild solutions of the equation 4.1, and assume that both generalized solutions are almost automorphic, let $z = x - y$. Then $z = [(z_\varepsilon)_\varepsilon] = (x_\varepsilon - y_\varepsilon)_\varepsilon + \mathcal{N}_{aa}$ is a well defined element of $\mathcal{G}_{aa}$, and satisfied

$$\frac{d}{dt}u(t) = A\, u(t).$$

Then we have $z_\varepsilon(t) = T_\varepsilon(t-s)w_\varepsilon(s) \qquad \forall t, s \in \mathbf{R} \quad t \geq s$. Also we hae

$$\| z_\varepsilon(t) \| \leq M\, e^{-\alpha(t-s)} \| z_\varepsilon(s) \| \qquad \forall t \geq s.$$

By the elementary properties of the exponential yields.

$$\lim_{n\longrightarrow\infty} \| z_\varepsilon(n\tau) \| = 0, \;\; for \;\; all\; \tau > 0.$$

Hence, for all $q \in \mathbf{N}$, there is a constant $c > 0$ such that $\| z_\varepsilon(t) \| \leq c\,\varepsilon^q$ as ε, then $(z_\varepsilon)_\varepsilon$, is negligeable element, because if $t \longmapsto T_\varepsilon(t)x_0,\;\; x_0 \in \mathbf{R}$ is almost automorphic, then we have the following alternative

$$\inf_{t\in\mathbf{R}} \| x(t) \| > 0, \;\; or \;\; x(t) = 0, \; \forall t \in \mathbf{R}.$$

Therefore $x = y$ in $\mathcal{G}_{aa}$. □

4.3. Example

We consider the differential equation

$$\left(\frac{d}{dt} + \lambda t\right) u(t) = 0. \tag{4.9}$$

Where $u(t) = [(u_\varepsilon(t))_\varepsilon] \in \mathcal{G}((R))$, and $\lambda = [(2/\varepsilon)] \in \tilde{\mathbf{C}}$ is a complex real number. It is well known that the Gaussian functions

$$g_{\varepsilon,s}(t) = \frac{1}{\sqrt{\pi s}} e^{-t^2/\varepsilon}, \quad t \in \mathbf{R}, \;\; and \;\; \varepsilon \in (0,1).$$

And let φ_ε the test function defined by

$$\varphi_\varepsilon(.) = \frac{1}{\varepsilon}\varphi(.),$$

where φ is a smooth function with compact support. The function $g_\varepsilon(t) * \varphi_\varepsilon(t)$ satisfy the analogous equation

$$\left(\frac{d}{dt} + \lambda t\right) u_\varepsilon(x) = 0,$$

on the real line $\mathbf{R}$. Therefore in seeking a solution to equation (4.9) we consider the functions

$$u_{\varepsilon,s}(t) = \sum_{k=-\infty}^{\infty} g_{\varepsilon,s}(t + 2\pi k),$$

that are parameterised by ε, to which we apply the operator $\frac{d}{dt} + \frac{2}{\varepsilon}t$ and obtain

$$\begin{aligned} U_{\varepsilon,s}(t) &= \left(\frac{d}{dt} + \frac{2}{\varepsilon}t\right) u_{\varepsilon,s}(t) = \sum_{k=-\infty}^{\infty} \left(\frac{d}{dt} + \frac{2}{\varepsilon}t\right) g_{\varepsilon,s}(t + 2\pi k) \\ &= \frac{1}{\sqrt{\pi s}} \sum_{k=-\infty}^{\infty} \left(-\frac{2}{\varepsilon}t - \frac{4}{\varepsilon}\pi k + \frac{2}{\varepsilon}t\right) e^{-(t+2\pi k)^2/\varepsilon} \\ &= -\frac{4\sqrt{\pi}}{\varepsilon\sqrt{s}} \sum_{k=-\infty}^{\infty} k e^{-(t+2\pi k)^2/\varepsilon}. \end{aligned}$$

Hence the function $U_{\varepsilon;s}(t)$ is bounded above by the estimate,

$$|U_{\varepsilon,s}(t)| \leq \frac{4\sqrt{\pi}}{\varepsilon\sqrt{s}} \left(\left| \sum_{k=-\infty}^{-1} k e^{-(t+2\pi k)^2/\varepsilon} \right| + \sum_{k=1}^{\infty} \left| k e^{-(t+2\pi k)^2/\varepsilon} \right| \right),$$

and on the interval $t \in [-\pi, \pi]$

$$|U_{\varepsilon,s}(t)| \leq \frac{8\sqrt{\pi}}{\varepsilon\sqrt{\varepsilon}} \sum_{k=1}^{\infty} k e^{-\pi^2(2k-1)^2/\varepsilon} = \frac{8\sqrt{\pi}}{\varepsilon\sqrt{\varepsilon}} e^{-\pi^2/\varepsilon} \sum_{k=1}^{\infty} k \left(e^{-\pi^2/s}\right)^{4k(k-1)},$$

where the first few terms in the final sum are given explicitly in the square brackets of the relation

$$|U_{\varepsilon,s}(t)| \leq \frac{8\sqrt{\pi}}{\varepsilon\sqrt{\varepsilon}} e^{-\pi^2/\varepsilon} \left[1 + 2\left(e^{-\pi^2/\varepsilon}\right)^8 + 3\left(e^{-\pi^2/\varepsilon}\right)^{24} + \ldots\ldots\right] = \mathcal{O}(\varepsilon^{-C}), \quad as \quad \varepsilon \to 0 \; with \;\; C > 0,$$

which implies that $(U_{\varepsilon,s}(t))$ is an element of the moderate space $\mathcal{M}_{aa}$. And therefore $u(t)$ is an almost automorphic generalized function, $u(t) \in \mathcal{G}_{aa}$.

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