Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 2023 (41)** : 1–20.
ISSN-0037-8712 in press
doi:10.5269/bspm.51233

Advances in Additive Number Theory *

José Alfonso López Nicolás

ABSTRACT: We obtain sufficient conditions to know if given a positive even integer number and a set of positive integer numbers being all even or all odd, such a number can be expressed as sum of two elements of this set. As consequence we obtain a result which would prove Goldbach´s Conjecture for sequences with contractive distribution functions, provided that certain conditions are satisfied. These hypotheses, in the context of prime numbers, include Prime Consecutive Conjecture, which is a generalized form of Twin Prime Conjecture. In addition, we extend these results to sets of positive real numbers, even for two different sets. We also obtain a recurrent approximation of $\pi(x)$ for enough large $x \in \mathbb{R}$, being π the distribution function of the prime number set, which uses whichever expression of x as product of enough large factors. We also state this approximation in a more general context, give upper and lower bounds for the error, and show that this approximation is asymptotically equivalent to $\pi(x)$.

Key Words: Partition, prime number, distribution function, factorization.

Contents

1. Introduction

We investigate sufficient conditions to determine if given a positive even integer number and a set of positive integer numbers being all even or all odd, such a number can be expressed as sum of two elements of this set. In our research, for the particular case of the prime numbers set, we obtain a result which may be used to prove (Strong) Goldbach´s Conjecture for sequences with contractive distribution functions, stated in fact in a more general setting (see Corollary 2.2 and Corollary 7.2). One of the hypothesis of this result is the consecutive prime conjecture, which includes Twin Prime Conjecture as a particular case. In addition, we generalize these results to the general setting of positive real numbers, for sums of two elements of the same set and for the ones of two elements belonging every one of them to different sets.

We also obtain an approximation of the distribution function of prime numbers set using whatever factorization of the argument whenever its factors are enough large, and we study the quality of this

* The author would like to express his gratitude to the University of Murcia for its institutional support.
2010 *Mathematics Subject Classification:* 11A41, 11A25, 11P32, 11P82.
Submitted November 30, 2019. Published June 26, 2020

approximation under certain hypotheses. This approximation is also stated in a more general context. For the case consisting of two factors this approximation is:

$$\pi(x\,y) \approx \frac{x\,y\,\pi(x)\,\pi(y)}{x\,\pi(y) + y\,\pi(x)} \text{ for each } x,\ y \in \mathbb{R},\ x, y \geq k_0, \tag{1.1}$$

and it is based on

$$\pi(x) \approx \frac{x}{\log x},\ x \to +\infty.$$

We give upper and lower bounds for the error in the generalized approximation (1.1) for $m \geq 2$ factors, prove that this approximation is asymptotic to $\pi(x)$, and apply these results to approximate $\pi(x)$ when x is the product of prime numbers.

We establish some notation. We denote $\mathbb{Z}^+ := \{m \in \mathbb{Z} \mid m > 0\}$. Given $A \subseteq \mathbb{R}^n$, we denote the indicator function of A with respect to $\mathbb{R}^n$ by χ_A, and define $xA := \{a\,x \mid a \in A\}$, $A + B := \{a + b \mid a \in A,\ b \in B\}$ for every $B \subseteq \mathbb{R}^n$. Given $m \in \mathbb{N}$, $a_1, ..., a_m \in \mathbb{R}$, we denote their product by $Prod_{i=1}^m a_i$. We denote by $\mathbb{P}$ the set of prime numbers, and $\mathbb{P}^* := \mathbb{P} \setminus \{2\}$. We denote the number of elements of a set A by $Card(A)$. Given $x \in \mathbb{R}$ we denote by $E(x)$ its integer part. We also denote the set of positive real numbers by $\mathbb{R}^+$.

1.1. Definitions

We work with the concepts of distribution function and discriminant function of a set of natural numbers $\mathcal{P}$. We also use other concepts which help us to determine whether a given even natural number is or not a sum of two elements of $\mathcal{P}$.

Definition 1.1. *Let $\mathcal{P} = (p_i)_{i\in\mathbb{Z}^+}$ be a strictly increasing sequence in $\mathbb{Z}^+$.*

1. $a := p_1 = \min\{p_i \mid i \in \mathbb{Z}^+\} \in \mathbb{Z}^+$.
2. $r_0 := \min\{p_{i+1} - p_i \mid i \in \mathbb{Z}^+\} = \min\{q - p \mid p,\ q \in \mathcal{P},\ p < q\} \geq 1$.
3. $\mathbb{N}_{\geq a} := \{n \in \mathbb{N} \mid n \geq a\} \subseteq \mathbb{Z}^+$, $2\mathbb{N}_{\geq a} := \{2n \mid n \in \mathbb{N},\ n \geq a\}$.
4. *The function $\pi := \pi_{\mathcal{P}} : (0,\ +\infty) \to \mathbb{R}$ defined by*

$$\pi(x) := Card\left(\{p \in \mathcal{P} \mid p \leq x\}\right) = Card\left(\mathcal{P} \cap (0,\ x]\right),$$

for each $x \in (0,\ +\infty)$, is called the distribution function of $\mathcal{P}$. π is monotonically increasing.

Remark 1.2. *Observe these consequences of the definition of π:*

1. *For every $n \in \mathbb{Z}^+$ we have $\{p_1, ..., p_{\pi(n)}\} \subseteq [1,\ n]$, and then $p_{\pi(n)} \leq n$.*
2. $\left(p \leq n \Leftrightarrow p \leq p_{\pi(n)}\right)$ *for every $n \in \mathbb{Z}^+$, $p \in \mathcal{P}$.*

Definition 1.3. *Let $\mathcal{P} = (p_i)_{i\in\mathbb{Z}^+}$ be a strictly increasing sequence in $\mathbb{Z}^+$ such that $\mathcal{P} \subseteq 2\mathbb{N}$ or $\mathcal{P} \subseteq 2\mathbb{N}+1$. Then r_0 is even and $r_0 \geq 2$.*

1. *Let $k \in 2\mathbb{N}$, $k \geq r_0$. Define the function $\pi_k : (0,\ +\infty) \to \mathbb{R}$ by*

$$\pi_k(x) := Card\left(\{p \in \mathcal{P} \mid\ (p + k \leq x,\ p + k \in \mathcal{P})\}\right) \leq \pi(x),$$

for each $x \in (0,\ +\infty)$.

2. *Let $x \in (0,\ +\infty)$. $Dif\,(\mathcal{P})\,(x) := \{p_{i+1} - p_i\ :\ i \in \mathbb{Z}^+,\ p_{i+1} \leq x\}$ is called the difference set of $\mathcal{P}$ until x. Notice that*

$$Dif\,(\mathcal{P})\,(x) \subseteq 2\mathbb{N}_{\geq \frac{r_0}{2}} = \{2s \mid\ (s \in \mathbb{N},\ 2s \geq r_0)\} = \{r_0, r_0 + 2, r_0 + 4, ...\}.$$

Observe that if $\mathcal{P} = \mathbb{P}$, then π_2 is the distribution function of the twin prime numbers.

Using Definition 1.1 we can formulate (Strong) Goldbach´s Conjecture:

Conjecture 1.4 ((Strong) Goldbach´s Conjecture)**.**

$$2\mathbb{N}_{\geq 3} \subseteq \mathbb{P}^* + \mathbb{P}^*.$$

In 2014 Harald Andres Helfgott proved the Ternary (also called Odd) Goldbach´s Conjecture, which we can formulate as follows:

Theorem 1.5 (Ternary Goldbach´s Conjecture (see [7]))**.**

$$2\mathbb{N}_{\geq 4} + 1 \subseteq \mathbb{P}^* + \mathbb{P}^* + \mathbb{P}^*.$$

This is, every odd integer greater or equal than 9 is the sum of three odd prime numbers.

Let $\mathcal{P} = (p_i)_{i\in\mathbb{Z}^+}$ be a strictly increasing sequence in $\mathbb{Z}^+$. Let $m \in \mathbb{N}_{\geq a}$, where $a = p_1$. We wonder if $2m \in \mathcal{P} + \mathcal{P}$. Obviously, if $m \in \mathcal{P}$ (the trivial case), then $2m = m + m \in \mathcal{P} + \mathcal{P}$ and the answer is affirmative. The question is what happens if $m \notin \mathcal{P}$. Suppose that $2m \in \mathcal{P} + \mathcal{P}$, with $m \notin \mathcal{P}$. There exist $p,\ q \in \mathcal{P}$, with $p \leq q$, such that $2m = p + q$. Then $2m \geq 2p$ and, consequently, $p \leq m$, or what is equivalent, $p \leq p_{\pi(m)}$. Thus we have that $p \in \mathcal{P}$, $2m-p \in \mathcal{P}$ and $p \leq p_{\pi(m)}$. So these last three conditions altogether are equivalent to $2m \in \mathcal{P} + \mathcal{P}$ when $m \notin \mathcal{P}$ (of course it is also true for $m \in \mathcal{P}$).

Definition 1.6 (Discriminant function)**.** *Let $\mathcal{P} = (p_i)_{i\in\mathbb{Z}^+}$ be a strictly increasing sequence in $\mathbb{Z}^+$, and define $a := p_1$. The function $\psi : \mathbb{N}_{\geq a} \to \mathbb{N}$ defined by*

$$\psi(m) := \sum_{i=1}^{\pi(m)} \pi(2m - p_i) = \sum_{p\in\mathcal{P},\ p\leq m} \pi(2m - p) \text{ for all } m \in \mathbb{N}_{\geq a},$$

is called the discriminant function of $\mathcal{P}$.

1.2. Results

Our main results are the following theorems.

Theorem 1.7. *Let $\mathcal{P} = (p_i)_{i\in\mathbb{Z}^+}$ be a strictly increasing sequence in $\mathbb{Z}^+$ such that $\mathcal{P} \subseteq 2\mathbb{N}$ or $\mathcal{P} \subseteq 2\mathbb{N}+1$. Suppose that there exists a constant $C > 0$ such that*

$$\pi(x) - \pi(y) \geq C\ \pi(x - y) \text{ for all } x,\ y \in [1,\ +\infty),\ x \geq y.$$

Define $a := p_1$, and consider the function $f : \mathbb{N}_{\geq a} \setminus \mathcal{P} \to \mathbb{N}$ defined by

$$f(m) := \sum_{k\in Dif(\mathcal{P})(m)} \pi(k + 2)\ \pi_k(m) \text{ for every } m \in \mathbb{N}_{\geq a} \setminus \mathcal{P}.$$

Then:

1. *$\psi(m)-\psi(m-1) \geq \pi(m)+C\ f(m)-\pi(2m-a-2) \geq \pi(m)+C\ f(m)-\pi(2m)$ for every $m \in \mathbb{N}_{\geq a}\setminus\mathcal{P}$.*
2. *Let $m \in \mathbb{N}_{\geq a} \setminus \mathcal{P}$. If $\pi(m) + C\ f(m) - \pi(2m - a - 2) > 0$, then $2m \in \mathcal{P} + \mathcal{P}$.*
3. *If $\liminf_{m\to+\infty,\ m\notin\mathcal{P}} \frac{\pi(m)+C\ f(m)}{\pi(2m)} \geq L \in (1,\ +\infty]$, then there exists $m_0 \in \mathbb{N}_{\geq a} \setminus \mathcal{P}$ such that $2m \in \mathcal{P} + \mathcal{P}$ for each $m \in \mathbb{N}_{\geq a} \setminus \mathcal{P}$, $m \geq m_0$.*

Remark 1.8. *Obviously in Theorem 3.5 we can replace the function f by a positive function $g : \mathbb{N}_{\geq a}\setminus\mathcal{P} \to \mathbb{Z}^+$ verifying that $f(m) \geq g(m)$ for all $m \in \mathbb{N}_{\geq a}\setminus\mathcal{P}$ whenever such a function exists. This is more suitable in practice as we will show in the next section (see Corollary 2.2).*

Given a sequence of real numbers greater or equal than 1, the following two theorems give us an approximation of the values of its distribution function, $\pi(x)$, depending on the factors of x, namely:

$$\pi(x) \approx \frac{x\ Prod_{j=1}^{m} \pi\left(x_j\right)}{\sum_{j=1}^{m} x_j\ Prod_{i=1,\ i\neq j}^{m} \pi\left(x_i\right)},$$

with $x = x_1 \cdot ... \cdot x_m$, $m \geq 2$. We may apply both of them to prime numbers set $\mathbb{P}$ with $A = 1$, $B = 1.1$, $x_0 = 60184$ (see [4], and also [3] p. 37).

Theorem 1.9. *Let $\mathcal{P} = (p_i)_{i\in\mathbb{Z}^+}$ be a strictly increasing sequence in $\mathbb{R}^+$, and let $\pi : (0, +\infty) \to \mathbb{R}$ be its distribution function. Suppose that there exist constants $x_0,\ A,\ B \in \mathbb{R}$, $x_0 \geq 1$, $0 < A \leq B < \log x_0$, such that*
$$\frac{x}{\log x - A} \leq \pi(x) \leq \frac{x}{\log x - B} \text{ for all } x \in [x_0,\ +\infty).$$
Let $m \in \mathbb{N}$, $m \geq 2$, $x_1, ..., x_m \in [x_0,\ +\infty)$. Define $x := Prod_{i=1}^m x_i \in [x_0,\ +\infty)$ and
$$g(x_1, ..., x_m, C, D) := \frac{Prod_{j=1}^m (\log x_j - D)}{\sum_{j=1}^m (\log x_j) - C} \sum_{j=1}^m \frac{1}{Prod_{i=1,\ i\neq j}^m (\log x_i - C)}$$
for all $C,\ D \in \mathbb{R}$. Then we have:

1. $g(x_1, ..., x_m, A, B) \leq \frac{\pi(x)}{\frac{x\ Prod_{j=1}^m \pi(x_j)}{\sum_{j=1}^m x_j\ Prod_{i=1,\ i\neq j}^m \pi(x_i)}} \leq g(x_1, ..., x_m, B, A).$
2. $\lim_{x_1\to+\infty,...,x_m\to+\infty} \frac{\pi(x)}{\frac{x\ Prod_{j=1}^m \pi(x_j)}{\sum_{j=1}^m x_j\ Prod_{i=1,\ i\neq j}^m \pi(x_i)}} = 1.$

Theorem 1.10. *Let $\mathcal{P} = (p_i)_{i\in\mathbb{Z}^+}$ be a strictly increasing sequence in $\mathbb{R}^+$, and let $\pi : (0, +\infty) \to \mathbb{R}$ be its distribution function. Suppose that there exist constants $x_0,\ A,\ B \in \mathbb{R}$, $x_0 \geq 1$, $0 < A \leq B < \log x_0$, such that*
$$\frac{x}{\log x - A} \leq \pi(x) \leq \frac{x}{\log x - B} \text{ for all } x \in [x_0,\ +\infty).$$
Let $m \in \mathbb{N}$, $m \geq 2$, $x_1, ..., x_m \in [x_0,\ +\infty)$. Define $x := Prod_{i=1}^m x_i \in [x_0,\ +\infty)$ and
$$h(x_1, ..., x_m, C, D) := \frac{1}{\sum_{j=1}^m \frac{Prod_{i=1}^m (\log x_i - C)}{Prod_{i=1,\ i\neq j}^m (\log x_i - D)}},$$
$$l(x_1, ..., x_m, C, D) := \frac{1}{\sum_{j=1}^m \log x_j - D} - h(x_1, ..., x_m, C, D)$$
for all $C,\ D \in \mathbb{R}$. Then we have:

1. $x\ l(x_1, ..., x_m, B, A) \leq \pi(x) - \frac{x\ Prod_{j=1}^m \pi(x_j)}{\sum_{j=1}^m x_j\ Prod_{i=1,\ i\neq j}^m \pi(x_i)} \leq x\ l(x_1, ..., x_m, A, B).$
2. $$\lim_{x_1\to+\infty,...,x_m\to+\infty} l(x_1, ..., x_m, B, A) = $$
$$= \lim_{x_1\to+\infty,...,x_m\to+\infty} l(x_1, ..., x_m, A, B) = 0$$
$$\lim_{x_1\to+\infty,...,x_m\to+\infty} x\ l(x_1, ..., x_m, B, A) = $$
$$= \lim_{x_1\to+\infty,...,x_m\to+\infty} x\ l(x_1, ..., x_m, A, B) = +\infty.$$
3. $\lim_{x_1\to+\infty,...,x_m\to+\infty} \frac{\pi(x) - \frac{x\ Prod_{j=1}^m \pi(x_j)}{\sum_{j=1}^m x_j\ Prod_{i=1,\ i\neq j}^m \pi(x_i)}}{\frac{x}{\sum_{1\leq i,j\leq m} (\log x_i)(\log x_j)}} = 1.$

The paper is structured as follows. Section 1 contains definitions and the main results. In section 2 we prove Theorem 1.7 and we obtain as consequence a result which may be applied to prove Goldbach´s Conjecture for sequences with contractive distribution functions (Corollary 2.2 and Corollary 7.2). In section 3 we generalize Theorem 1.7 to positive real numbers setting, obtaining Theorem 3.5. In section

4 we obtain sufficient conditions to determine, given two sets of positive real numbers $\mathcal{A}$, $\mathcal{B}$, when a given positive real number is a sum of an element of $\mathcal{A}$ and an element of $\mathcal{B}$. In section 5 we prove Theorem 1.9 and we obtain some consequences for prime numbers. Section 6 is devoted to the proof of Theorem 1.10. Finally, in section 7 we state slight generalizations of Theorem 1.7 and Corollary 2.2 for sequences of natural numbers whose distribution functions are quasi-contractive.

2. Proof of Theorem 1.7 and consequences.

In this section we will prove Theorem 1.7. First we need the following result which justifies the name of *discriminant fucntion*.

Lemma 2.1. *Let $\mathcal{P} = (p_i)_{i\in\mathbb{Z}^+}$ be a strictly increasing sequence in $\mathbb{Z}^+$ such that $\mathcal{P} \subseteq 2\mathbb{N}$ or $\mathcal{P} \subseteq 2\mathbb{N}+1$. Define $a := p_1$, and let $m \in \mathbb{N}_{\geq a}$, $m \notin \mathcal{P}$ be. Then:*

1. *$\psi(m) - \psi(m-1) = \sum_{p\in\mathcal{P},\, p\leq m} \chi_{\mathcal{P}}(2m-p) \geq 0$ is the number of times that $2m$ can be expressed as sum of two elements of $\mathcal{P}$ (considering the same form $p+q$ and $q+p$ for all $p,\, q \in \mathcal{P}$).*

2. *$\psi(m) - \psi(m-1) > 0 \Leftrightarrow 2m \in \mathcal{P}+\mathcal{P}$.*

Proof. Since $m \notin \mathcal{P}$, then $\pi(m) = \pi(m-1)$. Therefore:

$$\psi(m) - \psi(m-1) = \sum_{i=1}^{\pi(m)} \pi(2m-p_i) - \sum_{i=1}^{\pi(m-1)} \pi\left(2(m-1)-p_i\right) =$$
$$= \sum_{i=1}^{\pi(m-1)} \pi(2m-p_i) - \sum_{i=1}^{\pi(m-1)} \pi\left((2m-p_i)-2\right) =$$
$$= \sum_{i=1}^{\pi(m-1)} \left[\pi(2m-p_i) - \pi\left((2m-p_i)-2\right)\right] = \sum_{i=1}^{\pi(m-1)} \chi_{\mathcal{P}}(2m-p_i) =$$
$$= \sum_{i=1}^{\pi(m)} \chi_{\mathcal{P}}(2m-p_i) = \sum_{p\in\mathcal{P},\, p\leq m} \chi_{\mathcal{P}}(2m-p) \geq 0,$$

where the fifth equality, the key step, is because of $2m - p_i$ and $2m - p_i - 2$ are both even or both odd. □

Proof of Theorem 1.7.

1. Let $m \in \mathbb{N}_{\geq a} \setminus \mathcal{P}$. Then $\pi(m) = \pi(m-1)$. Therefore:

$$\psi(m) - \psi(m-1) = \sum_{i=1}^{\pi(m)} \pi(2m-p_i) - \sum_{i=1}^{\pi(m-1)} \pi\left(2(m-1)-p_i\right)$$
$$= \sum_{i=1}^{\pi(m)} \pi(2m-p_i) - \sum_{i=1}^{\pi(m-1)} \pi\left(2m-(p_i+2)\right)$$
$$= \sum_{i=1}^{\pi(m)} \pi(2m-p_i) - \sum_{i=1}^{\pi(m)} \pi\left(2m-(p_i+2)\right)$$
$$= \sum_{i=1}^{\pi(m)} \pi(2m-p_i) - \sum_{j=0}^{\pi(m)-1} \pi\left(2m-(p_{j+1}+2)\right)$$

$$
\begin{aligned}
&= \pi(2m - p_{\pi(m)}) - \pi\left(2m - (p_1 + 2)\right) \\
&\quad + \sum_{i=1}^{\pi(m)-1} \pi(2m - p_i) - \sum_{j=1}^{\pi(m)-1} \pi\left(2m - (p_{j+1} + 2)\right) \\
&= \pi(2m - p_{\pi(m)}) - \pi\left(2m - (p_1 + 2)\right) \\
&\quad + \sum_{i=1}^{\pi(m)-1} \left[\pi(2m - p_i) - \pi\left(2m - (p_{i+1} + 2)\right)\right].
\end{aligned}
$$

Since $p_{\pi(m)} \leq m$, then $2m - p_{\pi(m)} \geq m$, and thus $\pi(2m - p_{\pi(m)}) \geq \pi(m)$. On the other hand, let $i \in \{1, ..., \pi(m) - 1\}$ be. Then

$$
\begin{gathered}
\pi(2m - p_i) - \pi\left(2m - (p_{i+1} + 2)\right) \geq \\
\geq C\, \pi\left((2m - p_i) - (2m - (p_{i+1} + 2))\right) = \\
= C\, \pi\left(p_{i+1} - p_i + 2\right).
\end{gathered}
$$

Hence

$$
\psi(m) - \psi(m-1) \geq \pi(m) - \pi(2m - a - 2) + C \sum_{i=1}^{\pi(m)-1} \pi\left(p_{i+1} - p_i + 2\right).
$$

Since

$$
\begin{aligned}
\sum_{i=1}^{\pi(m)-1} \pi\left(p_{i+1} - p_i + 2\right) &= \sum_{p,\, q \in \mathcal{P} \text{ consecutive}, \, p < q \leq m} \pi(q - p + 2) = \\
&= \sum_{k \in Dif(\mathcal{P})(m)} \pi(k+2)\, \pi_k(m) = f(m),
\end{aligned}
$$

then we have

$$
\begin{gathered}
\psi(m) - \psi(m-1) \geq \pi(m) + C\, f(m) - \pi(2m - a - 2) \geq \\
\geq \pi(m) + C\, f(m) - \pi(2m).
\end{gathered}
$$

2. It is an immediate consequence of the previous item and the second item of Lemma 2.1.
3. Let $m \in \mathbb{N}_{\geq a} \setminus \mathcal{P}$. If $\pi(m) + C\, f(m) - \pi(2m) > 0$, then by the previous item we have $2m \in \mathcal{P} + \mathcal{P}$. From this fact we obtain the result.

$\square$

Corollary 2.2. *Let $\mathcal{P} = (p_i)_{i \in \mathbb{Z}^+}$ be a strictly increasing sequence in $\mathbb{Z}^+$ such that $\mathcal{P} \subseteq 2\mathbb{N}$ or $\mathcal{P} \subseteq 2\mathbb{N}+1$. Suppose that there exists a constant $C_1 > 0$ such that*

$$
\pi(x) - \pi(y) \geq C_1\, \pi(x - y) \text{ for all } x,\, y \in [1,\, +\infty),\; x \geq y.
$$

Define $a := p_1$. Let $\alpha \in \mathbb{R}^+$. Suppose that

1. *There exist constants $C_2 > 0$, $m_0 \in \mathbb{N}_{\geq a} \setminus \mathcal{P}$ such that*

$$
Card\left(Dif\left(\mathcal{P}\right)(m)\right) \geq C_2\, \log^{\alpha}(m)
$$

for every $m \in \mathbb{N}_{\geq a} \setminus \mathcal{P}$, $m \geq m_0$.

2. *There exist constants* $m_1 \in \mathbb{N}_{\geq a} \setminus \mathcal{P}$, A, B, $C_3 \in \mathbb{R}$, $C_3 > 0$ *such that*

$$\pi_k(m) \geq \frac{C_3\, m}{(\log(m) + A)^{\alpha+1} + B}$$

for every $m \in \mathbb{N}_{\geq a} \setminus \mathcal{P}$, $m \geq m_1$, *and every* $k \in \left\{2, 4, ..., 2\, E\left(\frac{C_2\, \log^{\alpha}(m)}{2}\right)\right\}$.

3. $\lim_{n\to+\infty,\, n\in\mathbb{N}} \frac{\pi(n)}{\frac{n}{\log n}} = 1$.

4. $C_1 \cdot C_2 \cdot C_3 > 1$.

Then there exists $m_2 \in \mathbb{N}_{\geq a} \setminus \mathcal{P}$ *such that* $2m \in \mathcal{P} + \mathcal{P}$ *for all* $m \in \mathbb{N}_{\geq a} \setminus \mathcal{P}$, $m \geq m_2$.

Proof. The assumption (3) implies that $\lim_{n\to+\infty,\, n\in\mathbb{N}} \frac{\pi(2n)}{\frac{2n}{\log(n)+\log 2}} = 1$.

Define $s_0 := \max\{m_0,\, m_1\}$. Let $x_0 \in \mathbb{Z}^+$ be an even integer such that $x_0 \geq 4$ and $\pi(x_0) \geq 1$. Let $m \in \mathbb{N}_{\geq a} \setminus \mathcal{P}$, $m \geq s_0$, be such that $Card\,(Dif\,(\mathcal{P})\,(m)) \geq \frac{x_0}{2} - 2$. Then

$$\begin{aligned} f(m) := \sum_{k\in Dif(\mathcal{P})(m)} \pi(k+2)\,\pi_k(m) &\geq \sum_{k\in Dif(\mathcal{P})(m),\, k\geq x_0-2} \pi(x_0)\,\pi_k(m) \geq \\ \geq \frac{C_3\, m}{(\log(m)+A)^{\alpha+1}+B} \cdot \Big(Card\,(Dif\,(\mathcal{P})\,(m)) - \frac{x_0}{2} + 2\Big) &\geq \\ \geq C_3\, m\, \frac{C_2\, \log^{\alpha}(m) - \frac{x_0}{2} + 2}{(\log(m)+A)^{\alpha+1}+B} &=: g(m). \end{aligned}$$

Hence

$$\begin{aligned} \liminf_{m\to+\infty} \frac{\pi(m) + C_1\, g(m)}{\pi(2m)} &= \liminf_{m\to+\infty} \frac{\frac{m}{\log m} + C_1\, C_3\, m\, \frac{C_2\, \log^{\alpha}(m) - \frac{x_0}{2} + 2}{(\log(m)+A)^{\alpha+1}+B}}{\frac{2m}{\log(m)+\log 2}} = \\ &= \frac{1 + C_1\, C_2\, C_3}{2} \in (1,\, +\infty]. \end{aligned}$$

We obtain the result as consequence of Theorem 1.7. □

Remark 2.3. *Observe that for* $\alpha = 1$, *and in the context of the prime numbers, the second hypothesis is called the Consecutive Prime Conjecture, and for* $k = 2$ *it is a Hardy and Littlewood´s conjecture for the distribution of the twin prime numbers (for example, see [6], [8] and [1]). In addition, the third hypothesis is Prime Number Theorem (see [5] and [2]).*

In the two following sections we generalize the results of the first section to positive real numbers.

3. On the existence of an expression of a positive real number as sum of two elements of a given set of positive real numbers.

3.1. Definitions

Definition 3.1. *Let* $\mathcal{P} = (p_i)_{i\in\mathbb{Z}^+}$ *be a strictly increasing sequence in* $\mathbb{R}^+$ *such that it is uniformly discrete (briefly, u.d.), this is,* $\inf_{i\in\mathbb{Z}^+}\{p_{i+1} - p_i\} > 0$.

1. $a := p_1 = \min\{p_i \mid i \in \mathbb{Z}^+\} \in \mathbb{R}^+$.

2. $\mathbb{R}_{\geq a} := \{x \in \mathbb{R} \mid x \geq a\} \subseteq \mathbb{R}^+$, $2\mathbb{R}_{\geq a} := \{2x \mid x \in \mathbb{R},\, x \geq a\}$.

3. *The function* $\pi := \pi_{\mathcal{P}} : (0,\, +\infty) \to \mathbb{R}$ *defined by*

$$\pi(x) := Card\,(\{p \in \mathcal{P} \mid p \leq x\}) = Card\,(\mathcal{P} \cap (0,\, x])\,,$$

for each $x \in (0,\, +\infty)$, *is called the distribution function of* $\mathcal{P}$. π *is monotonically increasing.*

Remark 3.2. *Notice these consequences of the definition of π:*

1. *For every $x \in \mathbb{R}^+$ we have $\{p_1, ..., p_{\pi(x)}\} \subseteq (0,\ x]$, and then $p_{\pi(x)} \leq x$.*
2. *$\left(p \leq x \Leftrightarrow p \leq p_{\pi(x)}\right)$ for every $x \in \mathbb{R}^+$, $p \in \mathcal{P}$.*

Definition 3.3. *Let $\mathcal{P} = (p_i)_{i\in\mathbb{Z}^+}$ be a strictly increasing and u.d. sequence in $\mathbb{R}^+$.*

1. *$Dif(\mathcal{P}) := \{p_{n+1} - p_n\ :\ n \in \mathbb{Z}^+\} \subseteq \mathbb{R}^+$ is called the difference set of $\mathcal{P}$.*
2. *Let $x \in (0,\ +\infty)$. $Dif(\mathcal{P})(x) := \{p_{n+1} - p_n\ :\ n \in \mathbb{Z}^+,\ p_{n+1} \leq x\}$ is called the difference set of $\mathcal{P}$ until x.*
3. *Let $p,\ q \in \mathcal{P},\ p < q$. We say that p and q are consecutive respect to $\mathcal{P}$ if $(p,\ q) \cap \mathcal{P} = \emptyset$.*
4. *Let $p,\ q \in \mathcal{P},\ p < q,\ k \in \mathbb{R}^+$. We say that p and q are k-consecutive if $q - p = k$.*
5. *Let $k \in Dif(\mathcal{P})$. We define the function $\pi_k := \pi_{k,\mathcal{P}} : (0,\ +\infty) \to \mathbb{R}$ by*

$$\pi_k(x) := Card\left(\{p \in \mathcal{P} \mid (p + k \leq x,\ p + k \in \mathcal{P})\}\right) \leq \pi(x),$$

   for each $x \in (0,\ +\infty)$. $\pi_k := \pi_{k,\mathcal{P}}$ is called the distribution function of the k-consecutive elements of $\mathcal{P}$, and it is also called the distribution function of the k-differences of elements of $\mathcal{P}$.

Let $\mathcal{P} = (p_i)_{i\in\mathbb{Z}^+}$ be a strictly increasing and u.d. sequence in $\mathbb{R}^+$. We define $a := p_1$. Let $(b_m)_{m\in\mathbb{Z}^+}$ be a strictly increasing sequence in $\mathbb{R}^+$ such that there exists $m \in \mathbb{N},\ m \geq 2$ verifying $b_m \geq a$. Let $m \in \mathbb{N},\ m \geq 2$ be, with $b_m \geq a$. We wonder if $2b_m \in \mathcal{P} + \mathcal{P}$. If $b_m \in \mathcal{P}$ (the trivial case), then $2b_m = b_m + b_m \in \mathcal{P} + \mathcal{P}$ and the answer is affirmative. The question is what happens if $b_m \notin \mathcal{P}$. Assume that $2b_m \in \mathcal{P} + \mathcal{P}$, with $b_m \notin \mathcal{P}$. There exist $p,\ q \in \mathcal{P}$, with $p \leq q$, such that $2b_m = p + q$. Then $2b_m \geq 2p$, and therefore $p \leq b_m$, or what is equivalent, $p \leq p_{\pi(b_m)}$. Thus we have that $p \in \mathcal{P},\ 2b_m - p \in \mathcal{P}$ and $p \leq p_{\pi(b_m)}$. So these last three conditions altogether are equivalent to $2b_m \in \mathcal{P} + \mathcal{P}$ when $b_m \notin \mathcal{P}$ (notice it is also true for $b_m \in \mathcal{P}$).

Definition 3.4 (Discriminant function)**.** *Let $\mathcal{P} = (p_i)_{i\in\mathbb{Z}^+}$ be a strictly increasing and u.d. sequence in $\mathbb{R}^+$, and define $a := p_1$. Let $(b_m)_{m\in\mathbb{Z}^+}$ be a strictly increasing sequence in $\mathbb{R}^+$ such that there exists $m \in \mathbb{N},\ m \geq 2$ verifying $b_m \geq a$. The function $\psi : \mathbb{Z}^+ \to \mathbb{N}$ defined by*

$$\psi(t) := \sum_{i=1}^{\pi(b_t)} \pi(2b_t - p_i) = \sum_{p\in\mathcal{P},\ p\leq b_t} \pi(2b_t - p) \text{ for all } t \in \mathbb{Z}^+,$$

is called the discriminant function of $\mathcal{P}$ respect to $(b_m)_{m\in\mathbb{Z}^+}$.

3.2. Results

We have the following result.

Theorem 3.5. *Let $\mathcal{P} = (p_i)_{i\in\mathbb{Z}^+}$ be a strictly increasing and uniformly discrete sequence in $\mathbb{R}^+$, and define $a := p_1$. Let $(b_m)_{m\in\mathbb{Z}^+}$ be a strictly increasing sequence in $\mathbb{R}^+$ such that there exists $m \in \mathbb{N},\ m \geq 2$ verifying $b_m \geq a$. Define $inf(b) := \inf_{m\in\mathbb{Z}^+}(b_{m+1} - b_m) \geq 0$. Let $m \in \mathbb{N},\ m \geq 2$ verifying $b_m \geq a$. Suppose that $b_m \notin \mathcal{P}$, $\pi(b_m) = \pi(b_{m-1})$, and*

$$\pi(2b_m - p_i) - \pi(2b_{m-1} - p_i) = \chi_{\mathcal{P}}(2b_m - p_i) \text{ for each } i \in \{1, ..., \pi(b_{m-1})\}.$$

Also assume that there exists a constant $C > 0$ such that

$$\pi(x) - \pi(y) \geq C\,\pi(x - y) \text{ for all } x,\ y \in (0,\ +\infty),\ x \geq y.$$

Define

$$f(m) := \sum_{k\in Dif(\mathcal{P})(m)} \pi\left(k + 2\left(b_m - b_{m-1}\right)\right)\ \pi_k(m) \geq$$
$$\geq \sum_{k\in Dif(\mathcal{P})(m)} \pi\left(k + 2\, inf(b)\right)\ \pi_k(m) =: g(m) \geq 0.$$

Then:

1.

$$\psi(m) - \psi(m-1) \geq \pi\left(b_m\right) + C\, f(m) - \pi\left(2b_{m-1} - a\right) \geq$$
$$\geq \pi\left(b_m\right) + C\, g(m) - \pi\left(2b_{m-1} - a\right).$$

2. *If* $\pi\left(b_m\right) + C\, f(m) - \pi\left(2b_{m-1} - a\right) > 0$*, then* $2b_m \in \mathcal{P} + \mathcal{P}$*.*

3. *If* $\pi\left(b_m\right) + C\, g(m) - \pi\left(2b_{m-1} - a\right) > 0$*, then* $2b_m \in \mathcal{P} + \mathcal{P}$*.*

We need the following lemma:

Lemma 3.6. *Let* $\mathcal{P} = (p_i)_{i\in\mathbb{Z}^+}$ *be a strictly increasing and u.d. sequence in* $\mathbb{R}^+$*, and define* $a := p_1$*. Let* $(b_m)_{m\in\mathbb{Z}^+}$ *be a strictly increasing sequence in* $\mathbb{R}^+$ *such that there exists* $m \in \mathbb{N}$*,* $m \geq 2$ *verifying that* $b_m \geq a$*. Let* $m \in \mathbb{N}$*,* $m \geq 2$ *be such that* $b_m \geq a$*. Suppose that* $b_m \notin \mathcal{P}$*,* $\pi\left(b_m\right) = \pi\left(b_{m-1}\right)$*, and*

$$\pi\left(2b_m - p_i\right) - \pi\left(2b_{m-1} - p_i\right) = \chi_{\mathcal{P}}\left(2b_m - p_i\right)\ \text{for each } i \in \left\{1, ..., \pi\left(b_{m-1}\right)\right\}.$$

Then:

1. $\psi(m) - \psi(m-1) = \sum_{p\in\mathcal{P},\, p\leq b_m} \chi_{\mathcal{P}}(2b_m - p) \geq 0$ *is the number of times that* $2b_m$ *can be expressed as sum of two elements of* $\mathcal{P}$ *(considering the same form* $p + q$ *and* $q + p$ *for all* p*,* $q \in \mathcal{P}$*).*

2. $\psi(m) - \psi(m-1) > 0 \Leftrightarrow 2b_m \in \mathcal{P} + \mathcal{P}$*.*

Proof.

$$\psi(m) - \psi(m-1) = \sum_{i=1}^{\pi(b_m)} \pi(2b_m - p_i) - \sum_{i=1}^{\pi(b_{m-1})} \pi\left(2b_{m-1} - p_i\right) =$$
$$= \sum_{i=1}^{\pi(b_{m-1})} \pi(b_m - p_i) - \sum_{i=1}^{\pi(b_{m-1})} \pi\left(2b_m - p_i\right) =$$
$$= \sum_{i=1}^{\pi(b_{m-1})} \left[\pi(2b_m - p_i) - \pi\left(2b_m - p_i\right)\right] = \sum_{i=1}^{\pi(b_{m-1})} \chi_{\mathcal{P}}(2b_m - p_i) =$$
$$= \sum_{i=1}^{\pi(b_m)} \chi_{\mathcal{P}}(2b_m - p_i) = \sum_{p\in\mathcal{P},\, p\leq b_m} \chi_{\mathcal{P}}(2b_m - p) \geq 0.$$

Therefore the next conditions are equivalent:

i) $\psi(m) - \psi(m-1) > 0$.

ii) There exists $p \in \mathcal{P}$, $p \leq b_m$, such that $\chi_{\mathcal{P}}(2b_m - p) > 0$.

iii) $2b_m \in \mathcal{P} + \mathcal{P}$.

∎

Proof of Theorem 3.5.

1. Since $\pi(b_m) = \pi(b_{m-1})$, then:

$$\psi(m) - \psi(m-1) = \sum_{i=1}^{\pi(b_m)} \pi(2b_m - p_i) - \sum_{i=1}^{\pi(b_{m-1})} \pi(2b_{m-1} - p_i) =$$
$$= \sum_{i=1}^{\pi(b_m)} \pi(2b_m - p_i) - \sum_{j=0}^{\pi(b_{m-1})-1} \pi(2b_{m-1} - p_{j+1}) =$$
$$= \sum_{i=1}^{\pi(b_m)} \pi(2b_m - p_i) - \sum_{j=0}^{\pi(b_m)-1} \pi(2b_{m-1} - p_{j+1}) =$$
$$= \pi(2b_m - p_{\pi(b_m)}) - \pi(2b_{m-1} - p_1) +$$
$$+ \sum_{i=1}^{\pi(b_m)-1} \pi(2b_m - p_i) - \sum_{j=1}^{\pi(b_m)-1} \pi(2b_{m-1} - p_{j+1}) =$$
$$= \pi(2b_m - p_{\pi(b_m)}) - \pi(2b_{m-1} - p_1) +$$
$$+ \sum_{i=1}^{\pi(b_m)-1} \left[\pi(2b_m - p_i) - \pi(2b_{m-1} - p_{i+1})\right].$$

Since $p_{\pi(b_m)} \leq b_m$, then $2b_m - p_{\pi(b_m)} \geq b_m$, and thus $\pi\left(2b_m - p_{\pi(b_m)}\right) \geq \pi(b_m)$. On the other hand, let $i \in \{1, ..., \pi(b_m) - 1\}$ be. Then

$$\pi(2b_m - p_i) - \pi(2b_{m-1} - p_{i+1}) \geq$$
$$\geq C\, \pi((2b_m - p_i) - (2b_{m-1} - p_{i+1})) =$$
$$= C\, \pi(p_{i+1} - p_i + 2(b_m - b_{m-1})),$$

where we have used that $2b_m - p_i \geq 2b_{m-1} - p_{i+1}$ because of $p_{i+1} - p_i > 0 > 2(b_{m-1} - b_m)$. Hence

$$\psi(m) - \psi(m-1) \geq \pi(b_m) - \pi(2b_{m-1} - p_1) +$$
$$+C \sum_{i=1}^{\pi(b_m)-1} \pi(p_{i+1} - p_i + 2(b_m - b_{m-1})).$$

Since

$$\sum_{i=1}^{\pi(b_m)-1} \pi(p_{i+1} - p_i + 2(b_m - b_{m-1})) =$$
$$= \sum_{p,\, q \in \mathcal{P} \text{ consecutive},\, p<q\leq m} \pi(q - p + 2(b_m - b_{m-1})) =$$
$$= \sum_{k \in Dif(\mathcal{P})(m)} \pi(k + 2(b_m - b_{m-1}))\, \pi_k(m) = f(m),$$

then we have

$$\psi(m) - \psi(m-1) \geq \pi(b_m) + C\, f(m) - \pi(2b_{m-1} - a) \geq$$
$$\geq \pi(b_m) + C\, g(m) - \pi(2b_{m-1} - a) \geq$$
$$\geq \pi(b_m) + C\, g(m) - \pi(2b_{m-1}).$$

2. It is an immediate consequence of the previous item and the second item of Lemma 3.6.

3. It is obvious from the previous item.

□

Analogously we can prove the following result:

Proposition 3.7. *Let* $\mathcal{P} = (p_i)_{i\in\mathbb{Z}^+}$ *be a strictly increasing and uniformly discrete sequence in* $\mathbb{R}^+$*, and define* $a := p_1$*. Let* $(b_m)_{m\in\mathbb{Z}^+}$ *be a strictly increasing sequence in* $\mathbb{R}^+$ *such that there exists* $m \in \mathbb{N}$*,* $m \geq 2$ *verifying* $b_m \geq a$*. Let* $m \in \mathbb{N}$*,* $m \geq 2$ *be such that* $b_m \geq a$*. Suppose that* $b_m \notin \mathcal{P}$*,* $\pi(b_m) = \pi(b_{m-1})$*, and*

$$\pi(2b_m - p_i) - \pi(2b_{m-1} - p_i) = \chi_{\mathcal{P}}(2b_m - p_i) \text{ for each } i \in \{1, ..., \pi(b_{m-1})\}.$$

Also assume that there exists a constant $D > 0$ *such that*

$$\pi(x) - \pi(y) \leq D\,\pi(x-y) \text{ for all } x,\ y \in (0, +\infty),\ x \geq y.$$

Define

$$f(m) := \sum_{k\in Dif(\mathcal{P})(m)} \pi(k + 2(b_m - b_{m-1}))\ \pi_k(m).$$

Then:

1. $$0 \leq \psi(m) - \psi(m-1) \leq \pi\left(2b_m - p_{\pi(b_m)}\right) + D\,f(m) - \pi(2b_{m-1} - a).$$

   Hence, if $\pi\left(2b_m - p_{\pi(b_m)}\right) + D\,f(m) - \pi(2b_{m-1} - a) = 0$*, then* $2b_m \notin \mathcal{P} + \mathcal{P}$*.*

2. *Assume that there exists* $E_m \in (0, 1)$ *such that* $(E_m \cdot b_m, b_m) \cap \mathcal{P} \neq \emptyset$*. Then:*

   $$0 \leq \psi(m) - \psi(m-1) \leq \pi((2 - E_m)\,b_m) + D\,f(m) - \pi(2b_{m-1} - a).$$

   Thus, if $\pi((2 - E_m)\,b_m) + D\,f(m) - \pi(2b_{m-1} - a) = 0$*, then* $2b_m \notin \mathcal{P} + \mathcal{P}$*.*

Proof.

1. The proof of this item is analogous to the proof of Theorem 3.5.

2. In this case $E_m \cdot b_m < p_{\pi(b_m)} < b_m$ (observe that $b_m \notin \mathcal{P}$ because $\pi(b_m) = \pi(b_{m-1})$). Then $2b_m - E_m \cdot b_m > 2b_m - p_{\pi(b_m)} > b_m$ and therefore

   $$\pi\left(2b_m - p_{\pi(b_m)}\right) \leq \pi((2 - E_m)\,b_m).$$

□

Obviously, the constant $D > 0$ must be minimum, and this minimum exists if the distribution function π is not constant.

4. A positive real number as sum of two elements where every one belongs to a given set of positive real numbers.

In this section we will need Definition 3.1, Remark 3.2 and Definition 3.3.

Let $\mathcal{A} = (a_i)_{i\in\mathbb{Z}^+}$, $\mathcal{B} = (b_i)_{i\in\mathbb{Z}^+}$ be strictly increasing and u.d. sequences in $\mathbb{R}^+$. We define $a' := a_1$, $b' := b_1$. Let $(c_m)_{m\in\mathbb{Z}^+}$ be a strictly increasing sequence in $\mathbb{R}^+$ such that there exists $m \in \mathbb{N}$, $m \geq 2$ verifying $2c_m \geq a' + b'$. Let $m \in \mathbb{N}$, $m \geq 2$ be, with $2c_m \geq a' + b'$. We wonder if $2c_m \in \mathcal{A} + \mathcal{B}$.
Obviously, if $c_m \in \mathcal{A} \cap \mathcal{B}$ (the trivial case), then $2c_m = c_m + c_m \in \mathcal{A} + \mathcal{B}$ and the answer is affirmative. The question is what happens if $c_m \notin \mathcal{A} \cap \mathcal{B}$. Suppose that $2c_m \in \mathcal{A} + \mathcal{B}$, with $c_m \notin \mathcal{A} \cap \mathcal{B}$. There exist $a \in \mathcal{A}$, $b \in \mathcal{B}$ such that $2c_m = a + b$.

- If $a \leq b$: then $2a \leq a + b = 2c_m$, thus $a \leq c_m$, what is equivalent to $a \leq a_{\pi_{\mathcal{A}}(c_m)}$.
- If $b \leq a$: then $2b \leq a + b = 2c_m$, therefore $b \leq c_m$, what is equivalent to $b \leq b_{\pi_{\mathcal{B}}(c_m)}$.

Observe that $a \neq b$. Indeed, if $a = b$, then $2c_m = 2a = 2b$, and therefore $c_m = a = b \in \mathcal{A} \cap \mathcal{B}$, which is a contradiction with our assumption.
Conclusion: the cases $a \leq b$ and $b \leq a$ are mutually exclusive because the equality $a = b$ is not true.

Definition 4.1 (Discriminant function). *Let $\mathcal{A} = (a_i)_{i\in\mathbb{Z}^+}$, $\mathcal{B} = (b_i)_{i\in\mathbb{Z}^+}$ and $\mathcal{C} = (c_t)_{t\in\mathbb{Z}^+}$ be strictly increasing sequences in $\mathbb{R}^+$. Assume that $\mathcal{A}$ and $\mathcal{B}$ are uniformly discrete. The function $\psi : \mathbb{Z}^+ \to \mathbb{N}$ defined by*

$$\psi(t) := \sum_{a\in\mathcal{A},\, a\leq c_t} \pi_{\mathcal{B}}(2c_t - a) + \sum_{b\in\mathcal{B},\, b\leq c_t} \pi_{\mathcal{A}}(2c_t - b) =$$

$$= \sum_{i_{\mathcal{A}}=1}^{\pi_{\mathcal{A}}(c_t)} \pi_{\mathcal{B}}(2c_t - a_{i_{\mathcal{A}}}) + \sum_{i_{\mathcal{B}}=1}^{\pi_{\mathcal{B}}(c_t)} \pi_{\mathcal{A}}(2c_t - b_{i_{\mathcal{B}}}) \text{ for all } t \in \mathbb{Z}^+,$$

is called the discriminant function of $\{\mathcal{A}, \mathcal{B}\}$ respect to $\mathcal{C}$.

We have the following result.

Theorem 4.2. *Let $\mathcal{A} = (a_i)_{i\in\mathbb{Z}^+}$, $\mathcal{B} = (b_i)_{i\in\mathbb{Z}^+}$ be strictly increasing and u.d. sequences in $\mathbb{R}^+$. We define $a' := a_1$, $b' := b_1$. Let $(c_m)_{m\in\mathbb{Z}^+}$ be a strictly increasing sequence in $\mathbb{R}^+$ such that there exists $m_0 \in \mathbb{N}$, $m_0 \geq 2$ verifying $2c_{m_0} \geq a' + b'$. Let $m \in \mathbb{N}$, $m \geq 2$ be, and verifying that $2c_m \geq a' + b'$. Suppose that $c_m \notin \mathcal{A} \cap \mathcal{B}$, $\pi_{\mathcal{A}}(c_m) = \pi_{\mathcal{A}}(c_{m-1})$, $\pi_{\mathcal{B}}(c_m) = \pi_{\mathcal{B}}(c_{m-1})$ and*

$$\pi_{\mathcal{B}}(2c_m - a_{i_{\mathcal{A}}}) - \pi_{\mathcal{B}}(2c_{m-1} - a_{i_{\mathcal{A}}}) = \chi_{\mathcal{B}}(2c_m - a_{i_{\mathcal{A}}}) \text{ for each } i_{\mathcal{A}} \in \{1, ..., \pi_{\mathcal{A}}(c_{m-1})\},$$

$$\pi_{\mathcal{A}}(2c_m - b_{i_{\mathcal{B}}}) - \pi_{\mathcal{A}}(2c_{m-1} - b_{i_{\mathcal{B}}}) = \chi_{\mathcal{A}}(2c_m - b_{i_{\mathcal{B}}}) \text{ for each } i_{\mathcal{B}} \in \{1, ..., \pi_{\mathcal{B}}(c_{m-1})\}.$$

Define $inf(c) := \inf_{i\in\mathbb{Z}^+}(c_{i+1} - c_i) \geq 0$.

Also assume that there exist constants C_A, $C_B > 0$ such that

$$\pi_{\mathcal{A}}(x) - \pi_{\mathcal{A}}(y) \geq C_{\mathcal{A}}\, \pi_{\mathcal{A}}(x - y),$$
$$\pi_{\mathcal{B}}(x) - \pi_{\mathcal{B}}(y) \geq C_{\mathcal{B}}\, \pi_{\mathcal{B}}(x - y),$$

for all x, $y \in (0, +\infty)$, $x \geq y$.

We define

$$f(m) := C_{\mathcal{B}} \sum_{k\in Dif(\mathcal{A})(m)} \pi_{\mathcal{B}}(k + 2(c_m - c_{m-1}))\, \pi_{k,\mathcal{A}}(m) +$$
$$+ C_{\mathcal{A}} \sum_{l\in Dif(\mathcal{B})(m)} \pi_{\mathcal{A}}(l + 2(c_m - c_{m-1}))\, \pi_{l,\mathcal{B}}(m) \geq$$
$$\geq C_{\mathcal{B}} \sum_{k\in Dif(\mathcal{A})(m)} \pi_{\mathcal{B}}(k + 2\, inf(c))\, \pi_{k,\mathcal{A}}(m) +$$
$$+ C_{\mathcal{A}} \sum_{l\in Dif(\mathcal{B})(m)} \pi_{\mathcal{A}}(l + 2\, inf(c))\, \pi_{l,\mathcal{B}}(m) =: g(m) \geq 0.$$

Then:

1.

$$\begin{gathered}\psi(m)-\psi(m-1)\geq\pi_{\mathcal{B}}\left(c_m\right)+\pi_{\mathcal{A}}\left(c_m\right)+\\+f(m)-\pi_{\mathcal{A}}\left(2c_{m-1}-b'\right)-\pi_{\mathcal{B}}\left(2c_{m-1}-a'\right)\geq\\\geq\pi_{\mathcal{B}}\left(c_m\right)+\pi_{\mathcal{A}}\left(c_m\right)+\\+g(m)-\pi_{\mathcal{A}}\left(2c_{m-1}-b'\right)-\pi_{\mathcal{B}}\left(2c_{m-1}-a'\right).\end{gathered}$$

2. If $\pi_{\mathcal{B}}\left(c_m\right)+\pi_{\mathcal{A}}\left(c_m\right)+f(m)-\pi_{\mathcal{A}}\left(2c_{m-1}-b'\right)-\pi_{\mathcal{B}}\left(2c_{m-1}-a'\right)>0$*, then* $2b_m\in\mathcal{A}+\mathcal{B}$*.*

3. If $\pi_{\mathcal{B}}\left(c_m\right)+\pi_{\mathcal{A}}\left(c_m\right)+g(m)-\pi_{\mathcal{A}}\left(2c_{m-1}-b'\right)-\pi_{\mathcal{B}}\left(2c_{m-1}-a'\right)>0$*, then* $2b_m\in\mathcal{A}+\mathcal{B}$*.*

We need the following result:

Lemma 4.3. *Let* $\mathcal{A}=(a_i)_{i\in\mathbb{Z}^+}$*,* $\mathcal{B}=(b_i)_{i\in\mathbb{Z}^+}$ *be strictly increasing and u.d. sequences in* $\mathbb{R}^+$*. We define* $a':=a_1$*,* $b':=b_1$*. Let* $(c_m)_{m\in\mathbb{Z}^+}$ *be a strictly increasing sequence in* $\mathbb{R}^+$ *such that there exists* $m_0\in\mathbb{N}$*,* $m_0\geq 2$ *verifying* $2c_{m_0}\geq a'+b'$*. Let* $m\in\mathbb{N}$ *be such that* $m\geq 2$*,* $2c_m\geq a'+b'$*. Suppose that* $c_m\notin\mathcal{A}\cap\mathcal{B}$*,* $\pi_{\mathcal{A}}\left(c_m\right)=\pi_{\mathcal{A}}\left(c_{m-1}\right)$*,* $\pi_{\mathcal{B}}\left(c_m\right)=\pi_{\mathcal{B}}\left(c_{m-1}\right)$ *and*

$$\begin{gathered}\pi_{\mathcal{B}}\left(2c_m-a_{i_{\mathcal{A}}}\right)-\pi_{\mathcal{B}}\left(2c_{m-1}-a_{i_{\mathcal{A}}}\right)=\chi_{\mathcal{B}}\left(2c_m-a_{i_{\mathcal{A}}}\right)\text{ for each } i_{\mathcal{A}}\in\left\{1,\dots,\pi_{\mathcal{A}}\left(c_{m-1}\right)\right\},\\\pi_{\mathcal{A}}\left(2c_m-b_{i_{\mathcal{B}}}\right)-\pi_{\mathcal{A}}\left(2c_{m-1}-b_{i_{\mathcal{B}}}\right)=\chi_{\mathcal{A}}\left(2c_m-b_{i_{\mathcal{B}}}\right)\text{ for each } i_{\mathcal{B}}\in\left\{1,\dots,\pi_{\mathcal{B}}\left(c_{m-1}\right)\right\}.\end{gathered}$$

Then:

1. $\psi(m)-\psi(m-1)=\sum_{a\in\mathcal{A},\,a\leq c_m}\chi_{\mathcal{B}}(2c_m-a)+\sum_{b\in\mathcal{B},\,b\leq c_m}\chi_{\mathcal{A}}(2c_m-b)\geq 0$ *is the number of times that* $2c_m$ *can be expressed as sum of one element of* $\mathcal{A}$ *and one element of* $\mathcal{B}$ *(considering the same form* $a+b$ *and* $b+a$ *for all* $a\in\mathcal{A}$*,* $b\in\mathcal{B}$*).*

2. $\psi(m)-\psi(m-1)>0\Leftrightarrow 2c_m\in\mathcal{A}+\mathcal{B}$*.*

Proof.

$$\begin{aligned}\psi(m)-\psi(m-1)&=\sum_{i_{\mathcal{A}}=1}^{\pi_{\mathcal{A}}(c_m)}\pi_{\mathcal{B}}(2c_m-a_{i_{\mathcal{A}}})+\sum_{i_{\mathcal{B}}=1}^{\pi_{\mathcal{B}}(c_m)}\pi_{\mathcal{A}}(2c_m-b_{i_{\mathcal{B}}})-\\&-\sum_{i_{\mathcal{A}}=1}^{\pi_{\mathcal{A}}(c_{m-1})}\pi_{\mathcal{B}}(2c_{m-1}-a_{i_{\mathcal{A}}})-\sum_{i_{\mathcal{B}}=1}^{\pi_{\mathcal{B}}(c_{m-1})}\pi_{\mathcal{A}}(2c_{m-1}-b_{i_{\mathcal{B}}})=\\&=\sum_{i_{\mathcal{A}}=1}^{\pi_{\mathcal{A}}(c_{m-1})}\left[\pi_{\mathcal{B}}\left(2c_m-a_{i_{\mathcal{A}}}\right)-\pi_{\mathcal{B}}\left(2c_{m-1}-a_{i_{\mathcal{A}}}\right)\right]+\\&+\sum_{i_{\mathcal{B}}=1}^{\pi_{\mathcal{B}}(c_{m-1})}\left[\pi_{\mathcal{A}}\left(2c_m-b_{i_{\mathcal{B}}}\right)-\pi_{\mathcal{A}}\left(2c_{m-1}-b_{i_{\mathcal{B}}}\right)\right]=\\&=\sum_{i_{\mathcal{A}}=1}^{\pi_{\mathcal{A}}(c_{m-1})}\chi_{\mathcal{B}}\left(2c_m-a_{i_{\mathcal{A}}}\right)+\sum_{i_{\mathcal{B}}=1}^{\pi_{\mathcal{B}}(c_{m-1})}\chi_{\mathcal{A}}\left(2c_m-b_{i_{\mathcal{B}}}\right)=\\&=\sum_{a\in\mathcal{A},\,a\leq c_m}\chi_{\mathcal{B}}(2c_m-a)+\sum_{b\in\mathcal{B},\,b\leq c_m}\chi_{\mathcal{A}}(2c_m-b)\geq 0.\end{aligned}$$

Therefore the next conditions are equivalent:

i) $\psi(m)-\psi(m-1)=0$.

ii) $\chi_{\mathcal{B}}\left(2c_m-a\right)=0$ for all $a\in\mathcal{A}$, $a\leq c_m$, and $\chi_{\mathcal{A}}\left(2c_m-b\right)=0$ for all $b\in\mathcal{B}$, $b\leq c_m$.

iii) $2c_m\notin\mathcal{A}+\mathcal{B}$.

From here we obtain the result. □

Proof of Theorem 4.2.

1.

$$\psi(m)-\psi(m-1)=\sum_{i_{\mathcal{A}}=1}^{\pi_{\mathcal{A}}(c_m)}\pi_{\mathcal{B}}(2c_m-a_{i_{\mathcal{A}}})+\sum_{i_{\mathcal{B}}=1}^{\pi_{\mathcal{B}}(c_m)}\pi_{\mathcal{A}}(2c_m-b_{i_{\mathcal{B}}})-$$

$$-\sum_{i_{\mathcal{A}}=1}^{\pi_{\mathcal{A}}(c_{m-1})}\pi_{\mathcal{B}}(2c_{m-1}-a_{i_{\mathcal{A}}})-\sum_{i_{\mathcal{B}}=1}^{\pi_{\mathcal{B}}(c_{m-1})}\pi_{\mathcal{A}}(2c_{m-1}-b_{i_{\mathcal{B}}})=$$

$$=\left(\sum_{i_{\mathcal{A}}=1}^{\pi_{\mathcal{A}}(c_m)}\pi_{\mathcal{B}}(2c_m-a_{i_{\mathcal{A}}})-\sum_{i_{\mathcal{A}}=1}^{\pi_{\mathcal{A}}(c_{m-1})}\pi_{\mathcal{B}}(2c_{m-1}-a_{i_{\mathcal{A}}})\right)+$$

$$+\left(\sum_{i_{\mathcal{B}}=1}^{\pi_{\mathcal{B}}(c_m)}\pi_{\mathcal{A}}(2c_m-b_{i_{\mathcal{B}}})-\sum_{i_{\mathcal{B}}=1}^{\pi_{\mathcal{B}}(c_{m-1})}\pi_{\mathcal{A}}\left(2c_{m-1}-b_{i_{\mathcal{B}}}\right)\right).$$

Define

$$R_{BA}:=\sum_{i_{\mathcal{A}}=1}^{\pi_{\mathcal{A}}(c_m)}\pi_{\mathcal{B}}(2c_m-a_{i_{\mathcal{A}}})-\sum_{i_{\mathcal{A}}=1}^{\pi_{\mathcal{A}}(c_{m-1})}\pi_{\mathcal{B}}(2c_{m-1}-a_{i_{\mathcal{A}}}),$$

$$R_{AB}:=\sum_{i_{\mathcal{B}}=1}^{\pi_{\mathcal{B}}(c_m)}\pi_{\mathcal{A}}(2c_m-b_{i_{\mathcal{B}}})-\sum_{i_{\mathcal{B}}=1}^{\pi_{\mathcal{B}}(c_{m-1})}\pi_{\mathcal{A}}\left(2c_{m-1}-b_{i_{\mathcal{B}}}\right).$$

As we did in the proof of Theorem 3.5 (Proof 3.2), using that $\pi_{\mathcal{A}}\left(c_m\right)=\pi_{\mathcal{A}}\left(c_{m-1}\right)$, we obtain:

$$R_{BA}\geq\pi_{\mathcal{B}}\left(c_m\right)-\pi_{\mathcal{B}}\left(2c_{m-1}-a'\right)+C_{\mathcal{B}}\sum_{i=1}^{\pi(c_m)-1}\pi_{\mathcal{B}}\left(a_{i+1}-a_i+2\left(c_m-c_{m-1}\right)\right)=$$

$$=\pi_{\mathcal{B}}\left(c_m\right)-\pi_{\mathcal{B}}\left(2c_{m-1}-a'\right)+C_{\mathcal{B}}\sum_{i=1}^{\pi(c_m)-1}\pi_{\mathcal{B}}\left(a_{i+1}-a_i+2\left(c_m-c_{m-1}\right)\right)=$$

$$=\pi_{\mathcal{B}}\left(c_m\right)-\pi_{\mathcal{B}}\left(2c_{m-1}-a'\right)+$$

$$+C_{\mathcal{B}}\sum_{a,\,\widetilde{a}\in\mathcal{A}\text{ consecutive},\,a<\widetilde{a}\leq c_m}\pi\left(\widetilde{a}-a+2\left(c_m-c_{m-1}\right)\right)=$$

$$=\pi_{\mathcal{B}}\left(c_m\right)-\pi_{\mathcal{B}}\left(2c_{m-1}-a'\right)+C_{\mathcal{B}}\sum_{k\in Dif(\mathcal{A})(m)}\pi_{\mathcal{B}}(k+2\left(c_m-c_{m-1}\right))\,\pi_{k,\mathcal{A}}(m).$$

Analogously, using that $\pi_{\mathcal{B}}\left(c_m\right)=\pi_{\mathcal{B}}\left(c_{m-1}\right)$, we have:

$$R_{AB}\geq\pi_{\mathcal{A}}\left(c_m\right)-\pi_{\mathcal{A}}\left(2c_{m-1}-b'\right)+$$

$$+C_{\mathcal{A}}\sum_{l\in Dif(\mathcal{B})(m)}\pi_{\mathcal{A}}(l+2\left(c_m-c_{m-1}\right))\,\pi_{l,\mathcal{B}}(m).$$

Hence

$$\psi(m)-\psi(m-1)=R_{BA}+R_{AB}\geq\pi_{\mathcal{B}}\left(c_m\right)-\pi_{\mathcal{B}}\left(2c_{m-1}-a'\right)+$$
$$+C_{\mathcal{B}}\sum_{k\in Dif(\mathcal{A})(m)}\pi_{\mathcal{B}}(k+2\left(c_m-c_{m-1}\right))\,\pi_{k,\mathcal{A}}(m)+$$
$$+\pi_{\mathcal{A}}\left(c_m\right)-\pi_{\mathcal{A}}\left(2c_{m-1}-b'\right)+$$
$$+C_{\mathcal{A}}\sum_{l\in Dif(\mathcal{B})(m)}\pi_{\mathcal{A}}(l+2\left(c_m-c_{m-1}\right))\,\pi_{l,\mathcal{B}}(m)=\pi_{\mathcal{B}}\left(c_m\right)+$$
$$+\pi_{\mathcal{A}}\left(c_m\right)+f(m)-\pi_{\mathcal{A}}\left(2c_{m-1}-b'\right)-\pi_{\mathcal{B}}\left(2c_{m-1}-a'\right).$$

2. It is an immediate consequence of the previous item and the second item of Lemma 4.3.

3. It is an immediate consequence of the previous item.

□

Analogously we can prove the next complementary result.

Proposition 4.4. *Let $\mathcal{A}=(a_i)_{i\in\mathbb{Z}^+}$, $\mathcal{B}=(b_i)_{i\in\mathbb{Z}^+}$ be strictly increasing and u.d. sequences in $\mathbb{R}^+$. We define $a':=a_1$, $b':=b_1$. Let $(c_m)_{m\in\mathbb{Z}^+}$ be a strictly increasing sequence in $\mathbb{R}^+$ such that there exists $m_0\in\mathbb{N}$, $m_0\geq 2$ verifying $2c_{m_0}\geq a'+b'$. Let $m\in\mathbb{N}$, $m\geq 2$ be such that $2c_m\geq a'+b'$. Suppose that $c_m\notin\mathcal{A}\cap\mathcal{B}$, $\pi_{\mathcal{A}}\left(c_m\right)=\pi_{\mathcal{A}}\left(c_{m-1}\right)$, $\pi_{\mathcal{B}}\left(c_m\right)=\pi_{\mathcal{B}}\left(c_{m-1}\right)$ and*

$$\pi_{\mathcal{B}}\left(2c_m-a_{i_{\mathcal{A}}}\right)-\pi_{\mathcal{B}}\left(2c_{m-1}-a_{i_{\mathcal{A}}}\right)=\chi_{\mathcal{B}}\left(2c_m-a_{i_{\mathcal{A}}}\right)\ \textit{for each}\ i_{\mathcal{A}}\in\left\{1,...,\pi_{\mathcal{A}}\left(c_{m-1}\right)\right\},$$

$$\pi_{\mathcal{A}}\left(2c_m-b_{i_{\mathcal{B}}}\right)-\pi_{\mathcal{A}}\left(2c_{m-1}-b_{i_{\mathcal{B}}}\right)=\chi_{\mathcal{A}}\left(2c_m-b_{i_{\mathcal{B}}}\right)\ \textit{for each}\ i_{\mathcal{B}}\in\left\{1,...,\pi_{\mathcal{B}}\left(c_{m-1}\right)\right\}.$$

Also assume that there exist constants D_A, $D_B>0$ such that

$$\pi_{\mathcal{A}}(x)-\pi_{\mathcal{A}}(y)\leq D_{\mathcal{A}}\ \pi_{\mathcal{A}}(x-y),$$
$$\pi_{\mathcal{B}}(x)-\pi_{\mathcal{B}}(y)\leq D_{\mathcal{B}}\ \pi_{\mathcal{B}}(x-y),$$

for all $x,\ y\in(0,\ +\infty),\ x\geq y$.
We define

$$f(m):=D_{\mathcal{B}}\sum_{k\in Dif(\mathcal{A})(m)}\pi_{\mathcal{B}}\left(k+2\left(c_m-c_{m-1}\right)\right)\ \pi_{k,\mathcal{A}}(m)+$$
$$+D_{\mathcal{A}}\sum_{l\in Dif(\mathcal{B})(m)}\pi_{\mathcal{A}}\left(l+2\left(c_m-c_{m-1}\right)\right)\ \pi_{l,\mathcal{B}}(m).$$

Then:

1.

$$0\leq\psi(m)-\psi(m-1)\leq\pi_{\mathcal{B}}\left(2c_m-a_{\pi_{\mathcal{A}}(c_m)}\right)+\pi_{\mathcal{A}}\left(2c_m-b_{\pi_{\mathcal{B}}(c_m)}\right)+$$
$$+f(m)-\pi_{\mathcal{A}}\left(2c_{m-1}-b'\right)-\pi_{\mathcal{B}}\left(2c_{m-1}-a'\right).$$

*2. If $\pi_{\mathcal{B}}\left(2c_m-a_{\pi_{\mathcal{A}}(c_m)}\right)+\pi_{\mathcal{A}}\left(2c_m-b_{\pi_{\mathcal{B}}(c_m)}\right)+$
$+\,f(m)-\pi_{\mathcal{A}}\left(2c_{m-1}-b'\right)-\pi_{\mathcal{B}}\left(2c_{m-1}-a'\right)=0$, then $2b_m\notin\mathcal{A}+\mathcal{B}$.*

As in the previous section, the constants $D_{\mathcal{A}}$, $D_{\mathcal{B}}>0$ must be minimum, and both minima exist if every distribution function $\pi_{\mathcal{A}}$, $\pi_{\mathcal{B}}$ is not constant. See also Proof 3.2.

5. Proof of Theorem 1.9 and consequences.

In this section we will prove Theorem 1.9.

Proof of Theorem 1.9.

1. It is consequence of the inequalities

$$\frac{x}{\sum_{j=1}^{m}\log x_j - A} \leq \pi(x) \leq \frac{x}{\sum_{j=1}^{m}\log x_j - B}, \tag{5.1}$$

and

$$\frac{x_j}{\log x_j - A} \leq \pi\left(x_j\right) \leq \frac{x_j}{\log x_j - B} \text{ for every } j \in \{1, ..., m\}. \tag{5.2}$$

Hence

$$\frac{x}{Prod_{j=1}^{m}\left(\log x_j - A\right)} \leq Prod_{j=1}^{m}\pi\left(x_j\right) \leq \frac{x}{Prod_{j=1}^{m}\left(\log x_j - B\right)},$$

and then

$$\frac{x^2}{Prod_{j=1}^{m}\left(\log x_j - A\right)} \leq x\; Prod_{j=1}^{m}\pi\left(x_j\right) \leq \frac{x^2}{Prod_{j=1}^{m}\left(\log x_j - B\right)}.$$

So we have

$$\frac{Prod_{j=1}^{m}\left(\log x_j - B\right)}{x^2} \leq \frac{1}{x\; Prod_{j=1}^{m}\pi\left(x_j\right)} \leq \frac{Prod_{j=1}^{m}\left(\log x_j - A\right)}{x^2}. \tag{5.3}$$

Let $j \in \{1, ..., m\}$. From inequalities (5.2) we obtain:

$$\frac{Prod_{i=1,\, i\neq j}^{m}\, x_i}{Prod_{i=1,\, i\neq j}^{m}\left(\log x_i - A\right)} \leq Prod_{i=1,\, i\neq j}^{m}\,\pi\left(x_i\right) \leq \frac{Prod_{i=1,\, i\neq j}^{m}\, x_i}{Prod_{i=1,\, i\neq j}^{m}\left(\log x_i - B\right)}$$

Multiplying by x_j we have

$$\frac{x}{Prod_{i=1,\, i\neq j}^{m}\left(\log x_i - A\right)} \leq x_j \cdot Prod_{i=1,\, i\neq j}^{m}\,\pi\left(x_i\right) \leq \frac{x}{Prod_{i=1,\, i\neq j}^{m}\left(\log x_i - B\right)}.$$

Thus

$$x\sum_{j=1}^{m}\frac{1}{Prod_{i=1,\, i\neq j}^{m}\left(\log x_i - A\right)} \leq \sum_{j=1}^{m} x_j\; Prod_{i=1,\, i\neq j}^{m}\pi\left(x_i\right) \leq$$
$$\leq x\sum_{j=1}^{m}\frac{1}{Prod_{i=1,\, i\neq j}^{m}\left(\log x_i - B\right)}.$$

Multiplying these inequalities and (5.1) we obtain

$$x^2\,\frac{1}{\sum_{j=1}^{m}\log x_j - A}\sum_{j=1}^{m}\frac{1}{Prod_{i=1,\, i\neq j}^{m}\left(\log x_i - A\right)} \leq$$
$$\leq \pi(x)\left[\sum_{j=1}^{m} x_j\; Prod_{i=1,\, i\neq j}^{m}\pi\left(x_i\right)\right] \leq$$
$$\leq x^2\,\frac{1}{\sum_{j=1}^{m}\log x_j - B}\sum_{j=1}^{m}\frac{1}{Prod_{i=1,\, i\neq j}^{m}\left(\log x_i - B\right)}.$$

Finally, multiplying these inequalities and the ones of (5.3) we have

$$g(x_1, ..., x_m, A, B) \leq \frac{\pi(x)}{\frac{x \; Prod_{j=1}^{m} \pi(x_j)}{\sum_{j=1}^{m} x_j \; Prod_{i=1,\, i\neq j}^{m} \pi(x_i)}} \leq g(x_1, ..., x_m, B, A)$$

2. It is an immediate consequence of the previous item.
3. It is an immediate consequence of the first item.

□

Corollary 5.1. *Consider the sequence of number primes*

$$\mathbb{P} = \{p_1 = 2,\; p_2 = 3,\; p_3 = 5, ...\}.$$

Let $x_0,\; A,\; B \in \mathbb{R},\; x_0 \geq 1,\; 0 < A < B < \log x_0$ *be constants such that*

$$\frac{x}{\log x - A} \leq \pi(x) \leq \frac{x}{\log x - B} \text{ for all } x \in [x_0,\; +\infty).$$

Let $n \in \mathbb{Z}^+$ *be such that* $p_n \geq x_0$, *and let* $m \in \mathbb{Z}^+$ *be. Then:*

$$\frac{n\, p_n^{m-1}}{m\; \log p_n - A} \frac{(\log p_n - B)^m}{(\log p_n - A)^{m-1}} \leq \pi\left(p_n^m\right) \leq \frac{n\, p_n^{m-1}}{m\; \log p_n - B} \frac{(\log p_n - A)^m}{(\log p_n - B)^{m-1}}.$$

Proof. It is an immediate consequence of Theorem 1.9 for $x = p_n^m$ and of the fact consisting of $\pi\left(p_k\right) = k$ for every $k \in \mathbb{Z}^+$. □

Corollary 5.2. *Consider the sequence of number primes*

$$\mathbb{P} = \{p_1 = 2,\; p_2 = 3,\; p_3 = 5, ...\}.$$

Let $x_0,\; A,\; B \in \mathbb{R},\; x_0 \geq 1,\; 0 < A < B < \log x_0$ *be constants such that*

$$\frac{x}{\log x - A} \leq \pi(x) \leq \frac{x}{\log x - B} \text{ for all } x \in [x_0,\; +\infty).$$

Let $m \in \mathbb{Z}^+$, *and let* $n_1, ..., n_m \in \mathbb{Z}^+$ *be such that* $p_{n_1}, ..., p_{n_m} \geq x_0$. *Then:*

$$g(p_{n_1}, ..., p_{n_m}, A, B) \leq \frac{\pi\left(p_{n_1} \cdot ... \cdot p_{n_m}\right)}{\frac{p_{n_1} \cdot ... \cdot p_{n_m} \cdot n_1 \cdot ... \cdot n_m}{\sum_{j=1}^{m} p_{n_j} \; Prod_{i=1,\, i\neq j}^{m} n_i}} \leq g(p_{n_1}, ..., p_{n_m}, B, A),$$

where the function g *is defined in Theorem 1.9.*

Proof. This is consequence of Theorem 1.9 for $x = p_{n_1} \cdot ... \cdot p_{n_m}$ and of the fact consisting of $\pi\left(p_k\right) = k$ for every $k \in \mathbb{Z}^+$. □

Corollary 5.3. *Consider the sequence of number primes*

$$\mathbb{P} = \{p_1 = 2,\; p_2 = 3,\; p_3 = 5, ...\}.$$

Then:

$$\lim_{n_1 \to +\infty, ..., n_m \to +\infty} \frac{\pi\left(p_{n_1} \cdot ... \cdot p_{n_m}\right)}{\frac{p_{n_1} \cdot ... \cdot p_{n_m} \cdot n_1 \cdot ... \cdot n_m}{\sum_{j=1}^{m} p_{n_j} \; Prod_{i=1,\, i\neq j}^{m} n_i}} = 1.$$

6. Proof of Theorem 1.10.

Proof of Theorem 1.10.

1. It is consequence of the inequalities

$$\frac{x}{\sum_{j=1}^{m}\log x_j - A} \leq \pi(x) \leq \frac{x}{\sum_{j=1}^{m}\log x_j - B}, \tag{6.1}$$

and

$$\frac{x_j}{\log x_j - A} \leq \pi\left(x_j\right) \leq \frac{x_j}{\log x_j - B} \text{ for every } j \in \{1, ..., m\}. \tag{6.2}$$

Indeed

$$\frac{x^2}{Prod_{j=1}^{m}\left(\log x_j - A\right)} \leq x\ Prod_{j=1}^{m}\pi\left(x_j\right) \leq \frac{x^2}{Prod_{j=1}^{m}\left(\log x_j - B\right)}. \tag{6.3}$$

Let $j \in \{1, ..., m\}$. From inequalities (5.2) we obtain:

$$\frac{Prod_{i=1,\, i\neq j}^{m}\, x_i}{Prod_{i=1,\, i\neq j}^{m}\left(\log x_i - A\right)} \leq Prod_{i=1,\, i\neq j}^{m}\,\pi\left(x_i\right) \leq \frac{Prod_{i=1,\, i\neq j}^{m}\, x_i}{Prod_{i=1,\, i\neq j}^{m}\left(\log x_i - B\right)}.$$

Multiplying by x_j we have

$$\frac{x}{Prod_{i=1,\, i\neq j}^{m}\left(\log x_i - A\right)} \leq x_j \cdot Prod_{i=1,\, i\neq j}^{m}\,\pi\left(x_i\right) \leq$$
$$\leq \frac{x}{Prod_{i=1,\, i\neq j}^{m}\left(\log x_i - B\right)}.$$

Thus

$$x\sum_{j=1}^{m}\frac{1}{Prod_{i=1,\, i\neq j}^{m}\left(\log x_i - A\right)} \leq \sum_{j=1}^{m} x_j\ Prod_{i=1,\, i\neq j}^{m}\pi\left(x_i\right) \leq$$
$$\leq x\sum_{j=1}^{m}\frac{1}{Prod_{i=1,\, i\neq j}^{m}\left(\log x_i - B\right)}.$$

Then

$$\frac{1}{x}\,\frac{1}{\sum_{j=1}^{m}\frac{1}{Prod_{i=1,\, i\neq j}^{m}\left(\log x_i - B\right)}} \leq \frac{1}{\sum_{j=1}^{m} x_j\ Prod_{i=1,\, i\neq j}^{m}\pi\left(x_i\right)} \leq$$
$$\leq \frac{1}{x}\,\frac{1}{\sum_{j=1}^{m}\frac{1}{Prod_{i=1,\, i\neq j}^{m}\left(\log x_i - A\right)}}.$$

Multiplying the inequalities (6.3) and the previous ones we obtain

$$x\ h(x_1, ..., x_m, A, B) \leq \frac{x\ Prod_{j=1}^{m}\pi\left(x_j\right)}{\sum_{j=1}^{m} x_j\ Prod_{i=1,\, i\neq j}^{m}\pi\left(x_i\right)} \leq x\ h(x_1, ..., x_m, B, A)$$

So

$$-x\ h(x_1, ..., x_m, B, A) \leq -\frac{x\ Prod_{j=1}^{m}\pi\left(x_j\right)}{\sum_{j=1}^{m} x_j\ Prod_{i=1,\, i\neq j}^{m}\pi\left(x_i\right)} \leq -x\ h(x_1, ..., x_m, A, B)$$

Finally we sum these inequalities to inequalities (6.1), and obtain

$$x\, l(x_1,...,x_m,B,A) \le \pi(x) - \frac{x\; Prod_{j=1}^{m} \pi\left(x_j\right)}{\sum_{j=1}^{m} x_j\; Prod_{i=1,\, i\neq j}^{m} \pi\left(x_i\right)} \le x\, l(x_1,...,x_m,A,B).$$

2. It is obvious.
3. It is obvious.
4. It is an immediate consequence of the first item.

□

7. Appendix

A slight generalization of Theorem 1.7 and Corollary 2.2 are the following results, with analogous proofs.

Theorem 7.1. *Let $\mathcal{P} = (p_i)_{i\in\mathbb{Z}^+}$ be a strictly increasing sequence in $\mathbb{Z}^+$ such that $\mathcal{P} \subseteq 2\mathbb{N}$ or $\mathcal{P} \subseteq 2\mathbb{N}+1$. Assume that there exist constants $C > 0$, $D \ge 0$ such that*

$$\pi(x) - \pi(y) \ge C\; \pi(x-y) - D \text{ for every } x,\; y \in [1,\, +\infty),\; x \ge y.$$

Define $a := p_1$, and consider the function $f : \mathbb{N}_{\ge a} \setminus \mathcal{P} \to \mathbb{N}$ defined by

$$f(m) := \sum_{k\in Dif(\mathcal{P})(m)} \pi(k+2)\; \pi_k(m) \text{ for every } m \in \mathbb{N}_{\ge a} \setminus \mathcal{P}.$$

Then:

1. *$\psi(m) - \psi(m-1) \ge (1-D)\; \pi(m) + D + C\; f(m) - \pi(2m-a-2)$ for all $m \in \mathbb{N}_{\ge a} \setminus \mathcal{P}$.*
2. *Let $m \in \mathbb{N}_{\ge a} \setminus \mathcal{P}$. If $(1-D)\; \pi(m) + D + C\; f(m) - \pi(2m-a-2) > 0$, then $2m \in \mathcal{P}+\mathcal{P}$.*
3. *If $\liminf_{m\to+\infty,\, m\notin\mathcal{P}} \frac{(1-D)\,\pi(m)+C\, f(m)}{\pi(2m)} \ge L \in (1,\, +\infty]$, then there exists $m_0 \in \mathbb{N}_{\ge a} \setminus \mathcal{P}$ such that $2m \in \mathcal{P}+\mathcal{P}$ for each $m \in \mathbb{N}_{\ge a} \setminus \mathcal{P}$, $m \ge m_0$.*

Corollary 7.2. *Let $\mathcal{P} = (p_i)_{i\in\mathbb{Z}^+}$ be a strictly increasing sequence in $\mathbb{Z}^+$ such that $\mathcal{P} \subseteq 2\mathbb{N}$ or $\mathcal{P} \subseteq 2\mathbb{N}+1$. Suppose that there exist constants $C_1 > 0$, $D \ge 0$ such that*

$$\pi(x) - \pi(y) \ge C_1\; \pi(x-y) - D \text{ for all } x,\; y \in [1,\, +\infty),\; x \ge y.$$

Define $a := p_1$. Let $\alpha \in \mathbb{R}^+$. Suppose that

1. *There exist constants $C_2 > 0$, $m_0 \in \mathbb{N}_{\ge a} \setminus \mathcal{P}$ such that*

$$Card\left(Dif\left(\mathcal{P}\right)(m)\right) \ge C_2\, \log^{\alpha}(m)$$

   for every $m \in \mathbb{N}_{\ge a} \setminus \mathcal{P}$, $m \ge m_0$.

2. *There exist constants $m_1 \in \mathbb{N}_{\ge a} \setminus \mathcal{P}$, A, B, $C_3 \in \mathbb{R}$, $C_3 > 0$ such that*

$$\pi_k(m) \ge \frac{C_3\, m}{(\log(m)+A)^{\alpha+1} + B}$$

   for every $m \in \mathbb{N}_{\ge a} \setminus \mathcal{P}$, $m \ge m_1$, and every $k \in \left\{2, 4, ..., 2\, E\left(\frac{C_2\, \log^{\alpha}(m)}{2}\right)\right\}$.

3. *$\lim_{n\to+\infty,\, n\in\mathbb{N}} \frac{\pi(n)}{\frac{n}{\log n}} = 1$.*
4. *$C_1 \cdot C_2 \cdot C_3 > 1 + D$.*

Then there exists $m_2 \in \mathbb{N}_{\ge a} \setminus \mathcal{P}$ such that $2m \in \mathcal{P}+\mathcal{P}$ for all $m \in \mathbb{N}_{\ge a} \setminus \mathcal{P}$, $m \ge m_2$.

References

1. Bateman, P. T., Diamond H. G., *Analytic Number Theory: An Introductory Course.* World Scientific Pub. Co. Inc., Monographs In Number Theory, 2004.
2. De La Vallèe Poussin, Ch. J., *Recherches analytiques sur la thèorie des nombres premiers*, Ann. Soc. Sci. Bruxelles, vol. **20**, pp. 183-256, 281-297, 1896.
3. Dusart, Pierre, *Autour de la fonction qui compte le nombre de nombres premiers*, Doctoral Thesis, Universitè de Limoges, 1998.
4. Dusart, Pierre, *Estimates of some functions over primes without the Riemann Hypothesis* arXiv:1002.0442v1[math.Nt], 2010.
5. Hadamard, Jacques, *Sur la distribution des zèros de la fonction* $\zeta(s)$ *et ses consèquences arithmètiques*, Bull. Soc. Math. France, vol. **24**, pp. 199-220, 1896.
6. Hardy, G. H., Littlewood, J. E., *Some problems of 'Partitio Numerorum'. III. On the Expression of a Number as a Sum of Primes.* Acta Math. **44**, 1-70, 1923.
7. Helfgott, H. A., *The ternary Goldbach problem* arXiv: 1501.05438v2[mathNT], 2014.
8. Ribenboim, P., *The new Book of Prime Number Records.* Springer-Verlag (New York), 259-265, 1996.

José Alfonso López Nicolás,
Universidad de Murcia,
Calle Campus Universitario, s/n, 30100 El Puntal,
Murcia, Spain.
E-mail address: jaln4@um.es