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On the Structure of Split Regular δ-Hom-Jordan-Lie Superalgebras

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ABSTRACT: In this paper we study the structure of arbitrary split regular δ-Hom-Jordan-Lie super algebras. By developing techniques of connections of roots for this kind of algebras, we show that such a split regular δ-Hom-Jordan-Lie superalgebra $\mathcal{L}$ is of the form $\mathcal{L} = \mathcal{H}_{[\alpha]} \oplus \sum_{[\alpha]\in\Lambda/\sim} \mathcal{V}_{[\alpha]}$, with $\mathcal{H}_{[\alpha]}$ a graded linear subspace of the graded abelian subalgebra $\mathcal{H}$ and any $\mathcal{V}_{[\alpha]}$, a well-described ideal of $\mathcal{L}$, satisfying $[\mathcal{V}_{[\alpha]}, \mathcal{V}_{[\beta]}] = 0$ if $[\alpha] \neq [\beta]$. Under certain conditions, in the case of $\mathcal{L}$ being of maximal length, the simplicity of the algebra is characterized and it is shown that $\mathcal{L}$ is the direct sum of the family of its minimal ideals, each one being a simple split regular δ-Hom-Jordan-Lie superalgebra.

Key Words: Split Hom-Lie superalgebras, structure theory.

Contents

1. Introduction

Hom-algebraic structures appeared first as a generalization of Lie algebras in [4], where the author studied $q-$deformations of Witt and Virasoro algebras. A general study and construction of Hom-Lie algebras were considered in [11,12]. Since then, other intersting Hom-type algebraic structures of many classical structures were studied Hom-associative algebras, Hom-Lie admissible algebras and Hom-Jordan algebras. Hom-algebraic structures were extended to Hom-Lie superalgebras in [3].

As a generalization of Lie superalgebras and Jordan Lie algebras, the notion of δ-Jordan Lie superalgebra was introduced in [10,15], which is intimately related to both Jordan-super and antiassociative algebras. The case of $\delta = 1$ yields the Lie superalgebra, and we call the other case of $\delta = -1$ a Jordan Lie superalgebra because it turns out to be a Jordan superalgebra. It is often convenient to consider both cases of $\delta = \pm 1$, and call δ-Jordan Lie superalgebras. The motivations to charactrize Hom-Lie structurers are related to physics and to deformations of Lie algebras, in particular Lie algebras of vector fields. Hom-Lie superalgebras are a generalization of Lie superalgebras, where the classical super Jacobi identity is twisted by a linear map. If the skew-super symmetric bracket of a Hom-Lie superalgebra is replaced by δ-Jordan-super symmetric, it is called a δ-Jordan-Hom-Lie superalgebra (see [14]).

As is well-known, the class of the split algebras is especially related to addition quantum numbers, graded contractions and deformations. For instance, for a physical system, which displays a symmetry of Lie algebra $\mathcal{L}$, it is interesting to know in detail the structure of the split decomposition, because its roots can be seen as certain eigenvalues which are the additive quantum numbers characterizing the state of such a system. Determining the inner structure of split algebras will become more and more meaningful in the area of research in mathematical physics. Recently, in [2,5,8,9,13,16], the inner structure of arbitrary split Lie algebras, arbitrary split Lie superalgebras, arbitrary split regular Hom-Lie algebras, arbitrary split regular Hom-Lie superalgebras, arbitrary split regular δ-Hom-Lie algebras, arbitrary split involutive regular Hom-Lie algebras and arbitrary split involutive regular Hom-Lie color algebras have been determined by the techniques of connection of roots.

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Our goal in this work is to study the structure of arbitrary split regular δ-Hom-Jordan-Lie superalgebras by the techniques of connection of roots. The results of this article are based on some works in [1,6,13].

Throughout this paper, split regular δ-Hom-Jordan-Lie super algebras $\mathcal{L}$ are considered arbitrary dimension and over an arbitrary base field $\mathbb{F}$, with characteristic zero.

To close this introduction, we briefly outline the contents of the paper. In Section 2, we begin by recalling the necessary background on split regular δ-Hom-Jordan-Lie superalgebras. Section 3 develops techniques of connections of roots for split regular δ-Hom-Jordan-Lie superalgebras. We also show that such an arbitrary split regular δ-Hom-Jordan-Lie superalgebra $\mathcal{L}$ with a root system Λ is of the form $\mathcal{L} = \mathcal{H}_{[\alpha]} \oplus \sum_{[\alpha]\in\Lambda/\sim} \mathcal{V}_{[\alpha]}$, with $\mathcal{H}_{[\alpha]}$ a graded linear subspace of the graded abelian subalgebra $\mathcal{H}$ and any $\mathcal{V}_{[\alpha]}$, a well-described ideal of $\mathcal{L}$, satisfying $[\mathcal{V}_{[\alpha]}, \mathcal{V}_{[\beta]}] = 0$ if $[\alpha] \neq [\beta]$. In section 4, we show that under certain conditions, in the case of $\mathcal{L}$ being of maximal length, the simplicity of the algebra is characterized and it is shown that $\mathcal{L}$ is the direct sum of the family of its minimal ideals, each one being a simple split regular δ-Hom-Jordan-Lie superalgebra.

2. Preliminaries

Let us begin with some definitions concerning δ-Hom-Jordan-Lie super algebras. For a detailed discussion of this subject, we refer the reader to the literature [14].

Definition 2.1. *[15] A* δ-Jordan-Lie superalgebra *is a $\mathbb{Z}_2$-graded algebra $\mathcal{L} = \mathcal{L}_{\bar{0}} \oplus \mathcal{L}_{\bar{1}}$ over a base field $\mathbb{F}$ endowed with an even bilinear map $[.,.] : \mathcal{L} \times \mathcal{L} \longrightarrow \mathcal{L}$ (i.e,. $[\mathcal{L}_{\bar{i}}, \mathcal{L}_{\bar{j}}] \subset \mathcal{L}_{\bar{i}+\bar{j}}$, $\bar{i}, \bar{j} \in \mathbb{Z}_2$) satisfying;*

(i) $[x, y] = -\delta(-1)^{\bar{i}\bar{j}}[y, x], \quad \delta = \pm 1$

(ii) $(-1)^{\bar{i}\bar{k}}[x, [y, z]] + (-1)^{\bar{i}\bar{j}}[y, [z, x]] + (-1)^{\bar{i}\bar{k}}[z, [x, y]] = 0$

for all homogeneous elements $x \in \mathcal{L}_{\bar{i}}, y \in \mathcal{L}_{\bar{j}}$ and $z \in \mathcal{L}_{\bar{k}},$ with $\bar{i}, \bar{j}, \bar{k} \in \mathbb{Z}_2$.

Definition 2.2. *[14] A* δ-Hom-Jordan-Lie superalgebra *is a quadruple $(\mathcal{L}, [.,.], \delta, \phi)$ consisting of a $\mathbb{Z}_2-$graded vector space $\mathcal{L} = \mathcal{L}_{\bar{0}} \oplus \mathcal{L}_{\bar{1}}$, an even bilinear map $[.,.] : \mathcal{L} \times \mathcal{L} \longrightarrow \mathcal{L}$ and a linear map $\phi : \mathcal{L} \longrightarrow \mathcal{L}$ satisfying;*

(i) $[x, y] = -\delta(-1)^{\bar{i}\bar{j}}[y, x], \quad \delta = \pm 1$

(ii) $(-1)^{\bar{i}\bar{k}}[\phi(x), [y, z]] + (-1)^{\bar{i}\bar{j}}[\phi(y), [z, x]] + (-1)^{\bar{i}\bar{k}}[\phi(z), [x, y]] = 0,$ *(δ-super Hom-jacobi identity)*

for all homogeneous elements $x \in \mathcal{L}_{\bar{i}}, y \in \mathcal{L}_{\bar{j}}$ and $z \in \mathcal{L}_{\bar{k}},$ with $\bar{i}, \bar{j}, \bar{k} \in \mathbb{Z}_2$.

When ϕ is an algebra automorphism it is said to be a regular δ-Hom-Jordan-Lie superalgebra. We recover δ-Jordan-Lie superalgebra when we have $\phi = id$.

Especially, for $\delta = 1$ one has a Hom-Lie superalgebra and for $\delta = -1$ a Hom-Jordan-Lie superalgebra.

The usual regularity conditions will be understood in the graded sense. For instance, a subalgebra A of $\mathcal{L}$ is a graded subspace $A = A_{\bar{0}} \oplus A_{\bar{1}}$ of $\mathcal{L}$ such that $[A, A] \subset A$ and $\phi(A) = A$. A graded subspace $I = I_{\bar{0}} \oplus I_{\bar{1}}$ of $\mathcal{L}$ is called *an ideal* if $[I, \mathcal{L}] \subset I$ and $\phi(I) = I$. We say that $\mathcal{L}$ is *graded simple* if $[\mathcal{L}, \mathcal{L}] \neq 0$ and its only (graded) ideals are (0) and $\mathcal{L}$.

Throught this paper we consider regular δ-Hom-Jordan-Lie superalgebra $\mathcal{L}$ and denote by $\mathbb{N}_0$ the set of all non-negative integers and by $\mathbb{Z}$ the set of all integers.

We introduce the class of split algebras in the fromwork of regular δ-Hom-Jordan-Lie superalgebra in an analogous way. We begin by considering a maximal abelian $\mathbb{Z}_2-$graded subalgebra $\mathcal{H} = \mathcal{H}_{\bar{0}} \oplus \mathcal{H}_{\bar{1}}$ among the abelian $\mathbb{Z}_2-$graded subalgebras of $\mathcal{L}$. For a linear functional

$$\alpha : \mathcal{H}_{\bar{0}} \longrightarrow \mathbb{F},$$

we define the root space of $\mathcal{L}$ (with respect to $\mathcal{H}$) associated to α as the subspace

$$\mathcal{L}_\alpha := \{x_\alpha \in \mathcal{L} \mid [h_{\bar{0}}, x_\alpha] = \alpha(h_{\bar{0}})\phi(x_\alpha) \text{ for all } h_{\bar{0}} \in \mathcal{H}_{\bar{0}}\}.$$

The elements $\alpha : \mathcal{H}_{\bar{0}} \longrightarrow \mathbb{F}$ satisfying $\mathcal{L}_\alpha \neq \{0\}$ are called *roots of* $\mathcal{L}$ with respect to $\mathcal{H}$. We denote by $\Lambda := \{\alpha \in (\mathcal{H}_{\bar{0}})^* \setminus \{0\} \mid \mathcal{L}_\alpha \neq \{0\}\}$.

Definition 2.3. *We say that* $\mathcal{L}$ *is a* split regular δ-Hom-Jordan-Lie superalgebra*, with respect to* $\mathcal{H}$*, if*

$$\mathcal{L} = \mathcal{H} \oplus (\bigoplus_{\alpha\in\Lambda} \mathcal{L}_\alpha).$$

We also say that Λ *is the* root system of $\mathcal{L}$. *We recall that a root system* Λ *of* $\mathcal{L}$ *is called symmetric if* $\Lambda = -\Lambda$.

As examples of split regular δ-Hom-Jordan-Lie superalgebras we have the split regular Hom-Lie superalgebras and split regular δ-Jordan-Lie algebras. Hence, the present paper extends the results in [1,6].

Lemma 2.4. *Let* $\mathcal{L} = \mathcal{L}_{\bar{0}} \oplus \mathcal{L}_{\bar{1}}$ *be a split regular* δ*-Hom-Jordan-Lie superalgebra, with root space decomposition* $\mathcal{L} = \mathcal{H} \oplus (\bigoplus_{\alpha\in\Lambda} \mathcal{L}_\alpha)$*. Then*

(1) $\mathcal{L}_0 = \mathcal{H}$,

(2) for any $\alpha \in \Lambda \cup \{0\}$*, we have* $\mathcal{L}_\alpha = \mathcal{L}_{\bar{0},\alpha} \oplus \mathcal{L}_{\bar{1},\alpha}$*, where* $\mathcal{L}_{\bar{i},\alpha} = \mathcal{L}_\alpha \cap \mathcal{L}_{\bar{i}},\ \forall\alpha \in \Lambda,\ \forall\bar{i} \in \mathbb{Z}_2$,

(3) $\mathcal{L}_{\bar{0}} = \mathcal{H}_{\bar{0}} \oplus (\bigoplus_{\alpha\in\Lambda} \mathcal{L}_{\bar{0},\alpha})$ *is a split regular* δ*-Hom-Lie algebra and* $\mathcal{L}_{\bar{1}} = \mathcal{H}_{\bar{1}} \oplus (\bigoplus_{\alpha\in\Lambda} \mathcal{L}_{\bar{1},\alpha})$ *is a split regular anti-Lie triple system.*

Proof. (1) It is clear that the root space associated to the zero root satisfies $\mathcal{H} \subset \mathcal{L}_0$. Coveresly, given any $x_0 \in \mathcal{L}_0$ we can write $x_0 = h + \sum_{i=1}^n x_{\alpha_i}$ with $h \in \mathcal{H}$ and $x_{\alpha_i} \in \mathcal{L}_{\alpha_i}$, for $i = 1, 2, ..., n$, bing $\alpha_i \in \Lambda$ with $\alpha_i \neq \alpha_j$ if $i \neq j$. Hence, for any $h_{\bar{0}} \in \mathcal{H}_{\bar{0}}$ we have

$$0 = [h_{\bar{0}}, h + \sum_{i=1}^n x_{\alpha_i}] = \sum_{i=1}^n [h_{\bar{0}}, x_{\alpha_i}] = \sum_{i=1}^n \alpha_i(h_{\bar{0}})\phi(x_{\alpha_i}).$$

Taking into account the direct character of the sum, that $\alpha_i \neq 0$ and ϕ is an automorphism, we have that any $x_{\alpha_i} = 0$ and then $x_0 = h \in \mathcal{H}$.

(2) By the grading of $\mathcal{L}$, for any $x_\alpha \in \mathcal{L}_\alpha,\ \alpha \in \Lambda \cup \{0\}$ may be expressed in the form $x_{\bar{0},\alpha} + x_{\bar{1},\alpha}$ with $x_{\bar{i},\alpha} \in \mathcal{L}_{\bar{i}},\ \bar{i} \in \mathbb{Z}_2$. Then for any $h_{\bar{0}} \in \mathcal{H}_{\bar{0}}$ we have

$$[h_{\bar{0}}, x_{\bar{i},\alpha}] = \alpha(h_{\bar{0}})\phi(x_{\bar{i},\alpha}),$$

From here, $\mathcal{L}_\alpha = \mathcal{L}_{\bar{0},\alpha} \oplus \mathcal{L}_{\bar{1},\alpha}$, where $\mathcal{L}_{\bar{i},\alpha} = \mathcal{L}_\alpha \cap \mathcal{L}_{\bar{i}},\ \forall\alpha \in \Lambda,\ \forall\bar{i} \in \mathbb{Z}_2$. In particular $\mathcal{H} = \mathcal{L}_{\bar{0},0} \oplus \mathcal{L}_{\bar{1},0}$.

(3) By part (2) we have

$$\mathcal{L}_{\bar{0}} = \mathcal{H}_{\bar{0}} \oplus (\bigoplus_{\alpha\in\Lambda} \mathcal{L}_{\bar{0},\alpha}) \text{ and } \ \mathcal{L}_{\bar{1}} = \mathcal{H}_{\bar{1}} \oplus (\bigoplus_{\alpha\in\Lambda} \mathcal{L}_{\bar{1},\alpha}).$$

Taking into acount this expression of the Hom-Lie algebra $\mathcal{L}_{\bar{0}}$, the direct character of the sum and the fact $\alpha \neq 0$ for any $\alpha \in \Lambda$, we have that $\mathcal{H}_{\bar{0}}$ is a maximal abelian subalgebra of $\mathcal{L}_{\bar{0}}$. Hence $\mathcal{L}_{\bar{0}}$ is a split regular δ-Hom-Lie algebra with respect to $\mathcal{H}_{\bar{0}}$(for mor details see [5]). In the sence of [7] $\mathcal{L}_{\bar{1}} = \mathcal{H}_{\bar{1}} \oplus (\bigoplus_{\alpha\in\Lambda} \mathcal{L}_{\bar{1},\alpha})$ is a split regular anti-Lie triple system with respect to $\mathcal{H}_{\bar{1}}$. □

Note that if $\mathcal{L}$ is a split regular δ-Hom-Jordan-Lie superalgebra, with root space decomposition $\mathcal{L} = \mathcal{H} \oplus (\bigoplus_{\alpha\in\Lambda} \mathcal{L}_\alpha)$, taking into account Lemma 2.4, we then write

$$\mathcal{L} = (\mathcal{H}_{\bar{0}} \oplus (\bigoplus_{\Lambda_{\bar{0}}} \mathcal{L}_{\bar{0},\alpha})) \oplus (\mathcal{H}_{\bar{1}} \oplus (\bigoplus_{\Lambda_{\bar{1}}} \mathcal{L}_{\bar{1},\alpha})), \tag{2.1}$$

where $\Lambda_{\bar{0}} := \{\alpha \in \Lambda : \mathcal{L}_{\bar{0},\alpha} \neq 0\}$ and $\Lambda_{\bar{1}} = \{\alpha \in \Lambda : \mathcal{L}_{\bar{1},\alpha} \neq 0\}$.

Lemma 2.5. *For any $\alpha, \beta \in \Lambda \cup \{0\}$ and any $\bar{i}, \bar{j} \in \mathbb{Z}_2$, the following assertions hold.*

(1) $\phi(\mathcal{L}_{\bar{i},\alpha}) \subset \mathcal{L}_{\bar{i},\alpha\phi^{-1}}$ *and* $\phi^{-1}(\mathcal{L}_{\bar{i},\alpha}) \subset \mathcal{L}_{\bar{i},\alpha\phi}$,

(2) $[\mathcal{L}_{\alpha,\bar{i}}, \mathcal{L}_{\bar{j},\beta}] \subset \mathcal{L}_{\bar{i}+\bar{j},\delta(\alpha+\beta)\phi^{-1}}$,

(3) If $\mathcal{L}_{\bar{i},\alpha} \neq 0$ *then* $\mathcal{L}_{\bar{i},\alpha\phi^{\mathbb{Z}}} \neq 0$, *for any* $z \in \mathbb{Z}$.

Proof. (1) By Lemma 2.4-(2) and the fact that ϕ is an automorphism, we have

$$\begin{aligned} \phi(\mathcal{L}_{\bar{i},\alpha}) &= \phi(\mathcal{L}_\alpha \cap \mathcal{L}_{\bar{i}}) \\ &= \phi(\mathcal{L}_\alpha) \cap (\mathcal{L}_{\bar{i}}. \end{aligned}$$

Now taking into account Lemma 1. 1 in [1], we get

$$\phi(\mathcal{L}_{\bar{i},\alpha}) \subset \mathcal{L}_{\alpha\phi^{-1}} \cap \mathcal{L}_{\bar{i}} = \mathcal{L}_{\bar{i},\alpha\phi^{-1}}.$$

In a similar way, one gets the second statement in (1).

(2) For any for any $h_{\bar{0}} \in \mathcal{H}_{\bar{0}}, x_{\bar{i},\alpha} \in \mathcal{L}_{\bar{i},\alpha}$ and $y_{\bar{j},\beta} \in \mathcal{L}_{\bar{j},\beta}$, by denoting $h' = \phi(h_{\bar{0}})$, from δ-super Hom-jacobi identity, we have

$$\begin{aligned} [h', [x_{\bar{i},\alpha}, y_{\bar{j},\beta}]] &= [\phi(h_{\bar{0}}), [x_{\bar{i},\alpha}, y_{\bar{j},\beta}]] \\ &= \delta[[h_{\bar{0}}, x_{\bar{i},\alpha}], \phi(y_{\bar{j},\beta})] + \delta[\phi(x_{\bar{i},\alpha}), [h_{\bar{0}}, y_{\bar{j},\beta}]] \\ &= \delta[\alpha(h_{\bar{0}})\phi(x_{\bar{i},\alpha}), \phi(y_{\bar{j},\beta})] + \delta[\phi(x_{\bar{i},\alpha}), \beta(h_{\bar{0}})\phi(y_{\bar{j},\beta})] \\ &= \delta(\alpha+\beta)(h_{\bar{0}})\phi([x_{\bar{i},\alpha}, y_{\bar{j},\beta}]) \\ &= \delta(\alpha+\beta)(\phi^{-1}(h')\phi([x_{\bar{i},\alpha}, y_{\bar{j},\beta}]). \end{aligned}$$

Therefore, we get $[x_{\bar{i},\alpha}, y_{\bar{j},\beta}] \in \mathcal{L}_{\bar{i}+\bar{j},\delta(\alpha+\beta)\phi^{-1}}$ and so we conclud that the result.

(3) The proof is similar to part (1). □

3. Connections of roots and decompositions

In the following, $\mathcal{L}$ denotes a split regular δ-Hom-Jordan-Lie superalgebra with a symmetric root system Λ and $\mathcal{L} = \mathcal{H} \oplus (\bigoplus_{\alpha\in\Lambda} \mathcal{L}_\alpha)$, the corresponding root space decomposition. We begin by developing the techniques of connections of roots in this section.

Definition 3.1. *Let α, β be two nonzero roots in Λ. We say that α* is connected to *β and denoted by $\alpha \sim \beta$ if there exists a family*

$$\{\alpha_1, \alpha_2, \alpha_3, ..., \alpha_k\} \subset \Lambda,$$

satisfying the following conditions;

If $k = 1$:

(1) $\alpha_1 \in \{\alpha\phi^{-n} : n \in \mathbb{N}_0\} \cap \{\pm\beta\phi^{-m} : m \in \mathbb{N}_0\}$.

If $k \geq 2$:

(1) $\alpha_1 \in \{\alpha\phi^{-n} : n \in \mathbb{N}_0\}$.

(2) $\delta\alpha_1\phi^{-1} + \delta\alpha_2\phi^{-1} \in \Lambda$,
$\delta^2\alpha_1\phi^{-2} + \delta^2\alpha_2\phi^{-2} + \delta\alpha_3\phi^{-1} \in \Lambda$,
$\delta^3\alpha_1\phi^{-3} + \delta^3\alpha_2\phi^{-3} + \delta^2\alpha_3\phi^{-2} + \delta\alpha_4\phi^{-1} \in \Lambda$,
...
$\delta^i\alpha_\phi^{-i} + \delta^i\alpha_2\phi^{-i} + \delta^{i-1}\alpha_3\phi^{-i+1} + ... + \delta\alpha_{i+1}\phi^{-1} \in \Lambda$,
...
$\delta^{k-2}\alpha_1\phi^{-k+2} + \delta^{k-2}\alpha_2\phi^{-k+2} + \delta^{k-3}\alpha_3\phi^{-k+3} + ... + \delta^{k-i}\alpha_i\phi^{-k+i} + ... + \delta\alpha_{k-1}\phi^{-1} \in \Lambda$.

(3) $\delta^{k-1}\alpha_1\phi^{-k+1}+\delta^{k-1}\alpha_2\phi^{-k+1}+\delta^{k-2}\alpha_3\phi^{-k+2}+...+\delta^{k-i+1}\alpha_i\phi^{-k+i-1}+...+\delta\alpha_k\phi^{-1}\in\{\pm\beta\phi^{-m}: m\in\mathbb{N}_0\}$.

The family $\{\alpha_1,\alpha_2,\alpha_3,...,\alpha_k\}\subset\Lambda$ *is called a* connection *from* α *to* β.

Note that the case $k=1$ in Definition 3.1 is equivalent to the fact $\beta=\varepsilon\alpha\phi^{z}$ for some $z\in\mathbb{Z}$ and $\varepsilon\in\{\pm1\}$.

Lemma 3.2. *The following assertions hold*

(1) *For any* $\alpha\in\Lambda$, *we have that* $\alpha\phi^{z_1}$ *is connected to* $\alpha\phi^{z_2}$ *for every* $z_1,z_2\in\mathbb{Z}$. *We also have that* $\alpha\phi^{z_1}$ *is connected to* $-\alpha\phi^{z_2}$ *in case* $-\alpha\phi^{z_2}\in\Lambda$.

(2) *Let* $\{\alpha_1,\alpha_2,\alpha_3,...,\alpha_k\}$ *be a connection from* α *to* β. *Suppose that* $\alpha_1=\alpha\phi^{-n}, n\in\mathbb{N}_0$. *Then for any* $r\in\mathbb{N}_0$ *such that* $r\geq n$, *there exists a connection* $\{\tilde{\alpha}_1,\tilde{\alpha}_2,...,\tilde{\alpha}_k\}$ *from* α *to* β *such that* $\tilde{\alpha}_1=\alpha\phi^{-r}$.

(3) *Let* $\{\alpha_1,\alpha_2,\alpha_3,...,\alpha_k\}$ *be a connection from* α *to* β. *Suppose that* $\alpha_1=\varepsilon\beta\phi^{-m}, m\in\mathbb{N}_0$ *in case* $k=1$ *or*

$$\delta^{k-1}\alpha_1\phi^{-k+1}+\delta^{k-1}\alpha_2\phi^{-k+1}+\delta^{k-2}\alpha_3\phi^{-k+2}+...+\delta^{k-i+1}\alpha_i\phi^{-k+i-1}+...+\delta\alpha_k\phi^{-1}=\varepsilon\beta\phi^{-m},$$

in case $k\geq 2$, *with* $\varepsilon\in\{\pm1\}$. *Then for any* $r\in\mathbb{N}_0$ *such that* $r\geq m$, *there exists a connection* $\{\tilde{\alpha}_1,\tilde{\alpha}_2,...,\tilde{\alpha}_k\}$ *from* α *to* β *such that* $\tilde{\alpha}_1=\varepsilon\beta\phi^{-r}$ *in case* $k=1$ *or*

$$\delta^{k-1}\tilde{\alpha}_1\phi^{-k+1}+\delta^{k-1}\tilde{\alpha}_2\phi^{-k+1}+\delta^{k-2}\tilde{\alpha}_3\phi^{-k+2}+...+\delta\tilde{\alpha}_k\phi^{-1}=\varepsilon\beta\phi^{-r},$$

in case $k\geq 2$.

Proof. They are proved in Lemma 3.2 and Lemma 3.3 in [6]. □

Proposition 3.3. *The relation* $\sim$ *in* Λ *defined by*

$$\alpha\sim\beta \text{ if and only if } \alpha \text{ is connected to } \beta,$$

is an equivalence relation.

Proof. The proof is vertically identical to the proof of Proposition 3.4 in [6]. □

By the above proposition, we can consider the equivalence relation in Λ by the connection relation $\sim$ in Λ. So we denote by

$$\Lambda/\sim:=\{[\alpha]:a\in\Lambda\},$$

where $[\alpha]$ denotes the set of nonzero roots of $\mathcal{L}$ which are connected to α. Clearly, if $\beta\in[\alpha]$ then $-\beta\in[\alpha]$ and by Proposition 3.3, if $\beta\notin[\alpha]$ then $[\alpha]\cap[\beta]=\emptyset$.

Our next goal in this section is to associate an adequate ideal $\mathcal{L}_{[\alpha]}$ of $\mathcal{L}$ to any $[\alpha]$. For a fixed $\alpha\in\Lambda$, we define

$$\mathcal{H}_{[\alpha]}:=\operatorname{span}_{\mathbb{F}}\{[\mathcal{L}_\beta,\mathcal{L}_{-\beta}]\ :\ \beta\in[\alpha]\}\cup\mathcal{L}_0.$$

Applying Lemma 2.5-(2), we obtain

$$\begin{aligned}\mathcal{H}_{[\alpha]} &= \sum_{\beta\in[\alpha]}([\mathcal{L}_{\bar{0},\beta},\mathcal{L}_{\bar{0},-\beta}]+[\mathcal{L}_{\bar{1},\beta},\mathcal{L}_{\bar{1},-\beta}])\oplus\sum_{\beta\in[\alpha]}([\mathcal{L}_{\bar{0},\beta},\mathcal{L}_{\bar{1},-\beta}]+[\mathcal{L}_{\bar{1},\beta},\mathcal{L}_{\bar{0},-\beta}])\\ &\subset \mathcal{L}_{\bar{0},0}\oplus\mathcal{L}_{\bar{1},0}=\mathcal{H}.\end{aligned}\tag{3.1}$$

Next, we define

$$\mathcal{V}_{[\alpha]}:=\bigoplus_{\beta\in[\alpha]}\mathcal{L}_\beta=(\bigoplus_{\beta\in[\alpha]}\mathcal{L}_{\bar{0},\beta})\oplus(\bigoplus_{\beta\in[\alpha]}\mathcal{L}_{\bar{1},\beta}).$$

Finally, we denote by $\mathcal{L}_{[\alpha]}$ the direct sum of the two graded subspaces above, that is,

$$\mathcal{L}_{[\alpha]}:=\mathcal{H}_{[\alpha]}\oplus\mathcal{V}_{[\alpha]}.$$

Proposition 3.4. *For any $\alpha \in \Lambda$, the graded subspace $\mathcal{L}_{[\alpha]}$ is a graded subalgebra of $\mathcal{L}$.*

Proof. First, we are going to check that $\mathcal{L}_{[\alpha]}$ satisfies $[\mathcal{L}_{[\alpha]}, \mathcal{L}_{[\alpha]}] \subset \mathcal{L}_{[\alpha]}$. By the fact $\mathcal{H}_{[\alpha]} \subset \mathcal{L}_0 = \mathcal{H}$ and Eq. (3.1), it is clear that $[\mathcal{H}_{[\alpha]}, \mathcal{H}_{[\alpha]}] = 0$, and we have

$$\begin{aligned} [\mathcal{L}_{[\alpha]}, \mathcal{L}_{[\alpha]}] &= [\mathcal{H}_{[\alpha]} \oplus \mathcal{V}_{[\alpha]}, \mathcal{H}_{[\alpha]} \oplus \mathcal{V}_{[\alpha]}] \\ &\subset [\mathcal{H}_{[\alpha]}, \mathcal{V}_{[\alpha]}] + [\mathcal{V}_{[\alpha]}, \mathcal{H}_{[\alpha]}] + [\mathcal{V}_{[\alpha]}, \mathcal{V}_{[\alpha]}]. \end{aligned} \tag{3.2}$$

Let us consider the first summand in (3.2). For $\beta \in [\alpha], \bar{i} \in \mathbb{Z}_2$, by Lemmas 2.5-(1) and3.2-(1), one gets $[\mathcal{H}_{[\alpha]}, \mathcal{L}_{\bar{i},\beta}] \subset \mathcal{L}_{\bar{i},\delta\beta\phi^{-1}}$, where $\delta\beta\phi^{-1} \in [\alpha]$. Hence,

$$[\mathcal{H}_{[\alpha]}, \mathcal{V}_{[\alpha]}] \subset \mathcal{V}_{[\alpha]}. \tag{3.3}$$

Similarly, we can also get

$$[\mathcal{V}_{[\alpha]}, \mathcal{H}_{[\alpha]}] \subset \mathcal{V}_{[\alpha]}. \tag{3.4}$$

Consider now the third summand in (3.2). Given $\beta, \gamma \in [\alpha]$ and $\bar{i}, \bar{j} \in \mathbb{Z}_2$ such that $0 \neq [\mathcal{L}_{\bar{i},\beta}, \mathcal{L}_{\bar{j},\gamma}]$. If $\gamma = -\beta$, we have

$$[\mathcal{L}_{\bar{i},\beta}, \mathcal{L}_{\bar{j},\gamma}] = [\mathcal{L}_{\bar{i},\beta}, \mathcal{L}_{\bar{j},-\beta}] = [\mathcal{L}_{\beta}, \subset \mathcal{H}_{[\alpha]}.$$

Suppose that $0 \neq \beta + \gamma$, taking into account the fact $0 \neq [\mathcal{L}_{\bar{i},\beta}, \mathcal{L}_{\bar{j},\gamma}]$ together with Lemma 2.5-(2), one gets $\delta(\beta+\gamma)\phi^{-1} \in \Lambda$, therefore, $\{\beta, \gamma\}$ a connection from β to $\delta(\beta+\gamma)\phi^{-1}$. The equivalence relation $\sim$ gives us $(\beta+\gamma)\phi^{-1} \in [\alpha]$ and so $[\mathcal{L}_{\bar{i},\beta}, \mathcal{L}_{\bar{j},\gamma}] \subset \mathcal{L}_{\bar{i}+\bar{j},\delta(\beta+\gamma)\phi^{-1}} \subset \mathcal{V}_{[\alpha]}$. Hence,

$$[\mathcal{V}_{[\alpha]}, \mathcal{V}_{[\alpha]}] = [\bigoplus_{\beta\in[\alpha]} \mathcal{L}_{\bar{i},\beta}, \bigoplus_{\beta\in[\alpha]} \mathcal{L}_{\bar{j},\beta}] \subset \mathcal{H}_{[\alpha]} \oplus \mathcal{V}_{[\alpha]}. \tag{3.5}$$

From Eqs. (3.3), (3.4), and (3.5), we conclude that $[\mathcal{L}_{[\alpha]}, \mathcal{L}_{[\alpha]}] \subset \mathcal{L}_{[\alpha]}$.

Second, we have to verify that $\phi(\mathcal{L}_{[\alpha]}) = \mathcal{L}_{[\alpha]}$. But this is a direct consequence of Lemma 2.5-(1) and Lemma 3.2-(1). □

Proposition 3.5. *If $[\alpha] \neq [\beta]$, then $\mathcal{L}_{[\alpha]}, \mathcal{L}_{[\beta]}] = 0$.*

Proof. We have

$$\begin{aligned} [\mathcal{L}_{[\alpha]}, \mathcal{L}_{[\beta]}] &= [\mathcal{H}_{[\alpha]} \oplus \mathcal{V}_{[\alpha]}, \mathcal{H}_{[\beta]} \oplus \mathcal{V}_{[\beta]}] \\ &\subset [\mathcal{H}_{[\alpha]}, \mathcal{V}_{[\beta]}] + [\mathcal{V}_{[\alpha]}, \mathcal{H}_{[\beta]}] + [\mathcal{V}_{[\alpha]}, \mathcal{V}_{[\beta]}]. \end{aligned} \tag{3.6}$$

Let us consider the third summand in Eq. (3.6) and suppose there exist $\eta \in [\alpha]$ and $\mu \in [\beta]$ and $\bar{i}, \bar{j} \in \mathbb{Z}_2$ such that $0 \neq [\mathcal{L}_{\bar{i},\eta}, \mathcal{L}_{\bar{j},\mu}]$. By condition $[\alpha] \neq [\beta]$, one gets $\eta \neq -\mu$, then $\delta(\eta+\mu)\phi^{-1} \in \Lambda$. Hence, $\{\eta, \mu, -\delta\eta\phi^{-1}\}$ is a connection from η to μ. By the transitivity of $\sim$, we have $\alpha \in [\beta]$, which is a contradiction. Therefor, $[\mathcal{L}_{\bar{i},\eta}, \mathcal{L}_{\bar{j},\mu}] = 0$ and so $[\bigoplus_{\eta\in[\alpha]} \mathcal{L}_{\bar{i},\eta}, \bigoplus_{\mu\in[\beta]} \mathcal{L}_{\bar{j},[\beta]}] = 0$. Hence,

$$[\mathcal{V}_{[\alpha]}, \mathcal{V}_{[\beta]}] = \{0\}. \tag{3.7}$$

Consider now the first summand in Eq. (3.6) and suppose there exist $\eta \in [\alpha]$ and $\mu \in [\beta], \bar{i} \in \mathbb{Z}_2$ such that $0 \neq [[\mathcal{L}_{\eta}, \mathcal{L}_{-\eta}], \mathcal{L}_{\bar{i},\mu}]$. By Lemma 2.5-(1) we have $\mathcal{L}_{\bar{i},\mu} = \phi(\mathcal{L}_{\bar{i},\mu\phi})$, so we obtain $0 \neq [[\mathcal{L}_{\eta}, \mathcal{L}_{-\eta}], \phi(\mathcal{L}_{\bar{i},\mu\phi})]$. Using the grading of $\mathcal{L}_{\eta}$, we get

$$[[\mathcal{L}_{\bar{0},\eta}, \mathcal{L}_{-\eta}], \phi(\mathcal{L}_{\bar{i},\mu\phi})] + [[\mathcal{L}_{\bar{1},\eta}, \mathcal{L}_{-\eta}], \phi(\mathcal{L}_{\bar{i},\mu\phi})] \neq 0.$$

therefore,

$$\text{either } [[\mathcal{L}_{\bar{0},\eta}, \mathcal{L}_{-\eta}], \phi(\mathcal{L}_{\bar{i},\mu\phi})] \neq 0 \ \text{ or } [[\mathcal{L}_{\bar{1},\eta}, \mathcal{L}_{-\eta}], \phi(\mathcal{L}_{\bar{i},\mu\phi})] \neq 0.$$

In the first case, by the δ-super Hom-jacobi identity, we get either $[\mathcal{L}_{-\eta}, \phi(\mathcal{L}_{\bar{i},\mu\phi})] \neq 0$ or $[\mathcal{L}_{\bar{i},\mu\phi}, \mathcal{L}_{\bar{0},\eta}] \neq 0$. From here $[\mathcal{V}_{[\alpha]}, \mathcal{V}_{[\beta]}] \neq 0$ in any case, what contradicts Eq. (3.7). In the second case we have

$$[[\mathcal{L}_{\bar{1},\eta}, \mathcal{L}_{\bar{0},-\eta}], \phi(\mathcal{L}_{\bar{i},\mu\phi})] + [[\mathcal{L}_{\bar{1},\eta}, \mathcal{L}_{\bar{1},-\eta}], \phi(\mathcal{L}_{\bar{i},\mu\phi})] \neq 0,$$

and so

$$\text{either } [[\mathcal{L}_{\bar{1},\eta}, \mathcal{L}_{\bar{1},-\eta}], \phi(\mathcal{L}_{\bar{i},\mu\phi})] \neq 0 \quad \text{or } [[\mathcal{L}_{\bar{1},\eta}, \mathcal{L}_{\bar{1},-\eta}], \phi(\mathcal{L}_{\bar{i},\mu\phi})] \neq 0.$$

Again by the δ-super Hom-jacobi identity to any of these summends we have as above that necessarity $[\mathcal{V}_{[\alpha]}, \mathcal{V}_{[\beta]}] \neq 0$ what contradicts Eq. (3.7). Hence,

$$[\mathcal{H}_{[\alpha]}, \mathcal{V}_{[\beta]}] = 0. \tag{3.8}$$

In a similar way, we also have

$$[\mathcal{V}_{[\alpha]}, \mathcal{H}_{[\beta]}] = 0. \tag{3.9}$$

Finally, from Eqs. (3.7), (3.8), and (3.9), we conclude that $[\mathcal{L}_{[\alpha]}, \mathcal{L}_{[\beta]}] = 0$. □

Theorem 3.6. *The following assertions hold*

(1) For any $\alpha \in \Lambda$, the graded subalgebra

$$\mathcal{L}_{[\alpha]} = \mathcal{H}_{[\alpha]} \oplus \mathcal{V}_{[\alpha]},$$

of $\mathcal{L}$ associated to $[\alpha]$ is a (graded) ideal of $\mathcal{L}$.

(2) If $\mathcal{L}$ is simple, then there exists a connection from α to β for any $\alpha, \beta \in \Lambda$ and $\mathcal{H} = \sum_{\alpha\in\Lambda}[\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}]$.

Proof.
(1) Since $[\mathcal{L}_{[\alpha]}, \mathcal{H}] = [\mathcal{L}_{[\alpha]}, \mathcal{L}_0] \subset \mathcal{L}_{[\alpha]}$, taking into account Proposition 3.4 and Proposition 3.5, we have

$$[I_{[\alpha]}, \mathcal{L}] = [\mathcal{L}_{[\alpha]}, \mathcal{H} \oplus (\bigoplus_{\beta\in[\alpha]} \mathcal{L}_\beta) \oplus (\bigoplus_{\gamma\notin[\alpha]} \mathcal{L}_\gamma)] \subset I_{[\alpha]}.$$

As we also have by Proposition 3.4 that $\phi(\mathcal{L}_{[\alpha]}) = \mathcal{L}_{[\alpha]}$, we conclude that $\mathcal{L}_{[\alpha]}$ is an ideal of $\mathcal{L}$.
(2) The simplicity of $\mathcal{L}$ implies $I_{[\alpha]} = \mathcal{L}$. From here, it is clear that $[\alpha] = \Lambda$ and $\mathcal{H} = \sum_{\alpha\in\Lambda}[\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}]$.
□

Theorem 3.7. *For a vector superspace complement $\mathcal{U}$ of span$_{\mathbb{F}}\{[\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}] \ : \ \alpha \in \Lambda\}$ in $\mathcal{H}$, we have*

$$\mathcal{L} = \mathcal{U} \oplus \sum_{[\alpha]\in\Lambda/\sim} \mathcal{L}_{[\alpha]},$$

where any $\mathcal{L}_{[\alpha]}$ is one of the (graded) ideals of $\mathcal{L}$ described in Theorem 3.6-(1), satisfying $[\mathcal{L}_{[\alpha]}, \mathcal{L}_{[\beta]}] = 0$, whenever $[\alpha] \neq [\beta]$.

Proof. Each $\mathcal{L}_{[\alpha]}$ is well defined and by Theorem 3.6-(1), an ideal of $\mathcal{L}$. It is clear that

$$\mathcal{L} = \mathcal{H} \oplus (\bigoplus_{\alpha\in\Lambda} \mathcal{L}_\alpha) = \mathcal{U} \oplus \sum_{[\alpha]\in\Lambda/\sim} \mathcal{L}_{[\alpha]}.$$

Finally, Proposition 3.5 gives us $[\mathcal{L}_{[\alpha]}, \mathcal{L}_{[\beta]}] = \{0\}$, if $[\alpha] \neq [\beta]$. □

Let us denote by $Z(\mathcal{L})$ the center of $\mathcal{L}$, that is, $Z(\mathcal{L}) = \{x \in \mathcal{L} \ : \ [x, \mathcal{L}] = 0\}$.

Definition 3.8. *A δ-Hom-Jordan-Lie superalgebra $\mathcal{L}$ is called* perfect *if $Z(\mathcal{L}) = 0$ and $[\mathcal{L}, \mathcal{L}] = \mathcal{L}$.*

Corollary 3.9. *If $\mathcal{L}$ is a perfect split regular δ-Hom-Jordan-Lie superalgebra, then $\mathcal{L}$ is the direct sum of the ideals given in Theorem 3.6-(1),*

$$\mathcal{L} = \bigoplus_{[\alpha]\in\Lambda/\sim} \mathcal{L}_{[\alpha]}.$$

Proof. From $[\mathcal{L}, \mathcal{L}] = \mathcal{L}$, it is clear that $\mathcal{L} = \sum_{[\alpha]\in\Lambda/\sim} \mathcal{L}_{[\alpha]}$. Now, by $Z(\mathcal{L}) = 0$ and Proposition 3.5, the direct character of the sum is clear. □

4. the simple components

In this section, we are intersted in studying under which conditions a split regular δ-Hom-Jordan-Lie superalgebra $\mathcal{L}$ decompose as the direct sum of its simole graded ideals. From now on we will assume Λ is symmetric.

Lemma 4.1. *Let $\mathcal{L} = \mathcal{H} \oplus (\bigoplus_{\alpha\in\Lambda} \mathcal{L}_\alpha)$ be a ssplit regular δ-Hom-Jordan-Lie superalgebra. If I is an ideal of $\mathcal{L}$ such that $I \subset \mathcal{H}$, then $I \subset Z(\mathcal{L})$.*

Proof. It is clear that the assersion is a consequence of $[I, \mathcal{H}] \subset [\mathcal{H}, \mathcal{H}] = 0$ and $[I, \bigoplus_{\alpha\in\Lambda} \mathcal{L}_\alpha] \subset (\bigoplus_{\alpha\in\Lambda} \mathcal{L}_\alpha) \cap \mathcal{H} = 0$. □

Taking into account the above lemma, observe that the grading of I together with Lemma 2.4-(2), allows us to assert that

$$I = I_{\bar{0}} \oplus I_{\bar{1}} = (I_{\bar{0}} \cap \mathcal{H}_{\bar{0}}) \oplus (\bigoplus_{\alpha\in\Lambda}(I_{\bar{0}} \cap \mathcal{L}_{\bar{0},\alpha})) \oplus (I_{\bar{1}} \cap \mathcal{H}_{\bar{1}}) \oplus (\bigoplus_{\alpha\in\Lambda}(I_{\bar{1}} \cap \mathcal{L}_{\bar{1},\alpha})). \tag{4.1}$$

Lemma 4.2. *(1) For any $\alpha, \beta \in \Lambda$ with $\alpha \neq \beta$ there exists $h_{\bar{0}} \in \mathcal{H}_{\bar{0}}$ such that $\alpha(h_{\bar{0}}) \neq 0$ and $\alpha(h_{\bar{0}}) \neq \beta(h_{\bar{0}})$.*

(2) If I is an ideal of $\mathcal{L}$ and $x = h + \sum_{i=1}^{n} x_{\alpha_i} \in I$ with $h \in \mathcal{H}, x_{\alpha_i} \in \mathcal{L}_{\alpha_i}$ and $\alpha_i \neq \alpha_j$. Then any $x_{\alpha_i} \in I$.

Proof. *See Lemmas 5. 2 and 5. 3 in [6].* □

Let us introduce the concepts of root-multiplicativity and maximal length in the framework of split regular δ-Hom-Jordan-Lie superalgebra, in a similar way to the ones for split regular δ-Hom-Lie algebra in [6].

Definition 4.3. *We say that a split regular δ-Hom-Jordan-Lie superalgebra $\mathcal{L}$ is* root-multiplicative *if given $\alpha \in \Lambda_{\bar{i}}$ and $\beta \in \Lambda_{\bar{j}}$, for $\bar{i}, \bar{j} \in \mathbb{Z}_2$, such that $\delta(\alpha + \beta)\phi^{-1} \in \Lambda_{\bar{i}+\bar{j}}$, then $[\mathcal{L}_{\bar{i},\alpha}, \mathcal{L}_{\bar{j},\beta}] \neq 0$.*

Definition 4.4. *We say that a split regular δ-Hom-Jordan-Lie superalgebra $\mathcal{L}$ is of* maximal length *if for any $\alpha \in \Lambda_{\bar{i}}$ with $\bar{i} \in \mathbb{Z}_2$, we have $\dim \mathcal{L}_{\bar{i},\alpha} \in \{0, 1\}$.*

Observe that for a split regular δ-Hom-Jordan-Lie superalgebra $\mathcal{L}$ of maximal length, Eq. (4.1) allows us assert that given any nonzero graded ideal I of $\mathcal{L}$ we can write

$$I = ((I_{\bar{0}} \cap \mathcal{H}_{\bar{0}}) \oplus (\bigoplus_{\alpha\in\Lambda^I_{\bar{0}}} \mathcal{L}_\alpha)) \oplus ((I_{\bar{1}} \cap \mathcal{H}_{\bar{1}}) \oplus (\bigoplus_{\alpha\in\Lambda^I_{\bar{1}}} \mathcal{L}_\alpha)), \tag{4.2}$$

where $\Lambda^I_{\bar{i}} := \{\alpha \in \Lambda_{\bar{i}} : I_{\bar{i}} \cap \mathcal{L}_{\bar{i},\alpha} \neq 0\}$ for each $\bar{i} \in \mathbb{Z}_2$.

Theorem 4.5. *Let $\mathcal{L}$ be a centerless split regular δ-Hom-Jordan-Lie superalgebra of maximal length and root-multiplicative. Then $\mathcal{L}$ is simple if and only if $\mathcal{L}$ has all of its nonzero roots connected and $\mathcal{H} = \sum_{\alpha\in\Lambda}[\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}]$.*

Proof. The necessary implication is Theorem 3.6-(2). To prove the convers, consider any nonzero ideal I of $\mathcal{L}$, by Eq. (4.2) and Lemma 4.2, we have

$$I = ((I_{\bar{0}} \cap \mathcal{H}_{\bar{0}}) \oplus (\bigoplus_{\alpha\in\Lambda^I_{\bar{0}}} \mathcal{L}_\alpha)) \oplus ((I_{\bar{1}} \cap \mathcal{H}_{\bar{1}}) \oplus (\bigoplus_{\alpha\in\Lambda^I_{\bar{1}}} \mathcal{L}_\alpha)),$$

with $\Lambda^I_{\bar{i}} \subset \Lambda$, for each $\bar{i} \in \mathbb{Z}_2$. and some $\Lambda^I_{\bar{i}} \neq \phi$. Let us fix some $\alpha_0 \in \Lambda^I_{\bar{i}}$ so that

$$0 \neq \mathcal{L}_{\bar{i},\alpha_0} \subset I. \tag{4.3}$$

Thus

$$\{\mathcal{L}_{\bar{i},\alpha_0\phi^z} : z \in \mathbb{Z}\} \subset I. \tag{4.4}$$

The fact that $\phi(I) = I$ together with Lemma 2.5-(2) allows us to assert that

$$\text{If } \ \alpha \in \Lambda_{\bar{i}}^I \ \text{ then } \ \{\alpha\phi^z \ : \ z \in \mathbb{Z}\} \subset \Lambda_{\bar{i}}^I. \tag{4.5}$$

Now, let us take any $\beta \in \Lambda$ satisfying $\beta \notin \{\pm\alpha_0\phi^z \ : \ z \in \mathbb{Z}\}$. Since α_0 and β are connected, we have a connection $\{\alpha_1, \alpha_2, ..., \alpha_k\}, \ k \geq 2$ from α_0 to β satisfying the following conditions;

$$\begin{aligned}
&\alpha_1 = \alpha_0\phi^{-n} \ \text{ for some } n \in \mathbb{N}_0, \text{ and}\\
&\delta\alpha_1\phi^{-1} + \delta\alpha_2\phi^{-1} \in \Lambda,\\
&\delta^2\alpha_1\phi^{-2} + \delta^2\alpha_2\phi^{-2} + \delta\alpha_3\phi^{-1} \in \Lambda,\\
&\delta^3\alpha_1\phi^{-3} + \delta^3\alpha_2\phi^{-3} + \delta^2\alpha_3\phi^{-2} + \delta\alpha_4\phi^{-1} \in \Lambda,\\
&...\\
&\alpha_\phi^{-i} + \alpha_2\phi^{-i} + \alpha_3\phi^{-i+1} + ... + \alpha_{i+1}\phi^{-1} \in \Pi,\\
&...\\
&\delta^i\alpha_\phi^{-i} + \delta^i\alpha_2\phi^{-i} + \delta^{i-1}\alpha_3\phi^{-i+1} + ... + \delta\alpha_{i+1}\phi^{-1} \in \Lambda,\\
&...\\
&\delta^{k-2}\alpha_1\phi^{-k+2} + \delta^{k-2}\alpha_2\phi^{-k+2} + \delta^{k-3}\alpha_3\phi^{-k+3} + ... + \delta^{k-i}\alpha_i\phi^{-k+i} + ... + \delta\alpha_{k-1}\phi^{-1} \in \Lambda.\\
&\delta^{k-1}\alpha_1\phi^{-k+1} + \delta^{k-1}\alpha_2\phi^{-k+1} + \delta^{k-2}\alpha_3\phi^{-k+2} + ... + \delta^{k-i+1}\alpha_i\phi^{-k+i-1} + ... + \delta\alpha_k\phi^{-1} \in \{\pm\beta\phi^{-m} :\\
&m \in \mathbb{N}_0\}.
\end{aligned}$$

Consider $\alpha_1 = \alpha_0 \in \Lambda_{\bar{i}}$. Since $\alpha_2 \in \Lambda$ it follows $\mathcal{L}_{\bar{j},\alpha_2} \neq 0$, for some $\bar{j} \in \mathbb{Z}_2$, and so $\alpha_2 \in \Lambda_{\bar{j}}$. We have $\alpha_1 \in \Lambda_{\bar{i}}$ and $\alpha_2 \in \Lambda_{\bar{j}}$ such that $\delta(\alpha_1 + \alpha_2)\phi^{-1} \in \Lambda_{\bar{i}+\bar{j}}$ by Lemma 2.5-(2). From here, the root-multiplicativity and maximal length of $\mathcal{L}$ allow us to get

$$0 \neq [\mathcal{L}_{\bar{i},\alpha_1}, \mathcal{L}_{\bar{j},\alpha_2}] = \mathcal{L}_{\bar{i}+\bar{j},\delta(\alpha_1+\alpha_2)\phi^{-1}}.$$

Since $0 \neq \mathcal{L}_{\bar{i},\alpha_1} \subset I$ as consequence of Eq. (4.3), we have

$$0 \neq \mathcal{L}_{\bar{i}+\bar{j},\delta(\alpha_1+\alpha_2)\phi^{-1}} \subset I.$$

We can argue in a similar way from $\delta(\alpha_1 + \alpha_2)\phi^{-1}, \ \alpha_3$ and $\delta^2\alpha_1\phi^{-1} + \delta^2\alpha_2\phi^{-1} + \delta\alpha_3\phi^{-1}$ to get

$$0 \neq \mathcal{L}_{\bar{k},\delta^2\alpha_1\phi^{-1}+\delta^2\alpha_2\phi^{-1})+\delta\alpha_3\phi^{-1}} \subset I, \quad \text{for some } \bar{k} \in \mathbb{Z}_2.$$

If we follow this process with the connection $\{\alpha_1, \alpha_2, ..., \alpha_k\}$, then we obtain that

$$0 \neq \mathcal{L}_{tr,\delta^{k-1}\alpha_1\phi^{-k+1}+\delta^{k-1}\alpha_2\phi^{-k+1}+\delta^{k-2}\alpha_3\phi^{-k+2}+...+\delta^{k-i+1}\alpha_i\phi^{-k+i-1}+...+\delta\alpha_k\phi^{-1}} \subset I,$$

for some $\bar{r} \in \mathbb{Z}_2$ and so

$$\text{either } 0 \neq \mathcal{L}_{\bar{r},\beta\phi^{-m}} \subset I \ \text{ or } 0 \neq \mathcal{L}_{\bar{r},-\beta\phi^{-m}} \subset I,$$

for any $\beta \in \Lambda \setminus \{\pm\alpha_0\phi^z : z \in \mathbb{Z}\}$ and some $m \in \mathbb{N}_0$. Now taking into account Eqs. (4.4) and (4.5), we get

$$\text{either } \{\mathcal{L}_{\bar{i},\alpha\phi^z} : z \in \mathbb{Z}\} \subset I \ \text{ or } \{\mathcal{L}_{\bar{i},-\alpha\phi^{-z}:z\in\mathbb{Z}\}} \subset I, \tag{4.6}$$

for any $\alpha \in \Lambda_{\bar{i}}$ and a fixed $\bar{i} \in \mathbb{Z}_2$. For given any $\alpha \in \Lambda$, Eq. (4.6) can be reformulated by asserting that

$$\text{either } \{\alpha\phi^z : z \in \mathbb{Z}\} \subset \Lambda_{\bar{i}}^I \ \text{ or } \{-\alpha\phi^{-z} : z \in \mathbb{Z}\} \subset \Lambda_{\bar{i}}^I. \tag{4.7}$$

Now, we consider two cases;

Case 1. If $\Lambda_{\bar{0}} \cap \Lambda_{\bar{1}} = \emptyset$. Eq. (4.7) shows that for any $\alpha \in \Lambda = \Lambda_{\bar{0}} \cup \Lambda_{\bar{1}}$ we have

$$\mathcal{L}_{\epsilon\alpha} \subset I, \quad \text{for some } \ \epsilon \in \{\pm 1\}. \tag{4.8}$$

Taking into account $\mathcal{H} = \sum_{\alpha\in\Lambda}[\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}]$ we obtain

$$\mathcal{H} \subset I. \tag{4.9}$$

Now, for any $\alpha \in \Lambda$, the facts $\delta\alpha\phi \neq 0, \mathcal{H} \subset I$ and the maximal length of $\mathcal{L}$ show that

$$[\mathcal{H}_{\bar{0}}, \mathcal{L}_{\delta\alpha\phi}] = \mathcal{L}_\alpha \subset I. \tag{4.10}$$

From Eqs. (4.8)-(4.10) we conclud $I = \mathcal{L}$.

Case 2. If $\Lambda_{\bar{0}} \cap \Lambda_{\bar{1}} \neq \emptyset$. We first claim;

$$\text{If there exists some } \ \alpha \in \Lambda_{\bar{0}} \cap \Lambda_{\bar{1}} \ \text{ satisfying } \ \mathcal{L}_{\bar{0},\alpha} \oplus \mathcal{L}_{\bar{1},\alpha} \subset I \ \text{ then } \ I = \mathcal{L}. \tag{4.11}$$

Indeed, the fact that $\mathcal{L}$ has all its nonzero roots connected, from $\alpha \in \Lambda_{\bar{0}}$ we get that for any $\beta \in \Lambda\setminus\{\pm\alpha\}$, we have $\mathcal{L}_{\bar{r},\epsilon\beta} \subset I$ for some $\epsilon \in \{\pm 1\}$ and some $\bar{r} \in \mathbb{Z}_2$. Similarly, from $\alpha \in \Lambda_{\bar{1}}$ we obtain $0 \neq \mathcal{L}_{\bar{R}+\bar{1},\epsilon\beta} \subset I$. So $\mathcal{L}_{\epsilon\beta} \subset I$. From here, the experession of $\mathcal{H}$ gives us

$$\mathcal{H} \subset I. \tag{4.12}$$

Given now any $\gamma \in \Lambda$, the facts $\gamma \neq 0$, Eq. (4.12) and the maximal length of $\mathcal{L}$ show that

$$[\mathcal{H}_{\bar{0}}, \mathcal{L}_{\bar{0},\gamma}] = \mathcal{L}_{\bar{0},\gamma} \subset I, \ \forall \bar{i} \in \mathbb{Z}_2.$$

Hence,

$$[\mathcal{H}_{\bar{0}}, \mathcal{L}_\gamma] = \mathcal{L}_\gamma \subset I. \tag{4.13}$$

From Eqs. (4.12) and (4.13) we conclud $I = \mathcal{L}$.

Second, for any $\alpha \in \Lambda$ such that $\alpha \in \Lambda_{\bar{0}} \cap \Lambda_{\bar{1}}$, by Eq. (4.7) we have $\epsilon\alpha\phi^{-m} \in \Lambda$, for certain $\epsilon \in \{\pm 1\}$ and some $m \in \mathbb{N}_0$, then $0 \neq [\mathcal{H}_{\bar{0}}, \mathcal{L}_{\bar{r},\epsilon\alpha\phi^{-m}}]$ for fixed $\bar{r} \in \mathbb{Z}_2$. Since $\mathcal{H}_{\bar{0}} = \sum_{\gamma\in\Lambda}([\mathcal{L}_{\bar{0},\gamma}, \mathcal{L}_{\bar{0},-\gamma}] + [\mathcal{L}_{\bar{1},\gamma}, \mathcal{L}_{\bar{1},-\gamma}])$, there exists $\gamma \in \Lambda$ such that

$$\text{either } 0 \neq [[\mathcal{L}_{\bar{0},\gamma}, \mathcal{L}_{\bar{0},-\gamma}], \mathcal{L}_{\bar{r},\epsilon\alpha\phi^{-m}}] \ \text{ or } 0 \neq [[\mathcal{L}_{\bar{1},\gamma}, \mathcal{L}_{\bar{1},-\gamma}], \mathcal{L}_{\bar{r},\epsilon\alpha\phi^{-m}}]. \tag{4.14}$$

Now, by Lemma 2.4-(1) we have $\phi(\mathcal{L}_{\bar{r},\epsilon\alpha\phi^{-m+1}}) = \mathcal{L}_{\bar{r},\epsilon\alpha\phi^{-m}}$ and so

$$0 \neq [[\mathcal{L}_{\bar{j},\gamma}, \mathcal{L}_{\bar{j},-\gamma}], \phi(\mathcal{L}_{\bar{r},\epsilon\alpha\phi^{-m+1}})], \tag{4.15}$$

for some $\bar{j} \in \mathbb{Z}_2$ and fixed $\bar{r} \in \mathbb{Z}_2, \ m \in \mathbb{N}_0$. By the δ-super Hom-jacobi identity either $0 \neq [[\mathcal{L}_{\bar{j},\gamma},]$ or $[\mathcal{L}_{-\alpha}^{-\eta}, \mathcal{L}_{\bar{r},\epsilon\alpha\phi^{-m+1}}]$ or $0 \neq [\mathcal{L}_{\bar{j},-\gamma}, \mathcal{L}_{\bar{r},\epsilon\alpha\phi^{-m+1}}]$ and so by the maximal length of $\mathcal{L}$ either $0 \neq \mathcal{L}_{\bar{j}+\bar{r},\epsilon\alpha\phi^{-m+1}+\gamma}$ or $0 \neq \mathcal{L}_{\bar{j}+\bar{r},\epsilon\alpha\phi^{-m+1}-\gamma}$. That is,

$$0 \neq \mathcal{L}_{\bar{j}+\bar{r},\epsilon\alpha\phi^{-m+1}+\kappa\gamma} \subset I, \tag{4.16}$$

for some $\kappa \in \{\pm 1\}$. From here there are two possibilities, if $0 \neq \mathcal{L}_{\bar{j}+\bar{1},-\kappa\gamma}$, by the root-multiplicativity and maximal length of $\mathcal{L}$, we obtain

$$0 \neq [\mathcal{L}_{\bar{j}+\bar{r},\epsilon\alpha\phi^{-m+1}+\kappa\gamma}, \mathcal{L}_{\bar{j}+\bar{1},-\kappa\gamma}] = \mathcal{L}_{\bar{1}\bar{r},\epsilon\alpha\phi^{-m+1}} \subset I. \tag{4.17}$$

Taking into account Eq. (4.7), and Eq. (4.11) give us $\mathcal{L} = I$. If $0 = \mathcal{L}_{\bar{j}+\bar{1},-\kappa\gamma}$, as by semmetry of Λ, $-\epsilon\alpha\phi^{-m} \in \Lambda$ and by Lemma 2.5-(3) we also obtain $-\epsilon\alpha\phi^{-m+1} \in \Lambda$. So $0 \neq \mathcal{L}_{\bar{k},-\epsilon\alpha\phi^{-m+1}}$ for some $\bar{k} \in \mathbb{Z}_2$. By the root-multiplicativity and maximal length of $\mathcal{L}$, we obtain

$$0 \neq [\mathcal{L}_{\bar{j}+\bar{r},\epsilon\alpha\phi^{-m+1}+\kappa\gamma}, \mathcal{L}_{\bar{k},-\epsilon\alpha\phi^{-m+1}}] = \mathcal{L}_{\bar{j}+\bar{r}+\bar{k},\kappa\gamma} \subset I. \tag{4.18}$$

Now, if $\bar{r} + \bar{k} = \bar{0}$, taking into account Eq. (4.15) gives us

$$0 \neq [[\mathcal{L}_{\bar{j},\gamma}, \mathcal{L}_{\bar{j},-\gamma}], \mathcal{L}_{\bar{r},\epsilon\alpha\phi^{-m}}].$$

The fact that $\alpha \in \Lambda_{\bar{0}} \cap \Lambda_{\bar{1}}$ and Eq. (4.18) let us assert

$$0 \neq [[\mathcal{L}_{\bar{j},\gamma}, \mathcal{L}_{\bar{j},-\gamma}], \mathcal{L}_{\bar{m},\alpha}] = \mathcal{L}_{\bar{m},\alpha} \subset I. \tag{4.19}$$

If $\bar{r} + \bar{k} = \bar{1}$, taking into account Eq. (4.15), the root-multiplicativity and maximal length of $\mathcal{L}$, that

$$0 \neq [\mathcal{L}_{\bar{j}+\bar{1},\kappa\gamma}, \mathcal{L}_{\bar{r},\epsilon\alpha\phi^{-m+1}}] = \mathcal{L}_{\bar{j}+\bar{r}+\bar{1},\kappa\gamma+\epsilon\alpha\phi^{-m+1}} \subset I. \tag{4.20}$$

From here Eqs. (4.11) and (4.16) give us $\mathcal{L} = I$. Hnce, $\mathcal{L}$ is simple. $\square$

Theorem 4.6. *Let $\mathcal{L}$ be a centerless split regular δ-Hom-Jordan-Lie superalgebra of maximal length and root-multiplicative and $\mathcal{H} = \sum_{\alpha\in\Lambda}[\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}]$. Then*

$$\mathcal{L} = \bigoplus_{[\alpha]\in\Lambda/\sim} \mathcal{L}_{[\alpha]},$$

where any $\mathcal{L}_{[\alpha]}$ is a simple split regular δ-Hom-Jordan-Lie superalgebra having all its nonzero roots connected.

Proof. By Corollary 3.9, we can write $\mathcal{L} = \bigoplus_{[\alpha]\in\Lambda/\sim} \mathcal{L}_{[\alpha]}$ as direct sum of the family of ideals

$$\mathcal{L}_{[\alpha]} = \mathcal{H}_{[\alpha]} \oplus \mathcal{V}_{[\alpha]} = \sum_{\beta\in[\alpha]} [\mathcal{L}_\beta, \mathcal{L}_\beta] \oplus (\bigoplus_{\beta\in[\alpha]} \mathcal{L}_\beta),$$

where each $\mathcal{L}_{[\alpha]}$ is a split regular δ-Hom-Jordan-Lie superalgebra having as root system $\Lambda_{\mathcal{L}_{[\alpha]}} = [\alpha]$, with all of its root connected. Taking into account the fact $[\alpha] = -[\alpha]$ and $\mathcal{L}_{[\alpha]}$ is a graded subalgebra of $\mathcal{L}$, we easily deduce that $[\alpha]$ has all of its root connected through roots in $[\alpha]$. We also get that any of the $\mathcal{L}_{[\alpha]}$ is root-multiplicative as consequence of the root-multiplicativity of $\mathcal{L}$. Clearly, $\mathcal{L}_{[\alpha]}$ is of maximal length, and finally $Z_{\mathcal{L}_{[\alpha]}}(\mathcal{L}_{[\alpha]}) = 0$, as consequence of Lemma 3.5, Theorem 4.5, and $Z(\mathcal{L}) = 0$. We can therefore apply Theorem 4.5 to any $\mathcal{L}_{[\alpha]}$ so as to conclude that $\mathcal{L}_{[\alpha]}$ is simple. It is clear that the decomposition $\mathcal{L} = \bigoplus_{[\alpha]\in\Lambda/\sim} \mathcal{L}_{[\alpha]}$ satisfies the assertions of the theorem. $\square$

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