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Hausdorff of a Cycle in Topological Graph

Ali Ameer Jabor and Ahmed Abd-Ali Omran

ABSTRACT: The main purpose of the paper is to bring together two areas in which strong relation, graph theory and topological spaces. And derive interesting formula for the set that contains all minimal dominating sets(MDS) and (γ-set). Some separation axioms are discussed in topological graph theory, especially in a cycle graph C_n.

Key Words: Minimal dominating sets (MDS), T_0-MDS graph, T_1-MDS graph, T_2-MDS graph.

Contents

1. Introduction

In graph theory a simple graph G is a non-empty finite set V(G) of elements called vertices (or nodes), and a finite set E(G) of distinct unordered pairs of distinct elements of V(G) called edges[1]. A set S ⊆V of vertices in graph G=(V,E) is called dominating set if every vertex v∈V is either an element of S or is adjacent to an element of S , S is called minimal set if no proper subset S” ⊆S is a dominating set [2]. Smallest cardinality of minimal dominating sets called minimum dominating set(γ-set).
Almost previous research papers studied the relation among vertices and edges in dominating set. But not take care the family of dominating sets and relations between any two of them. Although, this will give us image of how many time will get dominating set in the graph. Despite the importance of this research.But benefit of minimum dominating set is not possible in sometime especially in real life. So must be interest the family of all minimal dominating sets . In this paper we take the family of all minimal dominating sets of a graph G. And study some topological relation on that family. For that introduced some new definitions such T_0-MDS graph ,T_1-MDS graph , T_2-MDS graph. Also use minimum dominating sets (γ-sets) to satisfy hausdorff axiom of cycle C_n.

2. Definitions

Definition 2.1. *Minimal dominating set[2]: A set S ⊆V of vertices in graph G=(V,E) is called dominating set if every vertex v∈V is either an element of S or is adjacent to an element of S , S is called minimal dominating set if no proper subset S”⊆S is a dominating set.*

Definition 2.2. *T_0-topology[4]: A topological space X satisfy T_0 axiom if each one of any two points of X has a neighbourhood that dose not contain the other point. more formally* $\forall$ *x,y*$\in$*X , x* $\neq$*y* $\exists$ U_y : $x \notin U_y$, $y \in U_y$.

Definition 2.3. *T_1-topology [5]:A topological space X is said to be T_1 if for any two distinct points x and y of X there is neighbourhood of x which dose not contain y and neighbourhood of y which dose not contain x.*

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Definition 2.4. *T_2-topology (Hausdorff Topology)[5]:A topological space X is said to be T_2 if given any two distinct point x and y of X , there are open sets U and V such that $x \in U$, $y \in V$, and $U \bigcap V = \phi$.*

3. New Definitions

Introduce some new definitions and theorems related with that definitions:

Definition 3.1. *T_0-MDS: A graph G is called T_0-MDS if $\forall$ u,v $\in$ G $\exists$ D minimal dominating set such that $u \in D$, $v \notin D$ or $u \notin D$, $v \in D$.*

Definition 3.2. *T_1-MDS :A graph G is called T_1-MDS if $\forall$ u,v $\in$ G $\exists$ D_1,D_2 minimal dominating sets such that $u \in D_1$, $v \notin D_1$ and $v \in D_2$, $u \notin D_2$.*

Definition 3.3. *T_2-MDS (Hausdorff-MDS):A graph G is called T_2-MDS if $\forall$ u,v $\in$ G $\exists$ D_1,D_2 disjoint minimal dominating sets such that $u \in D_1$ and $v \in D_2$.*

4. Main Results

Theorem 4.1. *If a graph G is T_i-MDS then it is T_j-MDS when $i \geq j$.*

Proof. Let G is T_2-MDS then for all two distinct vertices u , v there are two disjoint MDS D_1, D_2 such that $u \in D_1$ and $v \in D_2$. To prove G is T_1-MDS it is clearly for all two vertices u , v there are two MDS D_1, D_2 such that $u \in D_1$, $v \notin D_1$ and $v \in D_2$, $u \notin D_2$, thus G is T_1-MDS. Similarly to prove G is T_0-MDS for all two vertices u , v there are two MDS D_1, D_2 such that $u \in D_1$, $v \notin D_1$, thus G is T_0-MDS. □

Proposition 4.1. *Converse of theorem 4.1 is not true.*

Proof. We will prove by counter example. A graph in figure 1 is not T_1-MDS but it is T_0-MDS,and to show that take the family of all MDS of G as follows:
$D_1 = \{v_1,v_4\}$ $D_2 = \{v_2,v_4\}$ $D_3 = \{v_3,v_4\}$,since v_4 belong to all MDS there is no MDS contains v_1 and not contains v_4 thus G is not G T_1-MDS graph and clearly G is T_0-MDS graph. □

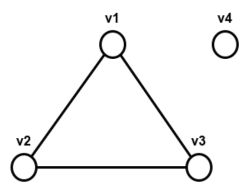

Figure 1: T_0-MDS but not T_1-MDS

Proposition 4.2. *If G be a graph and $|V(G)| \geq 3$ and G has only two disjoint MDS. then G is not T_0-MDS graph.*

Proof. Since G has only two disjoint MDS. (say D_1 , D_2 and $D_1 \cap D_2 = \phi$). Since $V(G) \geq 3$, at least two of these vertices belong to one MDS.thus exist two vertices belong to only one MDS. Hence G is not T_0-MDS graph. □

Remark 4.2. *S_3 is not T_0-MDS graph.*

Proof. Since S_3 has only two disjoint MDS and order of it equal 3, thus by proposition 4.2 we get the result. □

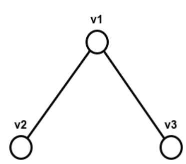

Figure 2: S_3 is not T_0-MDS.

Proposition 4.3. *For any graph G with $|V(G)| \geq 2$ and G has isolated vertex then G is not T_i-MDS graph , i=1,2.*

Proof. Let u be isolated vertex of graph G , u belong to all MDS of G. Since $|V(G)| \geq 2$, G contains more than one vertex.Then exist another vertex (say v) such that u , v $\in$ D MDS. And there is no MDS contains one of them. Hence G is not T_1-MDS graph. □

In the following theorem we deal with minimum dominating sets (γ-sets) in stead of minimal dominating sets (MDS).

Theorem 4.3. *If G is a cycle of order n, n≥4 then G is not T_0-MDS graph if n≡0(mod3) otherwise G is T_2-MDS graph.*

Proof. There are three cases depend on n modulo three as follows:

Case(1): If n≡ 0(mod3)

In this case there are exactly three γ-set D_0, D_1, D_2 where D_i contains all vertices that labelled equivalent to i modulo n (i=0,1,2)

it is clear that $D_i \cap D_j = \phi$ $\forall$ i≠j and $\forall$ D_i , i=0,1,2 contains at least two different vertices say u≠v . so, there is no open set contains u and not contains v.

Thus, C_n is not T_0-MDS and then not T_1-MDS and not T_2-MDS.

Case(2): If n≡ 1(mod3

Let v_i , v_j be two different vertices of C_n such that i<j, thus we have three cases depend on distance between v_i and v_j modulo 3 as follows:

(i) If d(v_i, v_j)= m ≡1(mod3)

There are three sub-cases depend on the index of v_i modulo 3 as follows:

(a)If i≡ 0(mod3)

We can take γ-sets D_i, D_j of G as follows :

$D_i = \{ v_i, v_{(i+2+3k)(modN)} \mid k=0,1,...,\lceil \frac{n}{3} \rceil-2 \}$

$D_j = \{v_j, v_{(j+2+3k)(modN)} \mid k=0,1,...,\frac{n-m}{3}-1\} \cup \{ v_{(i+1+3k)(modN)} \mid k=0,1,...,\frac{m-1}{3}-1\}$

now to prove $D_i \cap D_j = \phi$

let $v_k \in D_i$ we have the following cases :

if $0 \le k < i$ then $k \equiv 1(mod3)$
if $i < k < j$ then $k \equiv 2(mod3)$
if $j < k < 0$ then $k \equiv 2(mod3)$

let $v_k \in D_j$ we have the following cases :

if $0 \le k < i$ then $k \equiv 2(mod3)$
if $i < k < j$ then $k \equiv 1(mod3)$
if $j < k < 0$ then $k \equiv 0(mod3)$

Thus, from each cases to position of vertices above as shown in figure 3 . There is no any vertex in intersection, that means $D_i \cap D_j = \phi$.

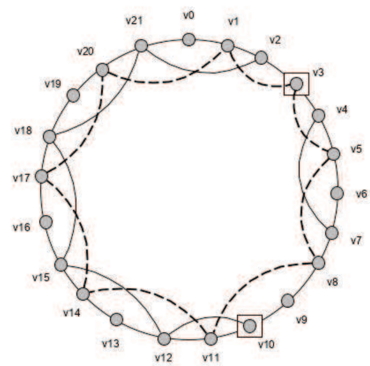

Figure 3: n=22, i=3, j=10, d(v_i,v_j)=7

(b)If i≡ 1(mod3)

We can take γ-sets D_i, D_j of G as follows :

$D_i = \{ v_i , v_{(i+2+3k)(modN)} \mid k=0,1,...,\lceil \frac{n}{3} \rceil-2 \}$

$D_j = \{ v_j , v_{(j+2+3k)(modN)} \mid k=0,1,...,\frac{n-m}{3}-1 \} \cup \{ v_{(i+1+3k)(modN)} \mid k=0,1,...,\frac{m-1}{3}-1\}$

now to prove $D_i \cap D_j = \phi$

let $v_k \in D_i$ we have the following cases :

if $0 \le k<i$ then $k \equiv 2(mod3)$
if $i < k < j$ then $k \equiv 0(mod3)$
if $j < k < 0$ then $k \equiv 0(mod3)$

let $v_k \in D_j$ we have the following cases :

if $0 \le k < i$ then $k \equiv 0(mod3)$
if $i < k < j$ then $k \equiv 2(mod3)$
if $j < k < 0$ then $k \equiv 1(mod3)$

Thus, from each cases to position of vertices above as shown in figure 4. There is no any vertex in intersection,that means $D_i \cap D_j = \phi$.

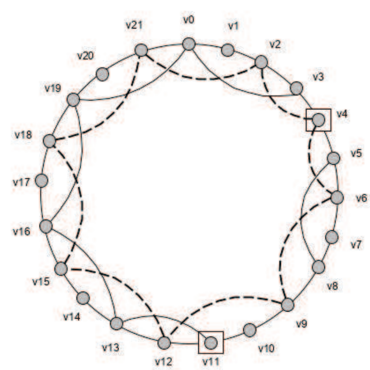

Figure 4: n=22, i=4, j=11, d(v_i,v_j)=7

(c) If i≡ 2(mod3)

We can take γ-sets D_i, D_j of G as follows :
$D_i = \{ v_i , v_{(i+2+3k)(modN)} \mid k=0,1,...,\lceil \frac{n}{3} \rceil -2 \}$
$D_j = \{ v_j , v_{(j+2+3k)(modN)} \mid k=0,1,...,\frac{n-m}{3} -1 \} \cup \{ v_{(i+1+3k)(modN)} \mid k=0,1,...,\frac{m-1}{3} -1\}$
now to prove $D_i \cap D_j = \phi$
let $v_k \in D_i$ we have the following cases :
if $0 \leq k < i$ then $k \equiv 0(mod3)$
if $i < k < j$ then $k \equiv 1(mod3)$
if $j < k < 0$ then $k \equiv 1(mod3)$
let $v_k \in D_j$ we have the following cases :
if $0 \leq k < i$ then $k \equiv 1(mod3)$
if $i < k < j$ then $k \equiv 0(mod3)$
if $j < k < 0$ then $k \equiv 2(mod3)$
Thus, from each cases to position of vertices above as shown in figure 5.
There is no any vertex in intersection, that means $D_i \cap D_j = \phi$.

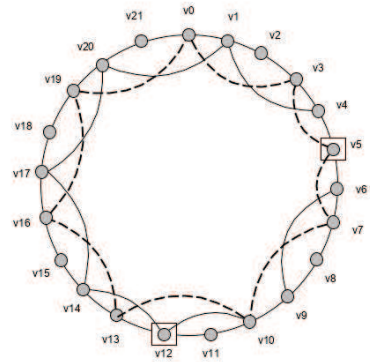

Figure 5: n=22, i=5, j=12, $d(v_i,v_j)=7$

(ii) If $d(v_i, v_j) = m \equiv 2(mod3)$
There are three sub-cases depend on the index of v_i modulo 3:
(a)If $i \equiv 0(mod3)$
We can take γ-sets D_i, D_j of G as follows :
$D_i = \{ v_{(i+3k)(modN)} \mid k=0,1,...,\lceil \frac{n}{3} \rceil -1 \}$
$D_j = \{ v_{(j+3k)(modN)} \mid k=0,1,...,\lceil \frac{n}{3} \rceil -1 \}$
now to prove $D_i \cap D_j = \phi$
let $v_k \in D_i$ we have the following cases :
if $0 \leq k < i$ then $k \equiv 2(mod3)$
if $i < k < j$ then $k \equiv 0(mod3)$
if $j < k < 0$ then $k \equiv 0(mod3)$
let $v_k \in D_j$ we have the following cases :
if $0 \leq k < i$ then $k \equiv 1(mod3)$
if $i < k < j$ then $k \equiv 1(mod3)$
if $j < k < 0$ then $k \equiv 2(mod3)$
Thus, from each cases to position of vertices above as shown in figure 6.
There is no any vertex in intersection, that means $D_i \cap D_j = \phi$.

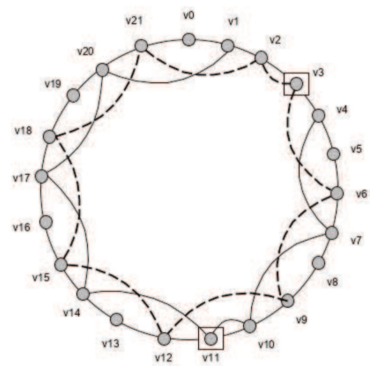

Figure 6: n=22, i=3, j=11, $d(v_i,v_j)=8$

(b)If $i \equiv 1(mod3)$
We can take γ-sets D_i, D_j of G as follows :
$D_i = \{ v_{(i+3k)(modN)} \mid k=0,1,...,\lceil \frac{n}{3} \rceil -1 \}$
$D_j = \{ v_{(j+3k)(modN)} \mid k=0,1,...,\lceil \frac{n}{3} \rceil -1 \}$
now to prove $D_i \cap D_j = \phi$

let $v_k \in D_i$ we have the following cases :
if $0 \le k < i$ then $k \equiv 0 (mod3)$
if $i < k < j$ then $k \equiv 1 (mod3)$
if $j < k < 0$ then $k \equiv 1 (mod3)$
let $v_k \in D_j$ we have the following cases :
if $0 \le k < i$ then $k \equiv 2 (mod3)$
if $i < k < j$ then $k \equiv 2 (mod3)$
if $j < k < 0$ then $k \equiv 0 (mod3)$
Thus, from each cases to position of vertices above as shown in figure 7.
There is no any vertex in intersection, that means $D_i \cap D_j = \phi$.

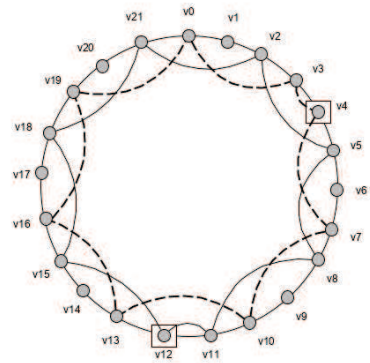

Figure 7: n=22, i=4, j=12, $d(v_i,v_j)$=8

(c)**If $i \equiv$ 2(mod3)**
We can take γ-sets D_i, D_j of G as follows :
$D_i = \{ v_{(i+3k)(modN)} \mid k=0,1,...,\lceil \frac{n}{3} \rceil -1 \}$
$D_j = \{ v_{(j+3k)(modN)} \mid k=0,1,...,\lceil \frac{n}{3} \rceil -1 \}$
now to prove $D_i \cap D_j = \phi$
let $v_k \in D_i$ we have the following cases :
if $0 \le k < i$ then $k \equiv 1 (mod3)$
if $i < k < j$ then $k \equiv 2 (mod3)$
if $j < k < 0$ then $k \equiv 2 (mod3)$
let $v_k \in D_j$ we have the following cases :
if $0 \le k < i$ then $k \equiv 0 (mod3)$
if $i < k < j$ then $k \equiv 0 (mod3)$
if $j < k < 0$ then $k \equiv 1 (mod3)$
Thus, from each cases to position of vertices above as shown in figure 8.
There is no any vertex in intersection, that means $D_i \cap D_j = \phi$.

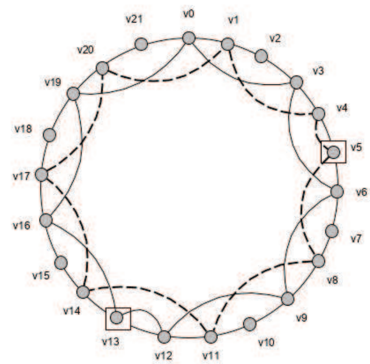

Figure 8: n=22, i=5, j=13, $d(v_i,v_j)$=8

(iii) If $d(v_i, v_j) = m \equiv 0(mod3)$
There are three sub-cases depend on the index of v_i modulo 3 :
(a)If $i \equiv$ 0(mod3)
We can take γ-sets D_i, D_j of G as follows :
$D_i = \{ v_i , v_{(i+2+3k)(modN)} \mid k=0,1,...,\lceil \frac{n}{3} \rceil -2 \}$
$D_j = \{ v_{(j+3k)(modN)} \mid k=0,1,...,\lceil \frac{n-m}{3} \rceil -1 \} \cup \{ v_{(i+1+3k)(modN)} \mid k=0,1,..., \frac{m}{3} -1 \}$
now to prove $D_i \cap D_j = \phi$
let $v_k \in D_i$ we have the following cases :
if $0 \le k < i$ then $k \equiv 1 (mod3)$
if $i < k < j$ then $k \equiv 2 (mod3)$
if $j < k < 0$ then $k \equiv 2 (mod3)$
let $v_k \in D_j$ we have the following cases :
if $0 \le k < i$ then $k \equiv 2 (mod3)$
if $i < k < j$ then $k \equiv 1 (mod3)$
if $j < k < 0$ then $k \equiv 0 (mod3)$
Thus, from each cases to position of vertices above as shown in figure 9.
There is no any vertex in intersection, that means $D_i \cap D_j = \phi$.

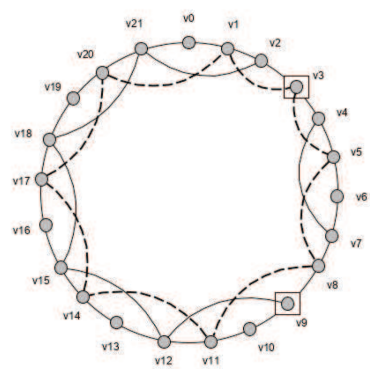

Figure 9: n=22, i=3, j=9, $d(v_i,v_j)$=6

(b)If i$\equiv$ 1(mod3)

We can take γ-sets D_i, D_j of G as follows :

$D_i = \{ v_i , v_{(i+1+3k)(modN)} |$ k=0,1,...,$\lceil \frac{n}{3} \rceil$-2 $\}$

$D_j = \{ v_{(j+3k)(modN)} |$ k=0,1,...,$\lceil \frac{n-m}{3} \rceil$ -1 $\} \cup \{ v_{(i+2+3k)(modN)} |$ k=0,1,..., $\frac{m}{3}$ -1 $\}$

now to prove $D_i \cap D_j = \phi$

let $v_k \in D_i$ we have the following cases :

if $0 \le k < i$ then $k \equiv 1(mod3)$

if $i < k < j$ then $k \equiv 2(mod3)$

if $j < k < 0$ then $k \equiv 2(mod3)$

let $v_k \in D_j$ we have the following cases :

if $0 \le k < i$ then $k \equiv 0(mod3)$

if $i < k < j$ then $k \equiv 0(mod3)$

if $j < k < 0$ then $k \equiv 1(mod3)$

Thus, from each cases to position of vertices above as shown in figure 10. There is no any vertex in intersection, that means $D_i \cap D_j = \phi$.

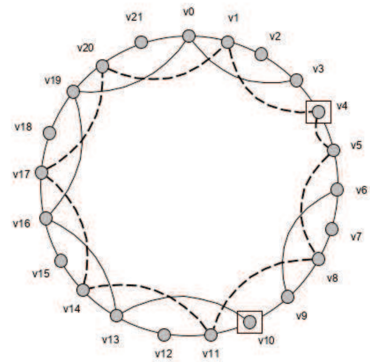

Figure 10: n=22,i=4,j=10, d(v_i,v_j)=6

(c)If i $\equiv$ 2(mod3)

We can take γ-sets D_i, D_j of G as follows :

$D_i = \{ v_i , v_{(i+1+3k)(modN)} |$ k=0,1,...,$\lceil \frac{n}{3} \rceil$-2 $\}$

$D_j = \{ v_{(j+3k)(modN)} |$ k=0,1,...,$\lceil \frac{n}{3} \rceil$ -1 $\}$

now to prove $D_i \cap D_j = \phi$

let $v_k \in D_i$ we have the following cases :

if $0 \le k < i$ then $k \equiv 2(mod3)$

if $i < k < j$ then $k \equiv 0(mod3)$

if $j < k < 0$ then $k \equiv 0(mod3)$

let $v_k \in D_j$ we have the following cases :

if $0 \le k < i$ then $k \equiv 1(mod3)$

if $i < k < j$ then $k \equiv 1(mod3)$

if $j < k < 0$ then $k \equiv 2(mod3)$

Thus, from each cases to position of vertices above as shown in figure 11. There is no any vertex in intersection, that means $D_i \cap D_j = \phi$.

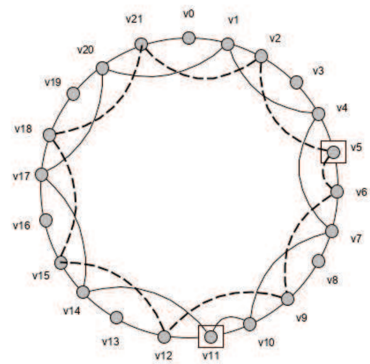

Figure 11: n=22,i=5,j=11, d(v_i,v_j)=6

Case(3): If n $\equiv$ 2(mod3)

Let v_i , v_j be and two different vertices of C_n such that i<j

thus we have three cases depend on distance between v_i and v_j modulo 3 as follows:

(i) If d(v_i, v_j)= m $\equiv$ 1(mod3)

There are three sub-cases depend on the index of v_i modulo 3 as follows:

(a) If i$\equiv$ 0(mod3)

We can take γ-sets D_i, D_j of G as follows :

$D_i = \{ v_{(i+3k)(modN)} |$ k=0,1,...,$\lceil \frac{n}{3} \rceil$-1 $\}$

$D_j = \{ v_{(j+3k)(modN)} |$ k=0,1,...,$\lceil \frac{n}{3} \rceil$ -2 $\}$

now to prove $D_i \cap D_j = \phi$

let $v_k \in D_i$ we have the following cases :
if $0 \leq k < i$ then $k \equiv 1(mod3)$
if $i < k < j$ then $k \equiv 0(mod3)$
if $j < k < 0$ then $k \equiv 0(mod3)$
let $v_k \in D_j$ we have the following cases :
if $0 \leq k < i$ then $k \equiv 2(mod3)$
if $i < k < j$ then $k \equiv 2(mod3)$
if $j < k < 0$ then $k \equiv 1(mod3)$
Thus, from each cases to position of vertices above as shown in figure 12. There is no any vertex in intersection, that means $D_i \cap D_j = \phi$.

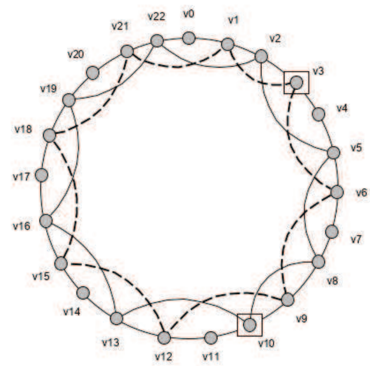

Figure 12: n=23,i=3,j=10, $d(v_i,v_j)=7$

(b) If i≡ 1(mod3)
We can take γ-sets D_i, D_j of G as follows :
$D_i = \{ v_{(i+3k)(modN)} \mid k=0,1,...,\lceil \frac{n}{3} \rceil -1 \}$
$D_j = \{ v_{(j+3k)(modN)} \mid k=0,1,...,\lceil \frac{n}{3} \rceil -1 \}$
now to prove $D_i \cap D_j = \phi$
let $v_k \in D_i$ we have the following cases :
if $0 \leq k < i$ then $k \equiv 2(mod3)$
if $i < k < j$ then $k \equiv 1(mod3)$
if $j < k < 0$ then $k \equiv 1(mod3)$
let $v_k \in D_j$ we have the following cases :
if $0 \leq k < i$ then $k \equiv 0(mod3)$
if $i < k < j$ then $k \equiv 0(mod3)$
if $j < k < 0$ then $k \equiv 2(mod3)$
Thus, from each cases to position of vertices above as shown in figure 13. There is no any vertex in intersection, that means $D_i \cap D_j = \phi$.

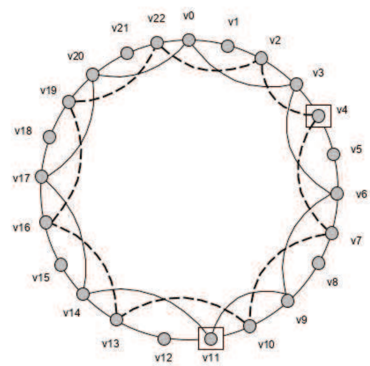

Figure 13: n=23,i=4,j=11, $d(v_i,v_j)=7$

(c) If i≡ 2(mod3)
We can take γ-sets D_i, D_j of G as follows :
$D_i = \{ v_{(i+3k)(modN)} \mid k=0,1,...,\lceil \frac{n}{3} \rceil -1 \}$
$D_j = \{ v_{(j+3k)(modN)} \mid k=0,1,...,\lceil \frac{n}{3} \rceil -1 \}$
now to prove $D_i \cap D_j = \phi$
let $v_k \in D_i$ we have the following cases :
if $0 \leq k < i$ then $k \equiv 0(mod3)$
if $i < k < j$ then $k \equiv 2(mod3)$
if $j < k < 0$ then $k \equiv 2(mod3)$
let $v_k \in D_j$ we have the following cases :
if $0 \leq k < i$ then $k \equiv 1(mod3)$
if $i < k < j$ then $k \equiv 1(mod3)$
if $j < k < 0$ then $k \equiv 0(mod3)$
Thus, from each cases to position of vertices above as shown in figure 14. There is no any vertex in intersection, that means $D_i \cap D_j = \phi$.

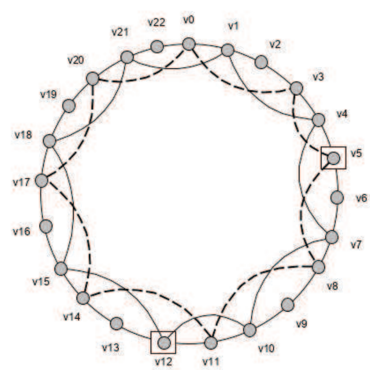

Figure 14: n=23,i=5,j=12, $d(v_i,v_j)=7$

(ii) If $d(v_i, v_j) = m \equiv 2(mod3)$
now we have three sub-cases depend on the index of v_i modulo 3 :

(a) If i≡ 0(mod3)
We can take γ-sets D_i, D_j of G as follows :
$D_i = \{ v_{(i+3k)(modN)} | k=0,1,...,\lceil \frac{n}{3} \rceil -1 \}$
$D_j = \{ v_{(j+2+3k)(modN)} | k=0,1,...,\lceil \frac{n}{3} \rceil -1 \}$
now to prove $D_i \cap D_j = \phi$
let $v_k \in D_i$ we have the following cases :
if $0 \leq k < i$ then $k \equiv 1(mod3)$
if $i < k < j$ then $k \equiv 0(mod3)$
if $j < k < 0$ then $k \equiv 0(mod3)$
let $v_k \in D_j$ we have the following cases :
if $0 \leq k < i$ then $k \equiv 2(mod3)$
if $i < k < j$ then $k \equiv 2(mod3)$
if $j < k < 0$ then $k \equiv 1(mod3)$
Thus, from each cases to position of vertices above as shown in figure 15. There is no any vertex in intersection, that means $D_i \cap D_j = \phi$.

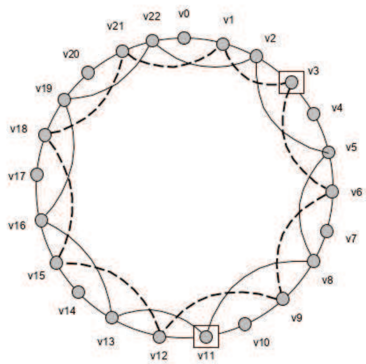

Figure 15: n=23,i=3,j=11, d(v_i,v_j)=8

(b) If i≡ 1(mod3)
We can take γ-sets D_i, D_j of G as follows :
$D_i = \{ v_{(i+3k)(modN)} | k=0,1,...,\lceil \frac{n}{3} \rceil -1 \}$
$D_j = \{ v_{(j+2+3k)(modN)} | k=0,1,...,\lceil \frac{n}{3} \rceil -1 \}$
now to prove $D_i \cap D_j = \phi$
let $v_k \in D_i$ we have the following cases :
if $0 \leq k < i$ then $k \equiv 2(mod3)$
if $i < k < j$ then $k \equiv 1(mod3)$
if $j < k < 0$ then $k \equiv 1(mod3)$
let $v_k \in D_j$ we have the following cases :
if $0 \leq k < i$ then $k \equiv 0(mod3)$
if $i < k < j$ then $k \equiv 0(mod3)$
if $j < k < 0$ then $k \equiv 2(mod3)$
Thus, from each cases to position of vertices above as shown in figure 16. There is no any vertex in intersection, that means $D_i \cap D_j = \phi$.

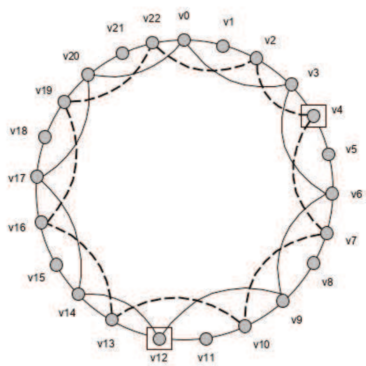

Figure 16: n=23,i=4,j=12, d(v_i,v_j)=8

(c) If i≡ 2(mod3)
We can take γ-sets D_i, D_j of G as follows :
$D_i = \{ v_{(i+3k)(modN)} | k=0,1,...,\lceil \frac{n}{3} \rceil -1 \}$
$D_j = \{ v_{(j+2+3k)(modN)} | k=0,1,...,\lceil \frac{n}{3} \rceil -1 \}$
now to prove $D_i \cap D_j = \phi$

let $v_k \in D_i$ we have the following cases :

if $0 \leq k < i$ then $k \equiv 0 (mod3)$
if $i < k < j$ then $k \equiv 2 (mod3)$
if $j < k < 0$ then $k \equiv 2 (mod3)$

let $v_k \in D_j$ we have the following cases :

if $0 \leq k < i$ then $k \equiv 1 (mod3)$
if $i < k < j$ then $k \equiv 1 (mod3)$
if $j < k < 0$ then $k \equiv 0 (mod3)$

Thus, from each cases to position of vertices above as shown in figure 17. There is no any vertex in intersection, that means $D_i \cap D_j = \phi$.

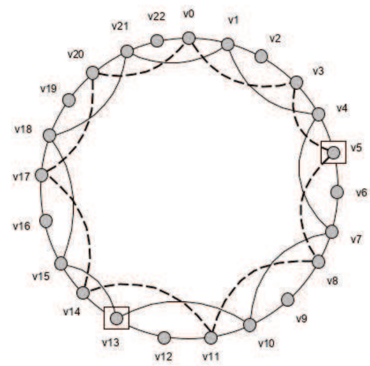

Figure 17: n=23,i=5,j=13, $d(v_i,v_j)=8$

(iii) If $d(v_i, v_j) = m \equiv 0(mod3)$

now we have three sub-cases depend on the index of v_i modulo 3 :

(a) If $i \equiv 0(mod3)$

We can take γ-sets D_i, D_j of G as follows :

$D_i = \{ v_i , v_{(i+2+3k)(modN)} \mid k=0,1,...,\lceil \frac{n}{3} \rceil -2 \}$

$D_j = \{ v_j , v_{(j+1+3k)(modN)} \mid k=0,1,...,\lceil \frac{n-m}{3} \rceil -1 \} \cup \{ v_{(i+1+3k)(modN)} \mid k=0,1,..., \frac{m}{3} -1 \}$

now to prove $D_i \cap D_j = \phi$

let $v_k \in D_i$ we have the following cases :

if $0 \leq k < i$ then $k \equiv 0 (mod3)$
if $i < k < j$ then $k \equiv 2 (mod3)$
if $j < k < 0$ then $k \equiv 2 (mod3)$

let $v_k \in D_j$ we have the following cases :

if $0 \leq k < i$ then $k \equiv 2 (mod3)$
if $i < k < j$ then $k \equiv 1 (mod3)$
if $j < k < 0$ then $k \equiv 1 (mod3)$

Thus, from each cases to position of vertices above as shown in figure 18. There is no any vertex in intersection, that means $D_i \cap D_j = \phi$.

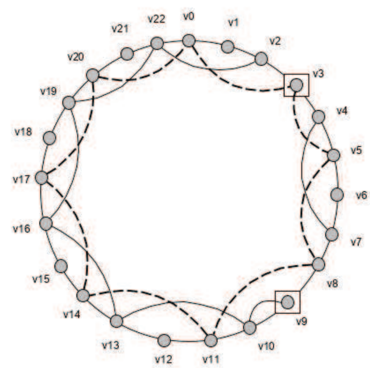

Figure 18: n=23,i=3,j=9, $d(v_i,v_j)=6$

(b) If $i \equiv 1(mod3)$

We can take γ-sets D_i, D_j of G as follows :

$D_i = \{ v_i , v_{(i+2+3k)(modN)} \mid k=0,1,...,\lceil \frac{n}{3} \rceil -2 \}$

$D_j = \{ v_j , v_{(j+1+3k)(modN)} \mid k=0,1,...,\lceil \frac{n-m}{3} \rceil -1 \} \cup \{ v_{(i+1+3k)(modN)} \mid k=0,1,..., \frac{m}{3} -1 \}$

now to prove $D_i \cap D_j = \phi$

let $v_k \in D_i$ we have the following cases :

if $0 \leq k < i$ then $k \equiv 1 (mod3)$
if $i < k < j$ then $k \equiv 0 (mod3)$
if $j < k < 0$ then $k \equiv 0 (mod3)$

let $v_k \in D_j$ we have the following cases :

if $0 \leq k < i$ then $k \equiv 0 (mod3)$
if $i < k < j$ then $k \equiv 2 (mod3)$
if $j < k < 0$ then $k \equiv 2 (mod3)$

Thus, from each cases to position of vertices above as shown in figure 19. There is no any vertex in intersection, that means $D_i \cap D_j = \phi$.

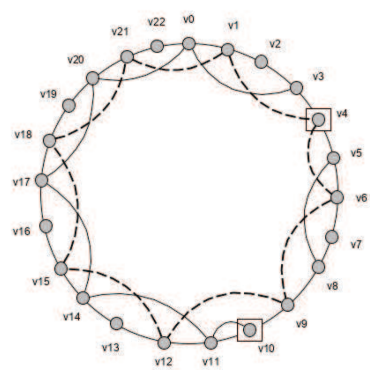

Figure 19: n=23,i=4,j=10, $d(v_i,v_j)=6$

(c) If i≡ 2(mod3)

We can take γ-sets D_i, D_j of G as follows :

$D_i = \{ v_i , v_{(i+2+3k)(modN)} \mid k=0,1,...,\lceil \frac{n}{3} \rceil -2 \}$

$D_j = \{ v_j , v_{(j+1+3k)(modN)} \mid k=0,1,...,\lceil \frac{n-m}{3} \rceil -1 \} \cup \{ v_{(i+1+3k)(modN)} \mid k=0,1,..., \frac{m}{3} -1 \}$

now to prove $D_i \cap D_j = \phi$

let $v_k \in D_i$ we have the following cases :

if $0 \leq k < i$ then $k \equiv 2(mod3)$

if $i < k < j$ then $k \equiv 1(mod3)$

if $j < k < 0$ then $k \equiv 1(mod3)$

let $v_k \in D_j$ we have the following cases :

if $0 \leq k < i$ then $k \equiv 1(mod3)$

if $i < k < j$ then $k \equiv 0(mod3)$

if $j < k < 0$ then $k \equiv 0(mod3)$

Thus, from each cases to position of vertices above as shown in figure 20. There is no any vertex in intersection, that means $D_i \cap D_j=\phi$.

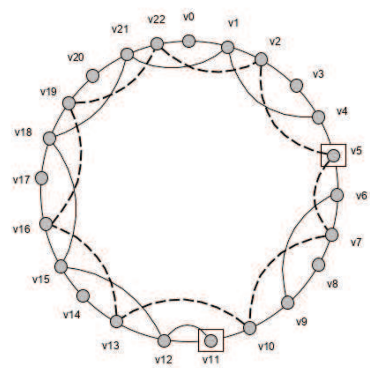

Figure 20: n=23,i=5,j=11, $d(v_i,v_j)=6$

from case(2) and case (3) we get for each two different vertices v_i and v_j there are two disjoint γ-sets D_i, D_j such that $v_i \in D_i$ and $v_j \in D_j$ thus C_n is T_2-MDS when n≡ 1(mod3) and n≡ 2(mod3) □

5. Conclusion

his paper discusses topological properties of the family dominating sets. And if the graph G satisfy T_0 property on that family is called T_0-MDS. In same manner G is called T_1-MDS , T_2-MDS if satisfy T_1 , T_2 properties respectively. And we give some condition on the graph to be T_0-MDS , T_1-MDS and T_2-MDS. Also we get the cycle C_n is not T_2-MDS, if $n \equiv 0(mod3)$.Otherwise, C_n is T_2-MDS when use γ-sets, but when using minimal dominating sets of cycle then C_n is T_2-MDS for all $n \geq 3$.

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Ali Ameer Jabor,
Department of Mathematics,Education College of Pure Sciences,
University of Babylon,
Iraq.
E-mail address: alialquzali89@gmail.com

and

Ahmed A. Omran,
Department of Mathematics,
College of Education for Pure Science, Babylon University,
Iraq.
E-mail address: pure.ahmed.omran@uobabylon.edu.iq